Higher Order Moments, Spectral and Bispectral Density Functions for INAR(1)

M. M. Gabr
Alexandria University
Department of Mathematics
Faculty of Science

B. S. El-Desouky
Mansoura University
Department of Mathematics
Faculty of Science

F. A. Shiha
Mansoura University
Department of Mathematics
Faculty of Science

Shimaa M. El-Hadidy
Mansoura University
Department of Mathematics
Faculty of Science

ABSTRACT

In this paper, some higher order moments, spectral and bispectral density functions for some integer autoregressive of order one (INAR(1)) models are calculated. These models are the new skew INAR(1) (NSINAR(1)), the shifted geometric INAR(1) type-II (SGINAR(1)-II) and the dependent counting geometric INAR(1) (DCGINAR(1)). The spectrum, bispectrum and normalized bispectrum are estimated using the one and two dimensional lag windows as in Subba Rao and Gabr(1984). A realization is generated for each model of size n=500 for estimation. Also, the bispectral density function and normalized bispectral density function are used for studying the linearity of integer valued time series models.

Keywords

INAR(1); NSINAR(1); SGINAR(1)-II; DCGINAR(1); Moments; Cumulants; Spectrum; Bispectrum; Normalized bispectrum; Parzen lag window; 2-dimensional Subba Rao and Gabr lag window

1. INTRODUCTION

Integer valued time series used in many fields in life such as medicine, economics, etc,. . . . McKenzie (1985) introduced integer valued autoregressive (INAR) models by replacing the scalar multiplication in the standard AR by thinning operator. The thinning operator defined as a probabilistic operation that can be applied to random counts when the thinning operator deletes some of these counts, the size of the shrinked counts is still integer-valued so, INAR is the best model for modelling large counting values instead of approximating it into continuous-valued models. The first and most popular thinning operator is the binomial thinning operator that introduced by Steutel and Van Harn (1979) based on the sum of Bernoulli counting series. After that, there are many types of thinning operators appears such as geometric and negative binomial thinning operators, thinning operators with dependent structure, mixed thinning operator, operators acting on true integer autoregressive, etc. . . . Through the last three decades statisticians tried to made the INAR models more realistic and flexible for practical purposes of modelling the observed data, thus they made some modifications on INAR. Some of them, modify the marginal distribution of INAR, others, modify the order of the model and others modify the thinning operators of the models. Al-Osh and Al-Zaid (1987) defined the INAR(1) based on the binomial thinning operator as

$$X_t = \alpha \circ X_t + \varepsilon_t,\ \ t \in Z,$$

where the operator '$\circ$' is defined as $\alpha \circ X = \sum_{i=1}^{X} Y_i$, $\alpha \in (0, 1)$, $\{Y_i\}$ is a sequence of i.i.d Bernoulli random variables with probability α. $\{\varepsilon_t\}$ is a sequence of i.i.d non-negative integer valued random variables with mean μ and variance σ^2, thinning operations are performed independently of each other and of $\{\varepsilon_t\}$ and $\{X_s\}_{s<t}$. The probability generating function (pgf) of ε_t is given by

$$\Phi_\varepsilon(s) = \frac{\Phi_X(s)}{\Phi_X(1-\alpha+\alpha s)}.$$

The first four moments of ε_t using the pgf are given by
$E(\varepsilon_t) = \Phi'_{\varepsilon_t}(1)$,
$E(\varepsilon_t^2) = \Phi'_{\varepsilon_t}(1) + \Phi''_{\varepsilon_t}(1)$,
$E(\varepsilon_t^3) = \Phi'_{\varepsilon_t}(1) + 3\Phi''_{\varepsilon_t}(1) + \Phi'''_{\varepsilon_t}(1)$,
$E(\varepsilon_t^4) = \Phi'_{\varepsilon_t}(1) + 7\Phi''_{\varepsilon_t}(1) + 6\Phi'''_{\varepsilon_t}(1) + \Phi''''_{\varepsilon_t}(1)$.
If $\{X_t\}$ is stationary process to k-th order, then the k-th order joint moment of
$X_t,\ X_{t+s_1}, \cdots, X_{t+s_{k-1}}$ is a function of k-1 parameters defined by

$$\mu_X(s_1, s_2, ..., s_{k-1}) = E(X_t X_{t+s_1} ... X_{t+s_{k-1}}),$$

with

$$\mu = E(X_t).$$

The k-th order joint centeral moments are given by
$C_k(t_1, t_2, ..., t_{k-1}) = E[(X_t - \mu)(X_{t+t_1} - \mu)(X_{t+t_2} - \mu)...(X_{t+t_{k-1}} - \mu)]$.
Leonov and Shiryaev (1959) derived relations between joint moments and joint cumulants of a stationary time series

$$Cum(X_t) = E(X_t), \quad (1)$$

$$C_2(t_1) = \mu_{(t_1)} - \mu^2, \quad (2)$$

$$C_3(t_1, t_2) = E[(X_t - \mu)(X_{t+t_1} - \mu)(X_{t+t_2} - \mu)], \quad (3)$$

From (2) and (3) the second- and third-cumulants are the same as the second- and third-order central moments. The second and third order cumulants satisfy the following symmetry relations

$$C_2(t) = C_2(-t),$$

$$\begin{aligned} C_3(0,t) &= C_3(t,0) = C_3(-t,-t) \\ &= C_3(0,-t) = C_3(-t,0) = C_3(t,t), \end{aligned}$$

$$\begin{aligned} C_3(s,s+t) &= C_3(s+t,s) = C_3(-s,t) \\ &= C_3(t,-s) = C_3(-s-t,-t) = C_3(-t,-t-s), \end{aligned}$$

for $s \geq 0$ and $t \geq 0$.

1.1 The spectral and bispectral density functions

The spectral density (second-order spectrum) , f(.), of $\{X_t\}$ can be expressed as the Fourier transform of autocovariance function $(C_2(.))$,

$$f_X(\omega) = \frac{1}{2\pi}\sum_{t_1=-\infty}^{\infty} C_2(t_1)\ e^{-it_1\omega},\ -\pi \leq \omega \leq \pi, \tag{4}$$

where $\sum_{t_1=-\infty}^{\infty} \mid C_2(t_1) \mid < \infty$.
The bispectral and normalized bispectral density functions are given respectively as

$$f_X(\omega_1,\omega_2) = \frac{1}{(2\pi)^2}\sum_{t_1=-\infty}^{\infty}\sum_{t_2=-\infty}^{\infty} C_3(t_1,t_2)\ e^{-i(t_1\omega_1+t_2\omega_2)}, \tag{5}$$

$$g(\omega_1,\omega_2) = \frac{f(\omega_1,\omega_2)}{\sqrt{f(\omega_1)f(\omega_2)f(\omega_1+\omega_2)}}, \tag{6}$$

where $-\pi \leq \omega_1,\omega_2 \leq \pi$. The bispectral density function exists for all ω_1,ω_2 if
$\sum_{t_1=-\infty}^{\infty}\sum_{t_2=-\infty}^{\infty} |C_3(t_1,t_2)| < \infty.$
The bispectral density function is a complex valued function takes the form

$$f_X(\omega_1,\omega_2) = r(\omega_1,\omega_2) + iq(\omega_1,\omega_2).$$

The modulus and phase of the bispectral density function are given respectively, by

$$|f_X(\omega_1,\omega_2)| = (r^2(\omega_1,\omega_2) + q^2(\omega_1,\omega_2))^{\frac{1}{2}},$$

$$phase = tan^{-1}(\frac{q(\omega_1,\omega_2)}{r(\omega_1,\omega_2)}).$$

The bispectral density function provides us useful information about the non-linearity of the process. For continuous non-Gaussian time series the modulus of the normalized bispectrum is flat.

