Testing Exponentiality against Exponential Better than Equilibrium Life in Convex based on Laplace Transformation

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ABSTRACT
This paper explores a new test statistic for testing exponentiality against exponential better than equilibrium life in convex (EBELC) class based on Laplace transformation. The selected critical values are tabulated for sample size 5(5)50. Pitman's asymptotic efficiencies of the test and Pitman's asymptotic relative efficiencies (PARE) are calculated. The powers of this test are estimated for some famous alternatives distributions in reliability such as Weibull, linear failure rate (LFR) and Gamma distributions. The problem in the case of right censored data is also touched. Finally, some applications to expound the usefulness of the proposed test in reliability analysis are discussed.

Keywords
Classes of life distributions; EBELC; Testing Exponentiality; U-statistic; Pitman asymptotic efficiency; censored data; Laplace transformation.

1. INTRODUCTION
Several classes of life distributions based on notions of ageing have been proposed and studies during the last few decades. The applications of classes of life distributions can be seen in reliability, engineering, biological science, maintenance and biometrics.

Note that the exponential distribution forms the backbone of statistical reliability theory and maintenance modeling see for example Barlow et.al. [1] and Zacks [2]. The most well-known classes of life distributions are IFR (increasing failure rate), IFRA (increasing failure rate in average), NBU (new better than used), NBUE (new better than used in expectation) and HNBUE (harmonic new better than used in expectation). For some properties and interrelationships of these criteria it is referred to Bryson and Siddiqui [3]. Cao and Wang [4] introduced a new class of life distribution named exponential better than equilibrium life in convex ordering (EBELC) and studied the relationship between this class and another classes.

Definition 1. A distribution function F with support $[0,\infty)$ and finite mean μ, where $\bar{F}=1-F$, is said to be EBELC (EWELC) if

$$\int_0^\infty \bar{\upsilon}(x+t)dx \le (\ge)\, \mu^2 e^{-\frac{t}{\mu}}, \qquad t \ge 0$$

Where

$\bar{\upsilon}(t)=\int_t^\infty \bar{F}(u)du$, and

$$\mu=\int_0^\infty \bar{F}(u)du.$$

Testing against EBELC based on moment inequalities has been studied by Abdul-Moniem [5], while testing based TTT-transform has been proposed by Abu-Youssef et. al. [6].

2. HYPOTHESIS TESTING PROBLEM AGAINST EBELC CLASS FOR NON-CENSORED DATA
Our goal in this section is to present a test statistic based on Laplace transformation for testing $H_0\colon F$ is exponential against an alternative that $H_1\colon F$ is belongs to EBELC class but not exponential.

The following lemma is essential for the development of our test statistic.

Lemma 1. If F belongs to EBELC class and X is a random variable with distribution function, then the measure of departure from the null hypothesis H_0 is $\Delta(s)>0$, where

$$\Delta(s)=\mu^3-\frac{1}{2}\mu\mu_2-\frac{1}{2s}\mu_2+\frac{1}{s}\mu^2+\frac{1}{s^2}\mu\zeta(s)+\frac{1}{s^3}\zeta(s)-\frac{1}{s^3} \qquad \textbf{(1)}$$

Where

$$\zeta(s)=\int_0^\infty e^{-sx}dF(x).$$

Proof. Recall Def.1, F is EBELC iff:

$$\int_t^\infty \bar{\upsilon}(x)dx \le \mu^2 e^{-\frac{t}{\mu}}, \qquad t\ge 0 \qquad (\mathbf{2})$$

Multiplying both sides in (2) by e^{-st}, and integrating over $[0,\infty)$ with respect to t, to get

$$\int_0^\infty\int_t^\infty \bar{\upsilon}(x)dx\ e^{-st}dt \le \mu^2\int_0^\infty e^{-st}\,e^{-\frac{t}{\mu}}dt, \qquad (\mathbf{3})$$

