Application of Differential Quadrature for Modeling Solitary Wave: Numerical Solution of KDV Equation

Debabrata Datta
Radiological Physics & Advisory Division
Bhabha Atomic Research Centre
Homi Bhabha National Institute,
Mumbai - 400085

Seema Jagtap
Civil Engineering Department
Thakur College of Engineering
Mumbai – 400001

Ugandhara Gaikwad
Civil Engineering Department
Thakur College of Engineering
Mumbai – 400001

ABSTRACT
Protection of near shore area by means of artificial structure is an important issue for coastal engineering communityA solitary wave is a wave which propagates without any temporal evolution in shape or size when viewed in the reference frame moving with the group velocity of the wave. The envelope of the wave has one global peak and decays far away from the peak solution of Korteweg de Vries (KdV) equation provides this solitary wave and the numerical solution of this equation is developed using differential quadrature which is an innovative numerical technique. Differential quadrature basically approximates partial derivatives of any order. Time derivative of KdV equation is discredited using classic finite difference method and space derivates are discredited using differential quadrature technique. KdV equation which is third order non linear partial differential equations, describe behavior of travelling wave, known as solution. Stability of numerical analysis is evaluated by computing L_2 norm and L_∞. Application of solitonic solutions are highlighted in the paper. Differential quadrature based numerical scheme is explored in detail in this paper.

Keywords
KdV equation, differential quadrature, stability

1. INTRODUCTION
The theory of models of waves over shallow water presents a paradoxical feature: while it is easy to write down in closed form a solitary wave solution for the simplest standard model, namely the Korteweg–de Vries (KdV) equation, it has proved quite difficult to obtain the existence of such solutions for the problems from which the KdV equation was derived as a first approximation. A solution is a nonlinear solitary wave with the additional property that the wave retains its permanent structure, even after interacting with another solution. For example, two solutions propagating in opposite directions effectively pass through each other without breaking. Solutions form a special class of solutions of model equations, including the Kortewegde-Vries (KdV) and the Nonlinear Schrodinger (NLS) equation. Solitary waves were first observed, long time ago by Russel[1]. Russel’s observation of long wave propagation on shallow water has been explained by Korteweg and deVries by forming a third order non linear partial differential equation, known as KdV equation. Wave solutions of such a non-linear partial differential equation(KdV equation) is known as solitary waves that possess the following properties, viz. (1) they proceed with their shapes interact, and (2) they move with constant velocity. From a detailed numerical study Zabusky and Kruskal[2] found that stable pulse like waves could exist in a system described by the KdV equation. A remarkable quality of these solitary waves as mentioned before yields their particle-like nature and led Zabusky and Kruskal to name such waves as solitons. The discovery of the remarkable interaction properties of solitary wave solutions toKdV by Zabusky and Kruskal [2] and the invention by Gardner, Greene, Kruskal and Miura [3] of the Inverse Spectral Transform for the solution of the Cauchy problem for KdV stand as two of the most far-reaching breakthroughs in the development of modern nonlinear mathematical science.As the prototypical integrable nonlinear system, KdV has also had enormous indirect impact on many parts of theoretical physics, pure mathematics, and the areas in between. Vast areas of mathematics, including ordinary differential equations, algebraic geometry, Lie group theory, differential geometry and asymptotics have been opened up 'on the back', as it were, of the solving of KdV, and brought to bear on issues in quantum field theory, string and conformal field theory, quantum gravity and classical general relatively, to say nothing of the myriad applications in concrete settings of other famous integrable systems including nonlinear Schr6dinger (NLS) and sine-Gordon (SG).These latter applications range from condensed matter and semiconductor physics through nonlinear optics and laser physics, hydrodynamics, meteorology and plasma physics to protein systems and neurophysiology. The first success of the soliton conceptwas explaining the recurrence in the Fermi-Pasta-Ulam system [4].Largely free of contention, however, are Russell's own experiments on solitary waves in water channels in which, as is now well known, he not only correctly extracted the propagation velocity from measurements, and convincingly showed that solitary waves of depression are impossible and that an arbitrary initial elevation would break up into a finite number of solitary waves, but also that the interaction between solitary waves had the particle-like property which was not picked up for more than a further century.KdV theory comes into play in several areas of geophysics. In conduit flows, buoyant fluid introduced below a layer of fluid of greater viscosity rises through a conduit which it creates, with buoyancy and viscous shear stress in balance for steady flow in a conduit of uniform area. If the supply rate varies, axisymmetric bulges propagate upward as conduit waves.The KdV equation represents the dynamic model of solitary waves. With a view to the large number of applications of the KdV equation in the field of engineering science, solving these equations numerically by non-traditional numerical technique (other than finite