1.2 Estimation of spectrum

There are many methods for estimating the spectrum and bispectrum, but in this paper we interested in estimating the spectrum and bispectrum using the smoothed periodogram using the Parzen lag window. The smoothed spectral and bispectral density functions are respectively given by

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{2\pi}\sum_{s=-n}^{n}\lambda(\frac{s}{M})e^{-is\omega}\hat{C}_2(s) \\ &= \frac{1}{2\pi}\sum_{s=-n}^{n}\lambda(\frac{s}{M})\hat{C}_2(s)\cos\omega s, \end{aligned} \tag{7}$$

and

$$\begin{aligned} \hat{f}(\omega_1,\omega_2) &= \frac{1}{4\pi^2}\sum_{s_1=-(N-1)}^{N-1}\sum_{s_2=-(N-1)}^{N-1}\lambda(\frac{s_1}{M},\frac{s_2}{M}) \\ &\quad\times\hat{C}_3(s_1,s_2)e^{-is_1\omega_1-is_2\omega_2}, \end{aligned} \tag{8}$$

where $\hat{C}_2(s), \hat{C}_2(s_1,s_2)$ are given by

$$\begin{aligned} \hat{C}_2(s) &= \frac{1}{N-s}\sum_{t=1}^{N-|s|}(X_t-\bar{X})(X_{t+|s|}-\bar{X}), \\ \bar{X} &= \frac{1}{N}\sum_{t=1}^{N}X_t, \end{aligned}$$

$$\hat{C}_3(s_1,s_2) = \frac{1}{N}\sum_{t=1}^{N-\gamma}(X_t-\bar{X})(X_{t+s_1}-\bar{X})(X_{t+s_2}-\bar{X}),$$

where $s_1,s_2 \geq 0$, $\gamma = \max(s1,s2)$, $s = 0,\pm1,\pm2,\ldots,\pm(N-1)$, $-\pi \leq \omega_1,\omega_2 \leq \pi$, "$\lambda(.)$" is one-dimensional lag window and "$\lambda(\frac{s_1}{M},\frac{s_2}{M})$" is two-dimensional lag window [Parzen, Daniell, Tukey Hamming ,...]. In this paper, we use the Parzen lag window. Parzen (1961b) proposed the Parzen lag window

$$\lambda(s) = \begin{cases} 1-6s^2+6\,|s|^3 & |s| \leq \frac{1}{2} \\ 2(1-|s|)^3 & \frac{1}{2} < |s| \leq 1 \\ 0 & |s| > 1 \end{cases}, \tag{9}$$

and $\lambda(s_1,s_2)$ is given by

$$\lambda(s_1,s_2) = \lambda(s_1)\lambda(s_2)\lambda(s_1-s_2). \tag{10}$$

The normalized bispectrum is estimated by

$$\hat{g}(\omega_1,\omega_2) = \frac{\hat{f}(\omega_1,\omega_2)}{\sqrt{\hat{f}(\omega_1)\hat{f}(\omega_2)\hat{f}(\omega_1+\omega_2)}}. \tag{11}$$

for more details about the estimation of spectrum and lag windows, see [16]
This paper is organized as follows. In Section 2, some higher order joint moments of the NSINAR(1) model up to order three, the spectral and bispectral and normalized bispectral density functions are calculated. Moreover, the estimates for the spectral, bispectral and normalized bispectral density functions using a simulated realization of size n=500 are calculated. Also, the linearity of the model is investigated. In Section 3, the same measures are calculated for the SGINAR(1)-II model. In Section 4, the same measures are calculated for the DCGINAR(1) model.

2. THE NEW SKEW INAR(1) MODEL

Bourguignon and Vasconcellos (2016) introduced the NSINAR(1) process. This type of models are called the true integer autoregressive models and appear since needing for modelling and analysing count data with positive and negative values. This model is defined as the difference between Poisson INAR(1) model that introduced by Al-Osh and Al-Zaid (1987) and the NGINAR(1) model that introduced by Ristiĉ et al.(2009) see [1] and [11]. They defined the NSINAR(1) model as

$$Z_t = \alpha \star Z_{t-1} + \zeta_t, \tag{12}$$

where $\alpha \star Z_{t-1} \stackrel{d}{=} \alpha * X_{t-1} - \alpha \circ Y_{t-1}$ is the difference between the negative binomial thinning and binomial thinning operators, where the counting series $\alpha * X_{t-1}$ and $\alpha \circ Y_{t-1}$ are independent random variables. $\alpha * X = \sum_{i=0}^{X} W_i$ and $\alpha \circ Y = \sum_{i=0}^{Y} U_i$, where $\{W_i\}$ is a sequence of i.i.d geometric random variables and $\{U_i\}$ is a sequence of i.i.d Bernoulli random variables independent of $\{W_i\}$, $Z_t = X_t - Y_t$ where $X_t \sim$ geometric $(\frac{\mu}{1+\mu})$, $Y_t \sim$ Poisson(λ), $\{ Z_t\}$ be a sequence of random variables has a Geometric-Poisson (μ, λ) and ζ_t has the distribution of $\epsilon_t - \varepsilon_t$, where $\{\epsilon_t\}$ and $\{\varepsilon_t\}$ are independent r.v's and Z_t and ζ_{t-l} are independent for all $l \geq 1$. $\{\varepsilon_t\}$ are i.i.d r.v's with common poisson($\lambda(1-\alpha)$) distribution and ϵ_t is a mixture of two random variables with geometric $(\mu/(1+\mu))$ and geometric $(\alpha/(1+\alpha))$ distributions. The condition of the stationarity of the process $\{Z_t\}$ is $0 \leq \alpha \leq \mu/(1+\mu)$ and the condition of the non-stationarity of the process $\{Z_t\}$ is $\mu/(1+\mu) \leq \alpha \leq 1$. Here, our study is restricted to the stationary case.
The moment generating function of ζ_t and Z_t are given, respectively, by (see [5])

$$M_\zeta(s) = \frac{[1+\alpha(1+\mu)(1-e^s)]e^{[\lambda(1-\alpha)(e^{-s}-1)]}}{[1+\mu(1-e^s)][1+\alpha(1-e^s)]},$$

$$M_Z(s) = \frac{e^{\lambda(e^{-s}-1)}}{[1+\mu(1-e^s)]}.$$

The mean and variance of Z_t and ζ_t are then given by

$$\begin{aligned} \mu_Z &= \mu - \lambda,\ \sigma_Z^2 = \mu^2 + \mu + \lambda, \\ \mu_\zeta &= (1-\alpha)[\mu - \lambda], \\ \sigma_\zeta^2 &= (1+\alpha)\mu[(1-\alpha)(1+\mu) - \alpha] + \lambda(1-\alpha). \end{aligned}$$

The thinning operator has the following properties

(1) $E(\alpha \star Z) = \alpha(\mu - \lambda)$.
(2) $E(\alpha \star Z)^2 = \alpha^2(\mu-\lambda)^2 + \alpha\mu(1+2\alpha+\alpha\mu) + \alpha\lambda$.
(3) $E(\alpha \star Z)^2 = \alpha^2 E(Z^2) + \alpha(1+\alpha)E(X) + \alpha(1-\alpha)E(Y) = \alpha^2 E(Z^2) - \alpha^2 E(Z) + \alpha(\mu+\lambda)$.

2.1 The higher order joint central moments (cumulants)

THEOREM 1. *Let $\{Z_t\}$ be a stationary process satisfying (12) then,*
the second- order central moment is calculated as
$C_2(s) = \alpha^s(\mu^2 + \mu + \lambda) = \alpha^s C_2(0)$.
The third- order central moment are calculated as
$C_3(0,0) = 2\mu^3 + 3\mu^2 + \mu - \lambda$,
$C_3(0,s) = \alpha^s(2\mu^3 + 3\mu^2 + \mu - \lambda) = \alpha^s C_3(0,0)$,
$C_3(s,s) = \alpha^{2s} C_3(0,0) - \alpha C_2(0)\frac{\alpha^s - \alpha^{2s}}{(1-\alpha)}$.