Setting

$$I=\int_0^\infty\int_t^\infty \bar{\upsilon}(x)dx\ e^{-st}dt,$$
$$=\frac{1}{s}\int_0^\infty \bar{\upsilon}(t)dt-\frac{1}{s}\int_0^\infty \bar{\upsilon}(t)e^{-st}dt\,,$$

where

$$\int_0^\infty \bar{\upsilon}(t)dt = \frac{1}{2}\mu_2,$$

and

$$\int_0^\infty \bar{\upsilon}(t)e^{-st}dt = \frac{1}{s^2}\left[\mu - \frac{1}{s}\left(1 - \zeta(s)\right)\right].$$

Therefore

$$I = \frac{1}{2s}\mu_2 - \frac{1}{s^2}\mu - \frac{1}{s^3}\zeta(s) + \frac{1}{s^3} \qquad (\mathbf{4})$$

Similarly

$$II = \mu^2 \int_0^\infty e^{-st}\, e^{-\frac{t}{\mu}}dt \; = \frac{\mu^3}{s\mu + 1} \qquad (\mathbf{5})$$

Substituting (4) and (5) into (3), the next equation is resulted:

$$\frac{1}{2}\mu\mu_2 + \frac{1}{2s}\mu_2 - \frac{1}{s}\mu^2 - \frac{1}{s^2}\mu\zeta(s) - \frac{1}{s^3}\zeta(s) + \frac{1}{s^3} \le \mu^3 \qquad (\mathbf{6})$$

To estimate the measure of departure from exponentiality $\delta(s)$, let $X_1, X_2, \dots, X_n$ be a random sample from a population with distribution function $F \in EBELC$ class. From Eq. (6), Eq. (1) could be calculated.

Not that under H_0: $\Delta(s) = 0$, and H_1: $\Delta(s)$ is positive.

2.1 Empirical Test Statistic for EBELC Alternative

The empirical estimate of $\Delta(s)$, can be rewritten as

$$\hat{\Delta}(s) = n^{-3}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\left[X_iX_jX_k - \frac{1}{2}X_iX_j{}^2 - \frac{1}{2s}X_i{}^2 + \frac{1}{s}X_iX_j + \frac{1}{s^2}X_ie^{-sX_j} + \frac{1}{s^3}e^{-sX_i} - \frac{1}{s^3}\right]$$

To make the test scale invariant under H_0, the following expression is used

$$\hat{\delta}(s) = \frac{\hat{\Delta}(s)}{\bar{X}^3}$$

where $\bar{X}^3 = \frac{1}{n^3}\sum_{i=1}^{n} X_i$ is the sample mean. Setting

$$\phi(X_1, X_2, X_3) = X_1X_2X_3 - \frac{1}{2}X_1X_2{}^2 - \frac{1}{2s}X_1{}^2 + \frac{1}{s}X_1X_2 + \frac{1}{s^2}X_1e^{-sX_2} + \frac{1}{s^3}e^{-sX_1} - \frac{1}{s^3}, \qquad (\mathbf{7})$$

and defining symmetric kernel

$$\psi(X_1, X_2,, X_3) = \frac{1}{3!}\sum \phi(X_1, X_2,, X_3),$$

where the summation is over all arrangements of $X_1, X_2, \dots, X_n$, then $\hat{\delta}(s)$ is equivalent to U-statistic

$$U_n^{(3)} = \frac{1}{\binom{n}{3}}\sum \phi(X_1, X_2,, X_3).$$

The following lemma summarizes the asymptotic properties of the test.

Lemma 2. $As\, n \to \infty$, $\sqrt{n}[\hat{\delta}(s) - \Delta(s)]$ is asymptotically normal with mean zero and variance

$$\sigma^2(s) = Var\left\{3X\mu^2 - \frac{1}{2}X\mu_2 - \frac{1}{2}\mu X^2 - \frac{1}{2}\mu\mu_2 - \frac{1}{2s}X^2 - \frac{1}{s}\mu_2 + \frac{2}{s}X\mu + \frac{1}{s}\mu^2 + e^{-sX}\left[\frac{\mu}{s^2} + \frac{1}{s^3}\right] + \zeta(s)\left[\frac{X}{s^2} + \frac{\mu}{s^2} + \frac{2}{s^3}\right] - \frac{3}{s^3}\right\} \qquad (\mathbf{8})$$

Under H_0 the variance tends to

$$\sigma_0^2(s) = \frac{10 + s(16 + s(9 + 2s))}{(1 + s)^2(1 + 2s)} \qquad (\mathbf{9})$$

Proof.