difference, finite element and finite volume method) is a challenging task. Hence, fast innovative numerical method of solution of KdV equation is developed in this paper. The numerical solution of the KdV equation is developed using differential quadrature. Computations are carried out by developing a computer code using MATLAB 7.0 because of its simplicity in data computing and 3D visualization of data. The paper presents the detail of the differential quadrature. Methodology of solving KdV equation numerically using differential quadrature is presented in detail in this paper.

2. MATHEMATICAL BACKGROUND OF DIFFERENTIAL QUADRATURE

The differential quadrature is a numerical technique used to solve the initial and boundary value problems. This method was originally developed by Bellman and Casti[5] using a simple analogy with integral quadrature which is based on interpolation functions. The basic philosophy behind differential quadrature method (DQM) is the concept that the partial derivative of a field variable at any discrete points in the computational domain can be approximated by a weighted linear sum of the values of field variable along the line that passes through that point, which is parallel with coordinate direction of the derivative. Mathematical formulation of this philosophy can be framed as

$$C_x{}^m(x_i) = \frac{\partial^m C}{\partial x^m}\Big|_{x=x_i} = \sum_{k=1}^{N} A_{ik}^{(m)} C(x_k), \quad i = 1,2,\dots,N, \quad m = 1,2,\dots,N-1 \quad (1)$$

where x_i are the discrete points of the coordinate system, m is the order derivative of the function, $C(x_k)$ are the function values at those points and $A_{ik}^{(m)}$ are the weighting coefficients for the mothered derivative of the function with respect to x and N is the number of spatial grid points. Two points are worth to mention in the formulation of DQM which are (1) how the weighting coefficients are determined and (2) how the grid points are selected. There are many approaches to compute these weighting coefficients such as Bellman's approaches [6], Quan and Chang's approach [7,8] and Shu's approach [8]. Most of the differential quadrature methods address various test functions such as Legendre polynomials, Lagrange interpolation polynomials, spline functions, to compute the weighting coefficients. However, the most frequently used methods are based on Lagrange interpolation polynomials. Weighting coefficients of first order spatial derivative using Lagrange polynomials as the test function are given by

$$A_{ik}^{(1)} = \begin{cases} \frac{1}{L}\frac{M^{(1)}(x_i)}{(x_i - x_k)M^{(1)}(x_k)}, & \text{for } i,k = 1,\dots,N \\ -\sum_{k=1,\ i\neq k}^{N} A_{ik}^{(1)}, & \text{for } i = k \end{cases} \quad (2)$$

where, $M(x) = \prod_{k=1}^{N}(x - x_k)$, $M^{(1)}(x_i) = \prod_{k=1, i\neq k}^{N}(x_i - x_k)$ and L is the length interval. The weighting coefficients of the second order derivative can be computed using

$$\left[A_{ik}^{(2)}\right] = \left[A_{ik}^{(1)}\right]\left[A_{ik}^{(1)}\right] \quad (3)$$

Weighting coefficients of higher order derivatives can be computed using the generalized formula

$$A_{ik}^{(m)} = \begin{cases} m\left(A_{ii}^{(m-1)}A_{ik}^{(1)} - \frac{A_{ik}^{(m-1)}}{(x_i - x_k)}\right), & \text{for } i \neq k,\ i,k = 1,2,\dots,N \\ & 2 \leq m \leq N-1 \\ -\sum_{k=1. i\neq k}^{N} A_{ik}^{(m)}, & \text{for } i = k \end{cases} \quad (4)$$

It is worth to mention that weighting coefficients in DQM are Centro symmetric. A N x N matrix Q = [qij] is Centro symmetric if qij = -qN+1-I, N+1-j, i, j = 1, 2, .., N.