PROOF. $\mu_{(s)}$ is calculated as
$\mu_{(s)} = E(Z_t Z_{t+s}) = E(Z_t(\alpha \star Z_{t+s-1} + \zeta_t))$
$= \alpha\mu_{(s-1)} + \mu_Z \mu_{\zeta_t} = \alpha^s \mu_{(0)} + \frac{1-\alpha^s}{1-\alpha}\mu_Z\mu_{\zeta_t}$
$= \alpha^s[2\mu^2 + \mu + \lambda^2 + \lambda - 2\mu\lambda] + (1-\alpha^s)(\mu-\lambda)^2$
$= \alpha^s[\mu^2 + \mu + \lambda] + (\mu - \lambda)^2$,
then, $C_2(s)$ is calculated as
$C_2(s) = \mu_{(s)} - \mu_Z^2$.
$C_3(0,0)$ is calculated as
$C_3(0,0) = \mu_{(0,0)} - 3\mu_Z\mu_{(0)} + 2\mu_Z^3$.
$\mu_{(s,s)}$ is calculated as
$\mu_{(s,s)} = E(Z_t Z_{t+s} Z_{t+s}) = E(Z_t[\alpha \star Z_{t+s-1} + \zeta_t]^2)$
$= \alpha^2 E(Z_t Z_{t+s-1} Z_{t+s-1}) - \alpha^2 E(Z_t Z_{t+s-1}) + \alpha(\mu+\lambda)E(Z_t) + 2\alpha E(Z_t Z_{t+s-1})E(\zeta_t) + E(\zeta_t^2)E(Z_t)$
$= \alpha^2\mu_{(s-1,s-1)} + [2\alpha\mu_\zeta - \alpha^2]\mu_{(s-1)} + \alpha(\mu+\lambda)\mu_Z + \mu_Z[\sigma_\zeta^2 + \mu_\zeta^2]$
$= \alpha^2\mu_{(s-1,s-1)} + [2\alpha\mu_\zeta - \alpha^2][\mu^2+\mu+\lambda]\alpha^{s-1} + (\mu-\lambda)^2[2\alpha\mu_\zeta - \alpha^2] + \alpha(\mu+\lambda)\mu_Z + \mu_Z[\sigma_\zeta^2 + \mu_\zeta^2]$
$= \alpha^{2s}\mu_{(0,0)} + [2\alpha\mu_\zeta - \alpha^2][\mu^2+\mu+\lambda]\sum_{i=0}^{s-1}\alpha^{s-(i+1)}\alpha^{2i} + [(\mu-\lambda)^2[2\alpha\mu_\zeta - \alpha^2] + \alpha(\mu+\lambda)\mu_Z + \mu_Z[\sigma_\zeta^2 + \mu_\zeta^2]]\sum_{j=0}^{s-1}\alpha^{2j}$
$= \alpha^{2s}\mu_{(0,0)} + ([2\alpha\mu_\zeta - \alpha^2][\mu^2+\mu+\lambda])\frac{\alpha^s - \alpha^{2s}}{\alpha(1-\alpha)} + [(\mu-\lambda)^2[2\alpha\mu_\zeta - \alpha^2] + \alpha(\mu+\lambda)\mu_Z + \mu_Z[\sigma_\zeta^2 + \mu_\zeta^2]]\frac{1-\alpha^{2s}}{1-\alpha^2}$,
then, $C_3(s,s)$ is calculated as
$C_3(s,s) = \mu_{(s,s)} - \mu[E(Z_{t+s}^2) + 2\mu_{(s)}] + 2\mu^3$
$= \alpha^2\mu_{(s-1,s-1)} - \alpha^2\mu_{(s-1)} + \alpha(\mu+\lambda)\mu_Z + 2\alpha\mu_{(s-1)}\mu_\zeta + \mu_Z[\sigma_\zeta^2 + \mu_\zeta^2] - 2\mu_Z[\alpha\mu_{(s-1)} + \mu_Z\mu_{\zeta_t}] - \mu_Z E((\alpha \star Z_{t+s-1} + \zeta_t)^2) + 2\mu^3$
$= \alpha^2\mu_{(s-1,s-1)} - \alpha^2\mu_{(s-1)} + \alpha(\mu+\lambda)\mu_Z + 2\alpha\mu_{(s-1)}\mu_\zeta + \mu_Z[\sigma_\zeta^2 + \mu_\zeta^2] - 2\mu_Z[\alpha\mu_{(s-1)} + \mu_Z\mu_{\zeta_t}] - \mu_Z[E(\alpha \star Z_{t+s-1})^2 + E(\zeta_t{}^2) + 2\alpha E(\zeta_t)E(Z_{t+s-1}) + 2\mu^3$ $=$
$\alpha^2\mu_{(s-1,s-1)} - \alpha^2\mu_{(s-1)} + \alpha(\mu+\lambda)\mu_Z + 2\alpha\mu_{(s-1)}\mu_\zeta + \mu_Z[\sigma_\zeta^2 + \mu_\zeta^2] - 2\mu_Z[\alpha\mu_{(s-1)} + \mu_Z\mu_{\zeta_t}] - \mu_Z[\sigma_\zeta^2 + \mu_\zeta^2] - 2\alpha\mu_Z^2\mu_\zeta - \mu_Z[\alpha^2 E(Z_{t+s-1}^2) - \alpha^2 E(Z_{t+s-1}) + \alpha(\mu+\lambda)] + 2\mu^3$
$= \alpha^2[\mu_{(s-1,s-1)} - \mu_Z E(Z_{t+s-1}^2) - 2\mu_Z\mu_{(s-1)} + 2\mu^3] - \alpha^2[\mu_{(s-1)} - \mu_Z^2] = \alpha^2 C_3(s-1,s-1) - \alpha^2 C_2(s-1)$
$= \alpha^2 C_3(s-1,s-1) - \alpha^2\alpha^{s-1}C_2(0)$
$= \alpha^{2s}C_3(0,0) - \alpha^2 C_2(0)\sum_{i=0}^{s-1}\alpha^{s-(i+1)}\alpha^{2i} = \alpha^{2s}C_3(0,0) - \alpha^2 C_2(0)\frac{\alpha^s - \alpha^{2s}}{\alpha(1-\alpha)}$.
$\mu_{(0,s)}$ is calculated as
$\mu_{(0,s)} = E(Z_t Z_t Z_{t+s}) = E(Z_t Z_t[\alpha \star Z_{t+s-1} + \zeta_t])$
$= \alpha E(Z_t Z_t Z_{t+s-1}) + E(Z_t^2)E(\zeta_t) = \alpha\mu_{(0,s-1)} + [2\mu^2 + \mu + \lambda^2 + \lambda - 2\mu\lambda][(1-\alpha)(\mu-\lambda)]$
$= \alpha^s\mu_{(0,0)} + \frac{1-\alpha^s}{1-\alpha}[2\mu^2 + \mu + \lambda^2 + \lambda - 2\mu\lambda][(1-\alpha)(\mu-\lambda)]$
$= \alpha^s[6\mu^3 - 6\lambda\mu^2 + 6\mu^2 + 3\mu\lambda^2 - 3\lambda^2 - \lambda^3 - \lambda + \mu] + (1-\alpha^s)(\mu-\lambda)[2\mu^2 + \mu + \lambda^2 + \lambda - 2\mu\lambda]$.
Then $C_3(0,s)$ is calculated as
$C_3(0,s) = \mu_{(0,s)} - \mu_Z[\mu_{(0)} + 2\mu_{(s)}] + 2\mu^3$
$= (\mu-\lambda)[2\mu^2 + \mu + \lambda^2 + \lambda - 2\mu\lambda] + \alpha^s\mu[1 + 5\mu - 2\lambda\mu + 4\mu^2] - \alpha^s\lambda[1+2\lambda] - (\mu-\lambda)[2\mu^2 + \mu + \lambda^2 + \lambda - 2\mu\lambda + 2(\alpha^s[\mu^2 + \mu + \lambda] + (\mu-\lambda)^2)] + 2(\mu-\lambda)^3$.
$\mu_{(s,\tau)}$ is calculated as
$\mu_{(s,\tau)} = \alpha^{\tau-s}\mu_{(s,s)} + (1-\alpha^{\tau-s})[\alpha^s[\mu^2+\mu+\lambda] + (\mu-\lambda)^2][\mu-\lambda]$.
$\mu_{(0)}$ and $\mu_{(0,0)}$ are given by Bourguignon and Vasconcellos (see [5]). ■

2.2 The spectral, bispectral and normalized bispctral density functions of the NSINAR(1)

The spectral density function is given by (see [5])

$$f_Z(\omega) = \frac{(1-\alpha^2)(\mu^2+\mu+\lambda)}{2\pi(1+\alpha^2 - 2\alpha\cos\omega)},\ -\pi \leq \omega \leq \pi. \tag{13}$$