Using standard U-statistics theory, see Lee [7], and by direct calculations the mean and the variance can be found as follows:

$$\sigma^2(s) = var\{E[\phi^{(1)}(X_1, X_2, X_3)] + E[\phi^{(2)}(X_1, X_2,, X_3)] + E[\phi^{(3)}(X_1, X_2,, X_3)]\}, \qquad (\mathbf{10})$$

Recall definition of $\phi(X_1, X_2, X_3)$ in Eq. (7), thus it is easy to show that

$$E[\phi^{(1)}(X_1, X_2, X_3)] = X\mu^2 - \frac{1}{2}X\mu_2 - \frac{1}{2s}X^2 + \frac{1}{s}X\mu + \frac{1}{s^2}X\zeta(s) + \frac{1}{s^3}e^{-sX} - \frac{1}{s^3}, \qquad (\mathbf{11})$$

$$E[\phi^{(2)}(X_1, X_2, X_3)] = X\mu^2 - \frac{1}{2}\mu X^2 - \frac{1}{2s}\mu_2 + \frac{1}{s}X\mu + \frac{1}{s^2}\mu e^{-sX} + \frac{1}{s^3}\zeta(s) - \frac{1}{s^3}, \qquad (\mathbf{12})$$

and

$$E[\phi^{(3)}(X_1, X_2, X_3)] = X\mu^2 - \frac{1}{2}\mu\mu_2 - \frac{1}{2s}\mu_2 + \frac{1}{s}\mu^2 + \frac{1}{s^2}\mu\zeta(s) + \frac{1}{s^3}\zeta(s) - \frac{1}{s^3}, \qquad (\mathbf{13})$$

Upon using (10), (11), (12) and (13) Eq. (8) is obtained.

Under H_0, (9) is obtained.

2.2 The Pitman Asymptotic Relative Efficiency

To access the quality of the test, Pitman asymptotic efficiencies (PAEs) are computed and compared with an old test for the following alternative:

i. The Weibull Family:

$$\bar{F}_1(x) = e^{-x^\theta}, \qquad x \ge 0, \theta \ge 1.$$

ii. The Linear Failure Rate Family:

$$\bar{F}_2(x) = e^{-x-\frac{\theta}{2}x^2}, \qquad x \ge 0, \theta \ge 0.$$

iii. The Makeham Family:

$$\bar{F}_3(x) = e^{[-x-\theta(x+e^{-x}+1)]}, \qquad x \ge 0, \theta \ge 0.$$

Note that for $\theta = 1, F_1$ goes to exponential distribution and for $\theta = 0, F_2$and F_3 reduce to the exponential distributions. The PAE is defined by

$$PAE\left(\hat{\delta}(s)\right) = \frac{1}{\sigma_0}\left[\frac{d}{d\theta}\Delta(s)\right]_{\theta \to \theta_0},$$

when $s = 0.55$, this leads to:

Pitman asymptotic efficiency of $\hat{\delta}(0.55)$

Efficiency	Weibull	LFR	Makeham
$\frac{1}{\sigma_0}[\frac{d}{d\theta}\delta(s))]_{\theta\to\theta_0}$	0.903	0.991	0.228

where $\sigma_0 = 2.08131$

The Pitman asymptotic relative efficiency (PARE) of our test $\hat{\delta}(s)$ comparing with δ, which represent the test statistic of EBELC class of life distribution based on moments inequalities, given in Abdul-Moniem [5] is calculated where:

$$PARE(T_1, T_2) = \frac{PAE(T_1)}{PAE(T_2)}.$$

Table 1. The asymptotic relative efficiencies of our test $\hat{\delta}(s)$

Test	Weibull	LFR	Makeham
PARE $(\hat{\delta}(s), \delta)$	1.142	1.044	1.152

It can be seen from Table 1 that our test statistic $\hat{\delta}(s)$ for EBELC is more effective than the other test.