Q can be characterized by J Q J = Q, where J denotes the Centra identity matrix with the properties JT = J and J2 = 1. A new, skew Centro symmetric matrix R = [rij]N x N can be defined as rij= -rN+1-I,N+1-j, I , j = 1, 2, …,N and R = -J R J. In a similar manner we can write the weighting coefficients matrix elements, Aij = -AN+1-I, N+1-j.

3. SELECTION OF GRID POINTS AND STABILITY

The proper selection of grid points provides the accuracy as well as the stability of any numerical method. Generally, grid points are based on the zeros of suitable orthogonal polynomials. Keeping in view of this, the non-uniform grid points are chosen using Chebyshev-Gauss-Lobatto (CGL) algorithm and the formulation of the grid points as per CGL is given by

$$x_i = \frac{1}{2}L\left(1 - \cos\frac{(i-1)\pi}{N-1}\right), \text{ for } i = 1,2,\dots\dots,N \quad (5)$$

where, L = length of the domain = (b – a) for a ≤ x ≤ b.

Numerical Solution of KdV Equation

Governing equation of KdV equation is given by

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0,\ x \in \Omega = (a,b)\subset R, t > 0 \quad (6)$$

Initial condition of Eq. (6) is given by

$$u(x,0) = 0.5\,V\,\mathrm{sech}^2(0.5\,\sqrt{V}x)$$

Boundary conditions are given by

$$u(a,t) = 0.5\,V\mathrm{sech}^2(0.5\sqrt{V}(a - Vt))\ ,$$

$$u(b,t) = 0.5V\mathrm{sech}^2(0.5\sqrt{V}(b - Vt)).$$

Travelling wave solution (solitonic solution) of Eq. (6) can be obtained analytically by substitution of a new variable $\xi = x - Vt$, where V signifies the speed with which the wave travels. Analytical solution of Eq. (6) can be obviously written as

$$u(x,t) = 0.5\,V\mathrm{sech}^2[0.5\sqrt{V}(x - Vt)] \quad (7)$$

The differential quadrature version of KdV equation (Eq. (6)) can be now written as

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -6u_i^n \sum_{k=1}^{N} A_{ik}^{(1)} u_k^n - \sum_{k=1}^{N} A_{ik}^{(3)} u_k^n \quad (8)$$

The discretization of initial and boundary conditions can be subsequently written as

$$u_i^{n=1} = 0.5\,V\mathrm{sech}^2(0.5\,\sqrt{V}\,x_i),\ \text{where, } i = 1, 2, \ldots, N$$