THEOREM 2. *The bispectral density function is calculated as*

$$
\begin{aligned}
f_Z(\omega_1,\omega_2) \;=\; & \frac{1}{(2\pi)^2}[C_3(0,0)+C_3(0,0)\{h_1(-\omega_1)+h_1(-\omega_2) \\
& +h_1(\omega_1+\omega_2)\}+(C_3(0,0)+\frac{\alpha^2C_2(0)}{\alpha(1-\alpha)})\{h_2(\omega_1) \\
& +h_2(\omega_2)+h_2(-\omega_1-\omega_2)\} \\
& -\frac{\alpha^2C_2(0)}{\alpha(1-\alpha)}\{h_1(\omega_1)+h_1(\omega_2)+h_1(-\omega_1-\omega_2)\} \\
& +(C_3(0,0)+\frac{\alpha^2C_2(0)}{\alpha(1-\alpha)})\{h_2(-\omega_1-\omega_2)h_1(-\omega_2) \\
& +h_2(-\omega_1-\omega_2)h_1(-\omega_1)+h_2(\omega_2)h_1(-\omega_1) \\
& +h_2(\omega_1)h_1(-\omega_2)+h_2(\omega_1)h_1(\omega_1+\omega_2) \\
& +h_2(\omega_2)h_1(\omega_1+\omega_2)\} \\
& -\frac{\alpha^2C_2(0)}{\alpha(1-\alpha)}\{h_1(-\omega_1-\omega_2)h_1(-\omega_2) \\
& +h_1(-\omega_1-\omega_2)h_1(-\omega_1)+h_1(\omega_2)h_1(-\omega_1) \\
& +h_1(\omega_1)h_1(-\omega_2)+h_1(\omega_1)h_1(\omega_1+\omega_2) \\
& +h_1(\omega_2)h_1(\omega_1+\omega_2)\}], \qquad (14)
\end{aligned}
$$

where $h_1(\omega_k)=\frac{\alpha e^{i\omega_k}}{1-\alpha e^{i\omega_k}}$ *and* $h_2(\omega_k)=\frac{\alpha^2 e^{i\omega_k}}{1-\alpha^2 e^{i\omega_k}}$*, k=1,2.*

PROOF. The proof is too long to included it here. ■

The normalized bispectral density function is calculated by **(6),** where $f_Z(\omega)$ and $f_Z(\omega_1,\omega_2)$ are respectively given by (13) and (14) .

2.3 Estimation of spectrum

Estimates of the spectral, bispectral and normalized bispectral density functions are calculated using the smoothed periodogram method using Parzen lag window and simulated series from the NSINAR(1) model.

The theoretical spectrum $f_Z(\omega)$, theoretical bispectral and normalized bispectral modulus of $f_Z(\omega_1,\omega_2)$ and $g_Z(\omega_1,\omega_2)$ are respectively computed by setting $\alpha = .25, \mu = 5$ *and* $\lambda = 3$ in (13), (14) and (6). Fig. 1 represent the simulated series of $\{Z_t, t = 1,...,500\}$ from the NSINAR(1) model with $\alpha = .25, \lambda = 3$ and $\mu = 5$ that defined by (12). Fig. 2 represent theoretical spectrum and the estimate of the spectral density using Parzen window with M=7 as in (7) and (9). Fig. 3 and Fig. 4 represent the theoretical bispectrum and normalized bispectrum modulus. Fig. 5 and Fig. 6 represent the estimate of the bispectrum and normalized bispectrum modulus with M=7 by Parzen window using (8), (10) and (9).

List of Tables

It is clear that the values of the normalized bispectrum modulus given in Table 2 show that the normalized bispectrum modulus of the NSINAR(1) model is more flat than the non-normalized one given in Table 1 **.** This indicates that the test of linearity given by Subba Rao and Gabr (1980) and its modification given by Hinich (1982) can be used for integer valued time series models.

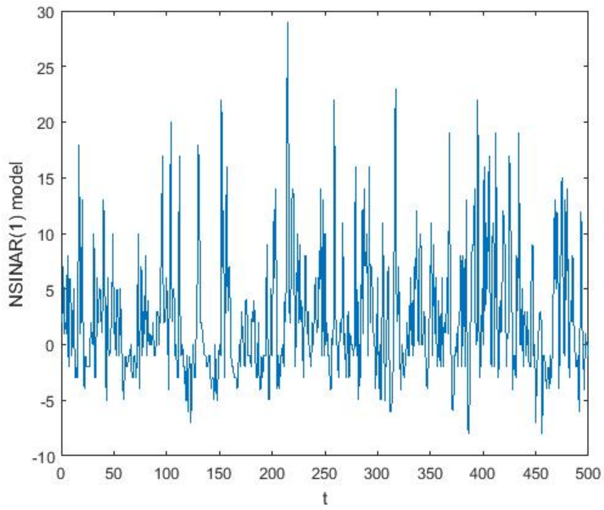

Fig. 1. Simulated series of NSINAR(1) model with $\alpha = .25, \lambda = 3$ and $\mu = 5$.

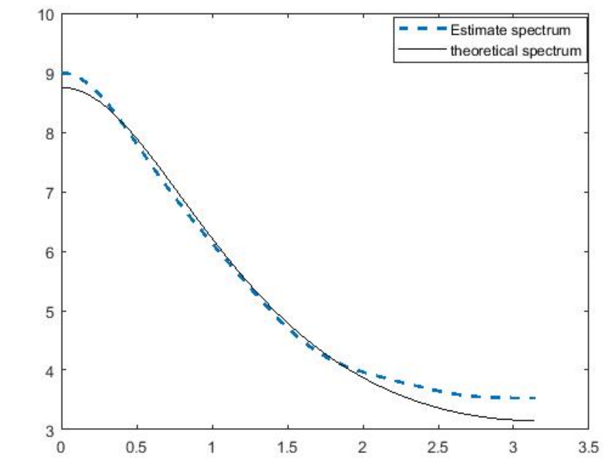

Fig. 2. The theoretical spectrum is represented by a solid line and estimated spectrum is represented by a dash line.

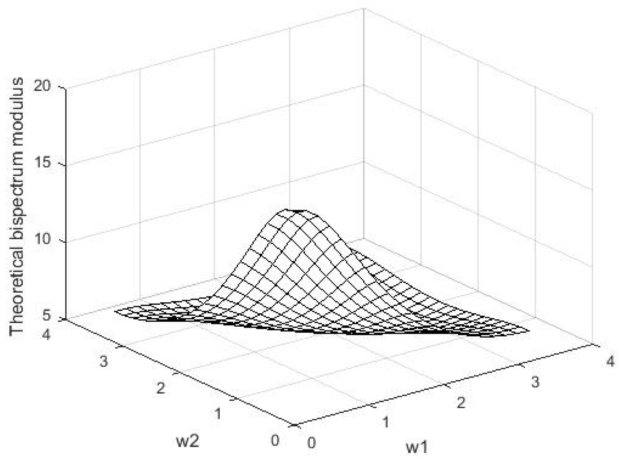

Fig. 3. Theoretical bispectral modulus of the NSINAR(1) with $\alpha = .25, \lambda = 3$ and $\mu = 5$.

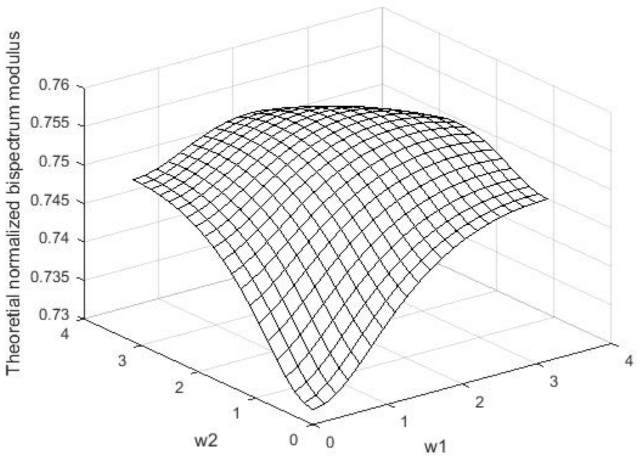

Fig. 4. Theoretical normalized bispectral modulus of NSINAR(1) with $\alpha = .25, \lambda = 3$ and $\mu = 5$.

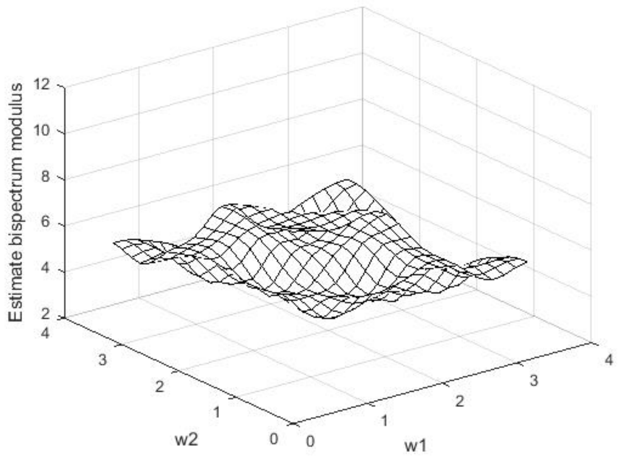

Fig. 5. Estimate bispectral modulus of NSINAR(1) at M=7.