3. MONTE CARLO NULL DISTRIBUTION CRITICAL POINTS

In this section, the upper percentile points of $\hat{\delta}(s)$ for $90\%, 95\%, 98\%$ and 99% are calculated based on 5000 simulated samples of sizes $n = 5(5)50$ and tabulated in Table 2.

Table 2. The upper percentile of $\hat{\delta}(s)$ with 5000 replications at $s = 0.55$

n	90%	95%	98%	99%
5	0.658369	0.968421	0.730337	0.744425
10	0.54724	0.589575	0.634248	0.649916
15	0.476897	0.525496	0.57004	0.596304
20	0.434967	0.484166	0.527666	0.560429
25	0.407672	0.458979	0.506581	0.534313
30	0.381872	0.434344	0.484093	0.503313
35	0.361822	0.406115	0.455381	0.484229
40	0.347605	0.387674	0.437475	0.46623
45	0.327163	0.372693	0.423803	0.449877
50	0.306556	0.350212	0.400525	0.429278

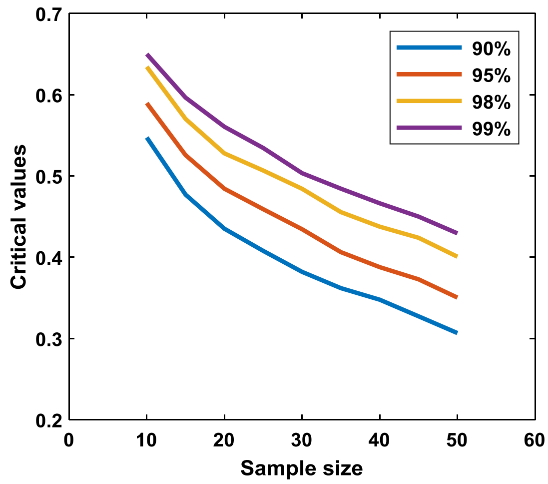

Fig 1: Relation between critical values, sample size and confidence levels.

It can be noticed from Table 2 and Fig.1 that the critical values are increasing as confidence level increasing and decreasing as the sample size increasing.

3.1 The Power Estimates

The power of proposed test will be estimated at $(1-\alpha)\%$ confidence level $\alpha = 0.05$ with suitable parameters values of θ at $n = 10{,}20$ and 30 for three commonly used distributions such as Weibull, linear failure rate and Gamma distributions based on 5000 simulated samples tabulated in Table 3.

Table 3. Power Estimates of the Statistic $\hat{\delta}(s)$ at $s = 0.55$

Distribution	Parameter θ	Sample size		
		$n = 10$	$n = 20$	$n = 30$
LFR	2	0.998	0.9998	0.9998
	3	0.9996	1.0000	1.0000
	4	0.9998	1.0000	1.0000
Weibull	2	1.0000	1.0000	1.0000
	3	1.0000	1.0000	1.0000
	4	1.0000	1.0000	1.0000
Gamma	2	0.9992	0.9998	1.0000
	3	1.0000	1.0000	1.0000
	4	1.0000	1.0000	1.0000

From Table 3, it is seen that our test $\hat{\delta}(s)$ has a good power for all alternatives.

3.2 Applications Using Complete (Uncensored) Data

Here, some of a good real example is presented to illustrate the use of our test statistics $\hat{\delta}(s)$ in the case of complete data at 95% confidence level.

Data-set #1.

Consider the data set given in Grubbs [8]. These data give the times between arrivals of 25 customers at a facility. It is easy to show that $\hat{\delta}(s) = 0.64335$ which is greater than the critical value of Table 2. Then H_1 the alternative hypotheses is accepted which show that the data set has EBELC property but not exponential.

Data-set #2.
Consider the data-set given in Edgman et. al.[9] which consist of 16 intervals in operating days between successive failures of air conditioning equipment in a Boeing 720 aircraft. In this case, the $\hat{\delta}(s) = 0.2708$ is gotten which is less than the critical value of the Table 2. Hence, the null hypothesis H_0 is accepted and rejecting the H_1. This means that this kind of data doesn't fit with EBELC property.