$$u_{i=1}^{n} = 0.5\,V\mathrm{sech}^2(0.5\,\sqrt{V}(x_{i=1} - Vt_n))$$

$$u_{i=N}^{n} = 0.5\,V\mathrm{sech}^2(0.5\,\sqrt{V}(x_{i=N} - Vt_n))$$

4. NUMERICAL RESULTS AND ANALYSIS

Numerical results of KdV equation are obtained with a = -15 cm, b = 15 cm, T = 0.5 s, V = 0.02 cm/s. Total number of temporal points, M = 60 and total number of spatial points, N = 20 are taken into account for computation. Numerical values of spatial points (x) within the complete domain of length (L = b – a = 30 cm) are calculated by Eq. (5) and hence these values are given as (0.0,0.20,0.81,1.81,3.16,4.84,5.79,8.97,11.31,13.75,16.23, 18.67, 21.01, 23.19, 25.15, 26.83, 28.18, 29.18, 29.79, 30.00).Numerical solution of KdV equation with its analytical solution at various spatial locations for a specific time (T = 0.5 s) are shown in Fig. 1. X-axis of Fig. 1 is labeled with respect to spatial node number, say NN =1 (X1 = 0.0), NN = 2 (X2 = 0.20), NN = 3 (X3 = 0.81), NN = 4 (X4 = 1.81), … , NN = 20 (X20 = 30.0) . It can be easily mentioned that numerical results are in good agreement with exact solution. Errors (L2 and L∞) are computed as approximately zero. A 3D profile of numerical solution of solitary wave is shown in Fig. 2 wherein labeling of X axis is based on temporal node number (TT = 1 to 60) and labeling of Y axis are based on the spatial nodal number (N = 1 to 20).

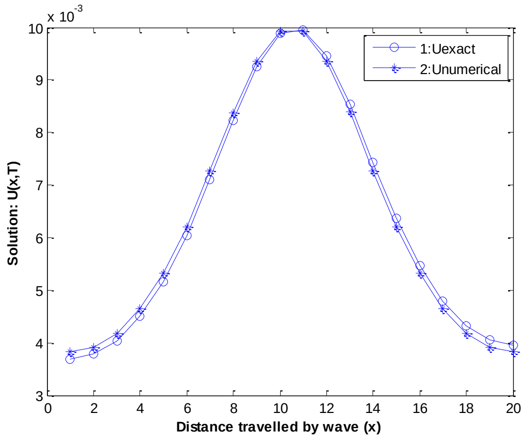

Fig. 1: Numerical and analytical solution of KdV equation at T = 0.5

5. APPLICATIONS TO WATER WAVES

A relationship between the phase speed and the wave number k is obtained by linearizing the governing equations of a physical system of interest, which yields a linearized dispersion relation.

For example, the linearized dispersion relation of surface water waves is given by

$$c^2/gh = [(1 + Bq^2)/q]\tan hq$$

where g is the acceleration due to gravity, h the constant depth of the water when it is unperturbed and q = kh the dimensionless wave number. The dimensionless number $B = \sigma/\rho g h^2$ is the Bond number, which measures the relative importance of surface tension and gravity, where σ the coefficient of surface is tension and ρ is the density of water. Solitary waves can exist provided that no real value of k satisfies the dispersion relation, meaning that khas a non-zero imaginary part. In the case of B= 0 (no surface tension),solitary waves can exist only when c2>gh. Solitary waves are found by reformulating the problem in the framework of a dynamical system.

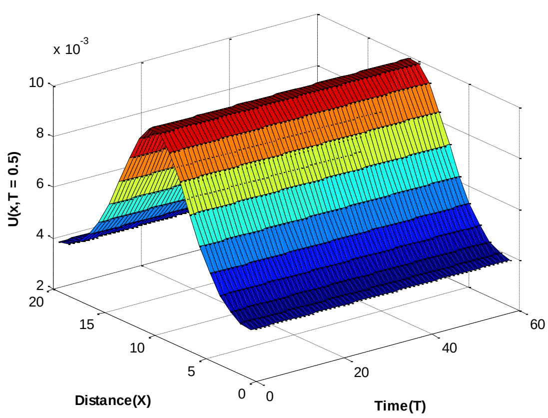

Fig. 1: Numerical and analytical solution of KdV equation at T = 0.5

6. CONCLUSIONS

Differential quadratutre based efficient innovative numerical method is developed to simulate KdV equation numerically. A detail of differential quadratutre based methodology of solving non linear partial differential equations has been discussed. Centro-symmetry property of the weighting coefficient matrix is also pointed out. Numerical stability of solutions is achieved by selecting spatial points on the basis of Chebyshev-Gauss-Lobatto algorithm, which is also known as Type-II sampling of collocation points. Numerical solutions are compared with exact solutions of the same equations by computing errors such as L2 and L∞. Results show that numerical and analytical solutions are in very good agreement, justifying that DQM can be a very good alternate computational scheme and efficient compared to traditional finite difference, finite element and finite volume element for solving non linear partial differential equations.

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