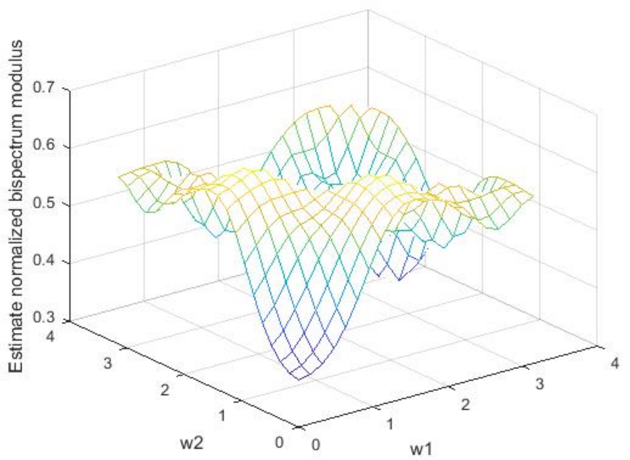

Fig. 6. Estimate normalized bispectral modulus of NSINAR(1) at M=7.

3. THE SHIFTED GEOMETRIC INAR(1) MODEL(SGINAR(1)-II)

Shifted geometric INAR(1) of type-II (in short, SGINAR(1)-II model introduced by Nastić (2012). It has the same approach that taken by Bakouch and Ristić (2010) for ZTPINAR(1) process but SGINAR(1)-II model is based on negative binomial thinning operator. A process $\{X_t\}$ is said to be SGINAR(1)-II if it satisfies

$$X_t = \begin{cases} \eta_t & \text{with probability } \frac{1}{1+\mu} \\ \alpha * X_{t-1} + \eta_t & \text{with probability } \frac{\mu}{1+\mu} \end{cases}, \quad (15)$$

where the operator "$*$" is defined as

$$\alpha * X_{t-1} = \sum_{i=1}^{X_{t-1}} Y_i, \ \ \alpha \in (0,1),$$

where $\{Y_i\}$ is a sequence of i.i.d random variables with geometric $(\frac{1}{1+\alpha})$ distribution independent of X_{t-1}, $\{X_t\}$ is a stationary process has shifted geometric $(\frac{1}{1+\mu})$ distribution and $\{\eta_t\}$ are i.i.d random variables independent of Y_i and of $X_{t-i}, i \geq 1$. The pgf of X_t and η_t are given by respectively (see[9])
$\phi_X(s) = \frac{s}{1+\mu-\mu s}$,
$\phi_\eta(s) = \frac{1}{1+\alpha-\alpha s}\left(\frac{\alpha(1+\mu)}{\mu}s + \frac{\mu-\alpha(1+\mu)}{\mu}\frac{s}{1+\mu-\mu s}\right)$.
The mean and variance of X_t and η_t are then given by respectively
$\mu_X = 1+\mu, \sigma^2_X = \mu(1+\mu)$,
$\mu_{\eta_t} = (1+\mu-\alpha\mu), \sigma^2_{\eta_t} = \mu(1-\alpha+\mu-\alpha^2\mu-2\alpha^2)$.
For more information about the model and the binomial thinning operator see [9] and [11].

3.1 Higher order joint moments and cumulants

THEOREM 3. *Let $\{X_t\}$ be a stationary process satisfying (15) then,*
The first order moment and cumulant for SGINAR(1)-II are given by μ_X .
The second order joint moments are calculated as
$\mu_{(0)} = (1+\mu)(1+2\mu)$,
$\mu_{(s)} = \mu(\mu+1)\left(\alpha\frac{\mu}{\mu+1}\right)^s + (1+\mu)^2$.
Then, the second order joint central moment is calculated as
$C_2(s) = [\frac{\alpha\mu}{1+\mu}]^s C_2(0) = (\frac{\alpha\mu}{1+\mu})^s \mu\,(\mu+1)$.
The third order joint moments are calculated as
$\mu_{(0,0)} = 6\mu^3 + 12\mu^2 + 7\mu + 1$,
$\mu_{(0,s)} = (\frac{\alpha\mu}{1+\mu})^s(4\mu^2+3\mu)\,(\mu+1) + (\mu+1)\,(3\mu+2\mu^2+1)$,
$\mu_{(s,s)} = \{(\alpha\frac{\mu}{\mu+1})^s(2\mu^2+3\mu-2\alpha\mu^2+\alpha\mu)(\frac{\mu+1}{1-\alpha}) + (\alpha^2\frac{\mu}{\mu+1})^s(2\mu^2-4\alpha\mu-2\alpha\mu^2)(\frac{\mu+1}{1-\alpha}) + (\frac{\mu+1}{1-\alpha})(1-2\alpha\mu^2+2\mu^2-3\alpha\mu+3\mu-\alpha)$,
$\mu_{(s,\tau)} = (\frac{\alpha\mu}{1+\mu})^{\tau-s}(\mu_{(s,s)} - \mu_{(s)}\mu_X) + \mu_{(s)}\mu_X$.

Then, the third order joint central moments are calculated as
$C_3(0,0) = (1+\mu)\mu(1+2\mu) = \mu\,(2\mu^2+3\mu+1)$,
$C_3(0,s) = (\frac{\alpha\mu}{1+\mu})^s C_3(0,0) = (\frac{\alpha\mu}{1+\mu})^s\mu\,(2\mu^2+3\mu+1)$,
$C_3(s,s) = (\frac{\alpha^2\mu}{1+\mu})^s C_3(0,0) + [\frac{\alpha\mu}{1+\mu}(1+\alpha) + 2(1+\mu)[\frac{\alpha^2\mu}{1+\mu} - (\frac{\alpha\mu}{1+\mu})^2]]C_2(0)\frac{(\frac{\alpha\mu}{1+\mu})^s - (\frac{\alpha^2\mu}{1+\mu})^s}{\frac{\alpha\mu}{1+\mu} - \frac{\alpha^2\mu}{1+\mu}}$
$= \left(\alpha^2\frac{\mu}{\mu+1}\right)^s(2\mu-4\alpha-2\alpha\mu)\mu(\frac{\mu+1}{1-\alpha}) + \left(\alpha\frac{\mu}{\mu+1}\right)^s(3\alpha+1)\mu(\frac{\mu+1}{1-\alpha})$,

$C_3(s,\tau) = (\frac{\alpha\mu}{1+\mu})^{\tau-s} C_3(s,s)$.
The fourth-joint moments are calculated as
$\mu_{(0,0,s)} = \frac{\alpha\mu}{1+\mu}\mu_{(0,0,s-1)} + \mu_\eta\mu_{(0,0)}$, $s > 0$,
$\mu_{(0,s,\tau)} = \frac{\alpha\mu}{1+\mu}\mu_{(0,s,\tau-1)} + \mu_{(0,s)}\mu_\eta$, $\tau > s > 0$,
$\mu_{(0,s,s)} = \frac{\alpha^2\mu}{1+\mu}\mu_{(0,s-1,s-1)} + \frac{\alpha\mu}{1+\mu}(1+\alpha)\mu_{(0,s-1)} + 2\frac{\alpha\mu}{1+\mu}\mu_\eta\mu_{(0,s-1)} + (1+3\mu-3\alpha\mu-2\mu\alpha^2+2\mu^2-2\alpha\mu^2)(1+3\mu+2\mu^2)$,
$\mu_{(s,\tau,u)} = \frac{\alpha\mu}{1+\mu}\mu_{(s,\tau,u-1)} + \mu_\eta\mu_{(s,\tau)}$, $u > \tau > s > 0$,
$\mu_{(s,s,v)} = \frac{\alpha\mu}{1+\mu}\mu_{(s,s,v-1)} + \mu_\eta\mu_{(s,s)}$, $v > s > 0$,
$\mu(s,s,s) = \frac{\alpha^3\mu}{1+\mu}\mu_{(s-1,s-1,s-1)} + 3\frac{\alpha^2\mu}{1+\mu}[(1+\alpha)+\mu_\eta]\mu_{(s-1,s-1)} + \frac{\alpha\mu}{1+\mu}[(1+\alpha)(1+2\alpha)+3\mu_\eta(1+\alpha)+3(1+3\mu-3\alpha\mu-2\mu\alpha^2+2\mu^2-2\alpha\mu^2)]\mu_{(s-1)} + (1+\mu)(1+7\mu-7\alpha\mu-12\mu\alpha^2+12\mu^2-12\alpha\mu^2-6\alpha^3\mu-6\alpha^2\mu^2+6\mu^3-6\mu^3\alpha)$.