Data-set #3.
Consider the data set given in Ghazal et.al. [10]. These data give the daily average wind speed from 1/3/2015 to 30/3/2015 for Cairo city in Egypt. It is easy to show that $\hat{\delta}(s) = 0.5617$ which is greater than the critical value of Table 2. Then the null hypotheses H_0 is rejected and data set has EBELC property.

4. TESTING AGAINST EBELC CLASS FOR CENSORED DATA

A test statistic is proposed to test H_0versus H_1 in case of randomly right-censored (RR-C) data in many practical experiments; the censored data are the only information available in a life-testing model or in a clinical study where patients may be lost (censored) before the completion of a study. This experimental situation can formally be modeled as follows: Suppose n units are put on test, and $X_1, X_2, \dots, X_n$ denote their true-life time which are independent, identically distributed (i.i.d.) according to continuous life distribution F. Let $Y_1, Y_2, \dots, Y_n$ be (i.i.d.) according to continuous life distribution G. X's and Y's are assumed to be independent. In the RR-C model, it is observed that the pairs (Z_j, δ_j), $j = 1,2,\dots,n$ where $Z_j = \min(X_j, Y_j)$ and

$$\delta_j = \begin{cases} 1 & if\ Z_j = X_j\ (j - th\ observation\ in\ uncensored) \\ 0 & if\ \ Z_j = Y_j\ (j - th\ observation\ in\ censored) \end{cases}$$

Let $Z_{(0)} = 0 < Z_{(1)} < Z_{(2)} < \cdots < Z_{(n)}$ denote the ordered $Z's$ and $\delta_{(j)}$ is the δ_j corresponding to $Z_{(j)}$. Using censored data $(Z_j, \delta_j), j = 1,2,\dots,n$. Kaplan et al. [11] proposed the product limit estimator,

$$\bar{F}_n(X) = \prod_{[j:\, Z_{(j)} \le X]} \{(n-j)(n-j+1)\}^{\delta_{(j)}}, \quad X \in [0, Z_{(j)}]$$

Now, for testing $H_0\colon \delta(s) = 0$ against $H_1\colon \delta(s) > 0$, using randomly right censored data, the following test statistic is proposed.

$$\hat{\delta}_c(s) = \mu^2\left(\mu + \frac{1}{s}\right) - \frac{1}{2}\mu_2\left(\mu + \frac{1}{s}\right) + \frac{1}{s^2}\zeta(s)\left(\mu + \frac{1}{s}\right) - \frac{1}{s^3}$$

For computational purposes, $\hat{\delta}_c(s)$ may be rewritten as

$$\hat{\delta}_c(s) = (\Phi + \frac{1}{s})\left[\Phi^2 - \frac{\Omega}{2} + \frac{\Theta}{s^2}\right] - \frac{1}{s^3},$$

Where

$$\Phi = \sum_{k=1}^{n}\left[\prod_{m=1}^{k-1} C_m^{\delta(m)}(Z_{(k)} - Z_{(k-1)})\right],$$

$$\Omega = 2\sum_{i=1}^{n}\left[\prod_{v=1}^{i-1} Z_{(i)} C_v^{\delta(v)}(Z_{(i)} - Z_{(i-1)})\right],$$

$$\Theta = \sum_{j=1}^{n} e^{-sZ_{(j)}}\left[\prod_{p=1}^{j-2} C_p^{\delta(p)} - \prod_{p=1}^{j-1} C_p^{\delta(p)}\right],$$

and

$$dx = (Z_{(j)} - Z_{(j-1)}), \qquad C_k = [n-k][n-k+1]^{-1}$$

Table 4 below gives the critical values percentiles of $\hat{\delta}_c(s)$ test for sample size $n = 5(5)30(10)70,81,86$.