Then, the fourth order joint cumulants are calculated as
$C_4(0,0,s) = \frac{\alpha\mu}{1+\mu}C_4(0,0,s-1)$, $s \geq 0$,
$C_4(0,s,s) = \frac{\alpha^2\mu}{1+\mu}C_4(0,s-1,s-1) + \frac{\alpha\mu}{1+\mu}(1+\alpha)C_3(0,s-1)$, $s \geq 0$,
$C_4(0,s,u) = \frac{\alpha\mu}{1+\mu}C_4(0,s,u-1)$, $u \geq s \geq 0$,
$C_4(s,\tau,u) = \frac{\alpha\mu}{1+\mu}C_4(s,\tau,u-1)$, $u \geq \tau \geq s \geq 0$,
$C_4(s,s,s) = \frac{\alpha^3\mu}{1+\mu}C_4(s-1,s-1,s-1) + 3\frac{\alpha^2\mu}{1+\mu}(1+\alpha)C_3(s-1,s-1) + \frac{\alpha\mu}{1+\mu}(1+\alpha)(1+2\alpha)$,
$C_4(s,s,u) = \frac{\alpha\mu}{1+\mu}C_4(s,s,u-1)$, $u \geq s \geq 0$.

PROOF. The proof of this theorem is similarly to the proof of theorem 1 in section 2, by using the definition of the SGINAR(1)-II and higher order moments and cumulants and the properties of the negative binomial thinning operator. ■

3.2 The spectral, bispectral and normalized bispectral density functions

The spectral density function is given by Nastić (2012) (see[9])

$$f_X(\omega) = \frac{\mu(1+\mu)(1+2\mu+2\mu^2(1-\alpha^2))}{2\pi(1+2\mu+\mu^2(1+\alpha^2)-2\alpha\mu(1+\mu)\cos\omega)}. \tag{16}$$

The normalized spectral density function is calculated as

$$g_X(\omega) = \frac{(1+2\mu+2\mu^2(1-\alpha^2))}{2\pi(1+2\mu+\mu^2(1+\alpha^2)-2\alpha\mu(1+\mu)\cos\omega)}.$$

THEOREM 4. *The bispectral density function is calculated as*

$$\begin{aligned} f_X(\omega_1,\omega_2) = & \frac{1}{(2\pi)^2}[C_3(0,0) + C_3(0,0)\{H_1(-\omega_1) \\ & +H_1(-\omega_2) + H_1(\omega_1+\omega_2)\} + (C_3(0,0) \\ & -\frac{[(1+\alpha)+2(1+\mu)\alpha(1-\frac{\mu}{1+\mu})]C_2(0)}{1-\alpha}) \\ & \{H_2(\omega_1) + H_2(\omega_2) + H_2(-\omega_1-\omega_2)\} \\ & +\frac{[(1+\alpha)+2(1+\mu)\alpha(1-\frac{\mu}{1+\mu})]C_2(0)}{1-\alpha} \\ & \{H_1(\omega_1) + H_1(\omega_2) + H_1(-\omega_1-\omega_2)\} \\ & +(C_3(0,0) \\ & -\frac{[(1+\alpha)+2(1+\mu)\alpha(1-\frac{\mu}{1+\mu})]C_2(0)}{1-\alpha}) \\ & \{H_1(-\omega_2)H_2(-\omega_1-\omega_2) \\ & +H_1(-\omega_1)H_2(-\omega_1-\omega_2) \\ & +H_1(-\omega_2)H_2(\omega_1) + H_1(-\omega_1)H_2(\omega_2) \\ & +H_1(\omega_1+\omega_2)H_2(\omega_1) + H_1(\omega_1+\omega_2)H_2(\omega_2)\} \\ & +\frac{[(1+\alpha)+2(1+\mu)\alpha(1-\frac{\mu}{1+\mu})]C_2(0)}{1-\alpha} \\ & \{H_1(-\omega_2)H_1(-\omega_1-\omega_2) \\ & +H_1(-\omega_1)H_1(-\omega_1-\omega_2) \\ & +H_1(-\omega_2)H_1(\omega_1) + H_1(-\omega_1)H_1(\omega_2) \\ & +H_1(\omega_1+\omega_2)H_1(\omega_1) \\ & +H_1(\omega_1+\omega_2)H_1(\omega_2)\}], \end{aligned} \tag{17}$$

where $H_1(\omega_k) = \frac{\frac{\alpha\mu}{1+\mu}e^{i\omega_k}}{1-\frac{\alpha\mu}{1+\mu}e^{i\omega_k}}$ *and* $H_2(\omega_k) = \frac{\frac{\alpha^2\mu}{1+\mu}e^{i\omega_k}}{1-\frac{\alpha^2\mu}{1+\mu}e^{i\omega_k}}$ *with* $k = 1, 2$.

PROOF. The proof is too long to included it here. ■
The normalized bispectral density function is calculated as (6), where $f_X(\omega_1,\omega_2)$ and $f_X(\omega_1)$ are defined in (17) and (16).

3.3 Estimation of spectrum

The estimates of the spectral, bispectral and normalized bispectral density functions are calculated using the smoothed periodogram method based on the Parzen lag window using simulated series $\{X_t, t = 1, 2, \ldots, 500\}$ from SGINAR(1)-II model.
The theoretical spectrum $f_X(\omega)$, theoretical bispectral and normalized bispectral modulus of $f_X(\omega_1,\omega_2)$ and $g_X(\omega_1,\omega_2)$ are respectively computed by setting $\mu = 2.6$ and $\alpha = .6$ in (17) and (6). Fig. 7 represents the simulated series of SGINAR(1)-II with $\mu = 2.6$ and $\alpha = .6$. Fig. 8 represents theoretical spectrum and the estimate spectrum by Parzen window with M=7 as in (7) and (9). Fig. 9 and Fig. 10 represent the theoretical bispectrum and normalized bispectrum modulus. Fig. 11 and Fig. 12 represent the estimate of the bispectrum and normalized bispectrum modulus with M=7 by Parzen window using (8), (11) and (10).
From Fig. 10, the normalized bispectrum modulus of the SGINAR(1)-II model is more flat than the non-normalized bispectrum modulus given in Fig. 9 , since the values of the normalized bispectrum modulus lies between (0,2) and the non-normalized bispectrum modulus lies between (0,15).

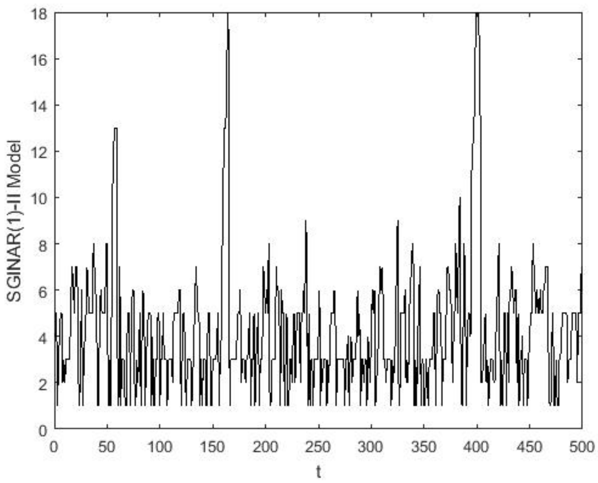

Fig. 7. Simulated series of SGINAR(1)-II with $\mu = 2.6$ and $\alpha = .6$.

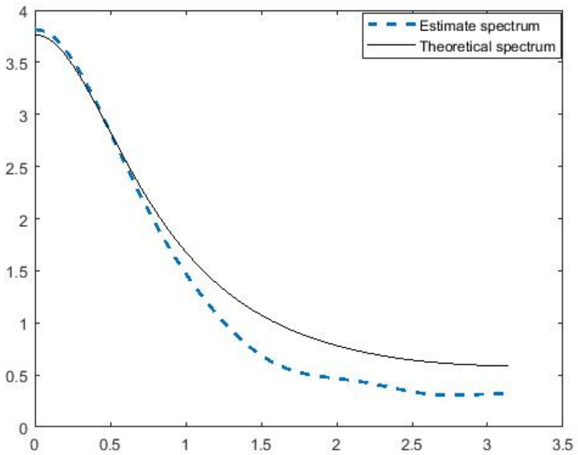

Fig. 8. Theoretical spectrum is represented by a solid line and estimated spectrum is represented by a dash line at M=7.

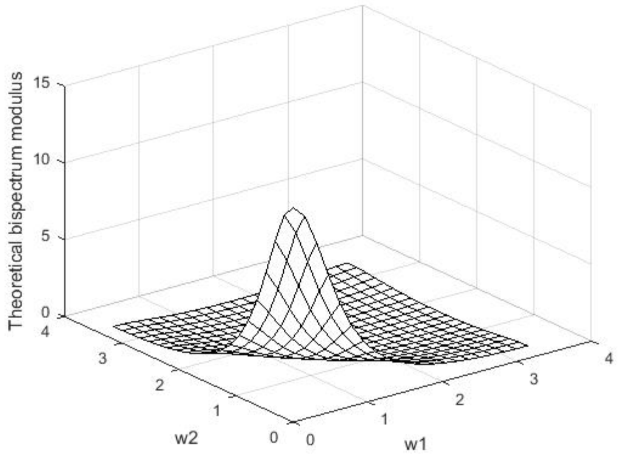

Fig. 9. The theoretical bispectral modulus of SGINAR(1)-II with $\mu = 2.6$ and $\alpha = .6$.

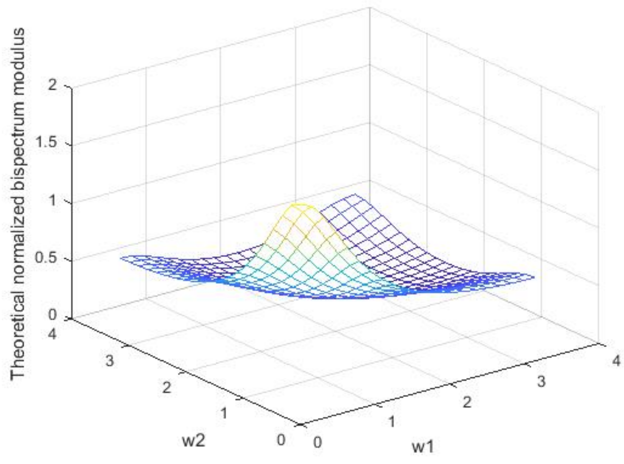

Fig. 10. The theoretical normalized bispectral modulus of SGINAR(1)-II with $\mu = 2.6$ and $\alpha = .6$.

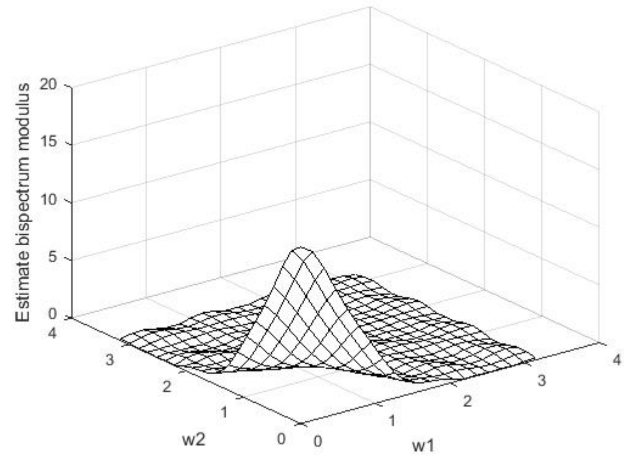

Fig. 11. Estimate bispectral modulus of SGINAR(1) at M=7.

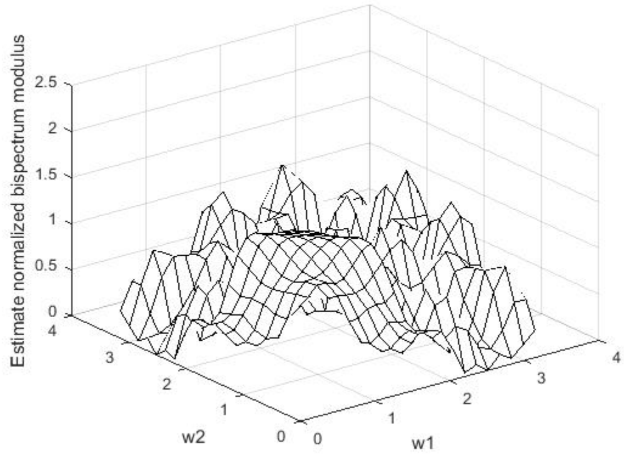

Fig. 12. Estimate normalized bispectral modulus of SGINAR(1) at M=7.

4. THE DEPENDENT COUNTING GEOMETRIC INAR(1) MODEL

Ristić et al. (2013) introduced the DCGINAR(1) model based on generalized binomial thinning operator type-I with a geometric marginal ($\bullet_\theta$). They defined the DCGINAR(1) as

$$X_t = \alpha \bullet_\theta X_{t-1} + \varepsilon_t,\ t \in Z,\ \alpha, \theta \in (0,1), \tag{18}$$

where the operator '$\bullet_\theta$' is defined as $\alpha \bullet_\theta X = \sum_{i=1}^{X} U_i$, $i \in N$, $\{U_i\}$ is a sequence of dependent Bernoulli(α) random variable defined as $U_i = (1-V_i)W_i + V_i Z$, $\{W_i\}$ is a sequence of i.i.d random variable with Bernoulli(α) distribution, $\{V_i\}$ is a sequence of i.i.d random variable with Bernoulli(θ) distribution, Z is a random variable with Bernoulli(α) distribution, W_i, V_j and Z are independent $\forall\, i, j \in N$ and $\{U_i\}$ are independent of X_l and ε_m for any i, l and m. $\{X_t\}$ has Geometric ($\frac{\mu}{1+\mu}$) distribution, $\mu > 0$ and $\{\varepsilon_t\}$ is a sequence i.i.d r.v's distributed as a mixture of zero and two geometrically random variables. This model satisfy these conditions,$\{\varepsilon_t\}$ is a sequence i.i.d random variables such that $Cov(\varepsilon_t, X_s) = 0$, $s < t.$, $\{U_i\}$ are independent of X_j and ε_k and $\{U_i\}$ used for generating X_s and X_t, representing the counting series of the process $\{X_t\}$ are mutually independent for $t \neq s$.
The pgf of U_i, ε_t and X_t are given by respectively (see[12])
$\phi_{U_i}(s) = 1 - \alpha + \alpha s$,
$\phi_\varepsilon(s) = \frac{(1+\alpha(1-\theta)\mu - \alpha(1-\theta)\mu s)(1+(\alpha+\theta-\alpha\theta)\mu - (\alpha+\theta-\alpha\theta)\mu s)}{(1+\mu-\mu s)(1+(\alpha+\theta-2\alpha\theta)\mu - (\alpha+\theta-2\alpha\theta)\mu s)}$,
$\phi_X(s) = \frac{1}{1+\mu-\mu s}$.
The mean and variance of X_t and ε_t are respectively given by
$\mu_X = \mu$, $\sigma_X^2 = \mu(1+\mu)$,
$\mu_{\varepsilon_t} = (1-\alpha)\mu$, $\sigma_{\varepsilon_t}^2 = (1-\alpha)\mu(1 + (1+\alpha-2\alpha\theta^2)\mu)$.
The second and third moments of ε_t are respectively calculated as

$$E(\varepsilon^2) = (1-\alpha)\mu + 2(\alpha-1)(\alpha\theta^2-1)\mu^2, \tag{19}$$

$$\begin{aligned} E(\varepsilon^3) &= (1-\alpha)\mu + 6(\alpha-1)(\alpha\theta^2-1)\mu^2 \\ &\quad -6(\alpha-1)(-\alpha^2\theta^2 \\ &\quad +2\alpha^2\theta^3 - \alpha\theta^3 - \alpha\theta^2 + 1)\mu^3. \end{aligned} \tag{20}$$

Some properties of the operator '$\bullet_\theta$':-
Let X, Y be any two random variables with finite first, second and third moments, $\alpha \in (0,1)$ and $\theta \in (0,1)$, then

(1) $E(\alpha \bullet_\theta X) = \alpha E(X)$,
(2) $E(\alpha \bullet_\theta X)^2 = \alpha(\alpha + (1-\alpha)\theta^2)E(X^2) + \alpha(1-\alpha)(1-\theta^2)E(X)$
(3) $E(Y(\alpha \bullet_\theta X)) = \alpha E(XY)$.
(4) $E(\alpha \bullet_\theta X)^3 = [\alpha^3 - 3\alpha^2\theta^3 + 2\alpha^3\theta^3 - 3\alpha^3\theta^2 + 3\alpha^2\theta^2 + \alpha\theta^3]E(X^3) + [9\alpha^3\theta^2 - 6\alpha^3\theta^3 - 12\alpha^2\theta^2 - 3\alpha\theta^3 + 3\alpha\theta^2 + 3\alpha^2 + 9\alpha^2\theta^3 - 3\alpha^3]E(X^2) + [9\alpha^2\theta^2 - 3\alpha^2 - 6\alpha^2\theta^3 - 6\alpha^3\theta^2 - 3\alpha\theta^2 + \alpha + 2\alpha^3 + 2\alpha\theta^3 + 4\alpha^3\theta^3]E(X)$.