Table 4. The upper percentile of $\hat{\delta}_c(s)$with 5000 replications at $s = 0.55$

n	90%	95%	98%	99%
5	0.250676	0.447784	0.652736	0.757158
10	0.0720935	0.371555	0.618182	0.812133
15	0.0428752	0.307588	0.570415	0.721806
20	0.0422512	0.289145	0.513928	0.660979
25	0.0314949	0.278151	0.480782	0.595484
30	0.0420123	0.256704	0.462524	0.621272
40	0.0400774	0.241116	0.432907	0.548008
50	0.0401263	0.216483	0.398589	0.489938
60	0.0295117	0.201165	0.384771	0.48702
70	0.0137789	0.174083	0.3377	0.446166
81	0.00665103	0.148971	0.30592	0.417849
86	0.00767289	0.164127	0.301698	0.403169

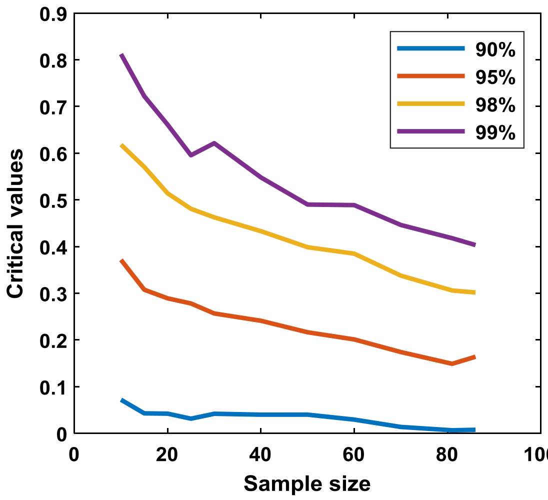

Fig 2: Relation between critical values, sample size and confidence levels.

From Table 4.and Fig 2 It can be observed that the critical values are increasing as confidence level increasing and decreasing as the sample size increasing.

4.1 The Power Estimates for $\hat{\delta}_c(s)$

The power of the statistic $\hat{\delta}_c(s)$ is considered at the significant level $\alpha = 0.05$ with suitable parameters values of θ at $n = 10,20$ and 30 for some commonly used distributions such as Weibull and linear failure rate distributions based on 5000 simulated samples tabulated in Table 5.

Table 5. Power Estimates of the Statistic $\hat{\delta}_c(s)$

Distribution	Parameter $\boldsymbol{\theta}$	Sample size		
		$\boldsymbol{n=10}$	$\boldsymbol{n=20}$	$\boldsymbol{n=30}$
LFR	2	1.0000	1.0000	1.0000
	3	1.0000	1.0000	1.0000
	4	1.0000	1.0000	1.0000
Weibull	2	0.999	0.9996	1.0000
	3	1.0000	1.0000	1.0000
	4	1.0000	1.0000	1.0000

From Table 5, it is seen that our test $\hat{\delta}_c(s)$ at $s = 0.55$ has a good power for all alternatives.

4.2 Applications for Censored Data

Two good real examples are presented to illustrate the use of our test statistics $\hat{\delta}_c(s)$ in case of censored data at 95% confidence level.

Data-set #4.

Consider the data-set in Susarla and Vanryzin [12]. These data represent 81 survival times of patients of melanoma. Out of these 46 represents whole times (non-censored data). The $\hat{\delta}_c(s) = -4.713 \times 10^{87}$ is gotten which is less than the tabulated value in Table 4. It is evident at the significant level $\alpha = 0.05$. This means that this kind of data doesn't fit with EBELC property.

Data-set #5.

Consider the data-set given in Pena [13] for lung cancer patients. These data consist of 86 survival times (in month) with 22 right censored. In this case, a $\hat{\delta}_c(s) = -1.679 \times 10^{86}$ is gotten which is less than the tabulated value in Table 5. Then, H_0 the null hypotheses are accepted which show that the data set has exponential property.

5. CONCLUSION

The EBELC is defined and a test statistic based on Laplace transformation for is presented. The Pitman asymptotic relative efficiencies (PARE) are calculated and it is noticed that the PAEs of our new test are better than an old test of the same class for all used alternatives. Monte Carlo null distribution critical points are simulated for sample size $n = 5(5)50$ and the power estimates of this test are also calculated for some common alternatives distribution followed by some numerical examples. The problem in case of right censored data is also handled and selected critical values are tabulated, the power estimates for censor data of this test are tabulated also the paper discussed some applications to educate the usefulness of the proposed test in reliability analysis for censored data.

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