For more details about the model and the properties of the generalized binomial thinning type-I see [12].

4.1 Higher order joint moments and cumulants

THEOREM 5. *Let $\{X_t\}$ be a stationary process satisfying (18) then,*
The first-order moment and first cumulant are given by μ_X.
The second-order joint moment is calculated as
$\mu_{(s)} = \alpha^s[\mu_{(0)} - \mu^2] + \mu^2 = \alpha^s\mu(1+\mu) + \mu^2$, $s \geq 0$ *and* $\mu_{(0)=\mu(1+2\mu)}$.
Then, the second order joint central moment is calculated as
$C_2(s) = \alpha^s C_2(0) = \alpha^s\mu(1+\mu)$, $s \geq 0$.
The third-order moments are calculated as
$\mu_{(0,0)} = \mu(1 + 6\mu + 6\mu^2)$,
$\mu_{(0,s)} = \alpha^s[\mu_{(0,0)} - \mu\mu_{(0)}] + \mu\mu_{(0)} = \mu^2(1+2\mu) + \alpha^s\mu(1 + 5\mu + 4\mu^2)$, $s \geq 0$,
$\mu_{(s,s)} = [\alpha(\alpha + (1-\alpha)\theta^2)]^s\mu_{(0,0)} + [\alpha(1-\alpha)(1-\theta^2) + 2\alpha\mu_\varepsilon](\mu_{(0)} - \mu^2)[\frac{\alpha^s - (\alpha(\alpha+(1-\alpha)\theta^2))^s}{\alpha(1-[\alpha(\alpha+(1-\alpha)\theta^2)]}] + ([\alpha(1-\alpha)(1-\theta^2) + 2\alpha\mu_\varepsilon]\mu^2 + \mu[\mu_\varepsilon^2 + \sigma_\varepsilon^2])[\frac{1-[\alpha(\alpha+(1-\alpha)\theta^2)]^s}{1-[\alpha(\alpha+(1-\alpha)\theta^2)]}]$,
$\mu_{(s,\tau)} = \alpha^{\tau-s}(\mu_{(s,s)} - \mu_{(s)}\mu) + \mu_{(s)}\mu$, $\tau > s$.
Then, the third-order joint central moments are calculated as
$C_3(0,0) = \mu(1 + 3\mu + 2\mu^2)$,
$C_3(0,u) = \alpha^u C_3(0,0) = \alpha^u\mu(1 + 3\mu + 2\mu^2)$, $u \geq 0$,
$C_3(s,s) = [\alpha(\alpha + (1-\alpha)\theta^2)]^s C_3(0,0) + [\alpha(1-\alpha)(1-\theta^2) + 2\mu\alpha(1-\alpha)\theta^2]C_2(0)[\frac{\alpha^s(1-(\alpha(\alpha+(1-\alpha)\theta^2))^s)}{\alpha(1-[\alpha(\alpha+(1-\alpha)\theta^2)])}]$, $s \geq 0$,
$C(s,\tau) = \alpha^{\tau-s}C_3(s,s)$.
The fourth-order moments are calculated as
$\mu_{(0,0,s)} = \alpha\mu_{(0,0,s-1)} + \mu_{(0,0)}\mu_\varepsilon$,
$\mu_{(0,s,s)} = \alpha(\alpha + (1-\alpha)\theta^2)\mu_{(0,s-1,s-1)} + (2\alpha(1-\alpha)\mu + \alpha(1-\alpha)(1-\theta^2))\mu_{(0,s-1)} + \mu_{(0)}(E(\varepsilon^2))$,
$\mu_{(0,s,\tau)} = \alpha\mu_{(0,s,\tau-1)} + \mu_{(0,s)}\mu_\varepsilon$,
$\mu_{(s,\tau,v)} = \alpha\mu_{(s,\tau,v-1)} + \mu_{(s,\tau)}\mu_\varepsilon$,
$\mu_{(s,s,s)} = [\alpha^3 - 3\alpha^2\theta^3 + 2\alpha^3\theta^3 - 3\alpha^3\theta^2 + 3\alpha^2\theta^2 + \alpha\theta^3]\mu_{(s-1,s-1,s-1)} + [9\alpha^3\theta^2 - 6\alpha^3\theta^3 - 12\alpha^2\theta^2 - 3\alpha\theta^3 + 3\alpha\theta^2 + 3\alpha^2 + 9\alpha^2\theta^3 - 3\alpha^3]\mu_{(s-1,s-1)} + [9\alpha^2\theta^2 - 3\alpha^2 - 6\alpha^2\theta^3 - 6\alpha^3\theta^2 - 3\alpha\theta^2 + \alpha + 2\alpha^3 + 2\alpha\theta^3 + 4\alpha^3\theta^3]\mu_{(s-1)} + \mu E(\varepsilon^3) + 3\mu_\varepsilon[\alpha(\alpha + (1-\alpha)\theta^2)\mu_{(s-1,s-1)} + \alpha(1-\alpha)(1-\theta^2)\mu_{(s-1)}] + 3\alpha E(\varepsilon^2)\mu_{(s-1)}$,
where $E(\varepsilon^2)$ and $E(\varepsilon^3)$ are given by (19) and (20) respectively.
The fourth-order joint cumulants are calculated as
$C_4(v,v,v) = [\alpha^3 - 3\alpha^2\theta^3 + 2\alpha^3\theta^3 - 3\alpha^3\theta^2 + 3\alpha^2\theta^2 + \alpha\theta^3]C_4(v-1, v-1, v-1) + [9\alpha^3\theta^2 - 6\alpha^3\theta^3 - 12\alpha^2\theta^2 - 3\alpha\theta^3 + 3\alpha\theta^2 + 3\alpha^2 + 9\alpha^2\theta^3 - 3\alpha^3]C_3(v-1, v-1) + [9\alpha^2\theta^2 - 3\alpha^2 - 6\alpha^2\theta^3 - 6\alpha^3\theta^2 - 3\alpha\theta^2 + \alpha + 2\alpha^3 + 2\alpha\theta^3 + 4\alpha^3\theta^3]C_2(v-1)$,
$C_4(0,v,v) = \alpha(\alpha + (1-\alpha)\theta^2)C_4(0, v-1, v-1) + \alpha(1-\alpha)(1-\theta^2)C_4(0, v-1)$,
$C_4(0,\tau,v) = \alpha C_4(0,\tau,v-1)$, $v \geq \tau \geq 0$,
$C_4(\tau,\tau,s) = \alpha C_4(\tau,\tau,s-1)$, $s \geq \tau \geq 0$,
$C_4(s,\tau,v) = \alpha C_4(s,\tau,v-1)$, $v \geq \tau \geq s \geq 0$.

PROOF. The proof is similar to the proof of theorem 1 using the definitions of the DCGINAR(1) and the properties of the generalized thinning operator. ■

4.2 The spectral, bispectral and normalized bispectral density functions

The non-normalized spectral density function $f_X(\omega)$ of DCGINAR(1) is calculated as

$$f_X(\omega) = \frac{\mu(1+\mu)(1-\alpha^2)}{2\pi(1+\alpha^2-2\alpha\cos\omega)}, \quad -\pi \leq \omega \leq \pi. \tag{21}$$

The normalized spectral density function $g_X(\omega)$ is calculated as

$$g_X(\omega) = \frac{(1-\alpha^2)}{2\pi(1+\alpha^2-2\alpha\cos\omega)}.$$

THEOREM 6. *The bispectral density function $f_X(\omega_1, \omega_2)$ of DCGINAR(1) is calculated as*

$$\begin{aligned} f_X(\omega_1,\omega_2) \;=\; & \frac{1}{(2\pi)^2}[C_3(0,0)\{1+F_1(-\omega_1)+F_1(-\omega_2) \\ & +F_1(\omega_1+\omega_2)\} \\ & +(C_3(0,0) \\ & -\frac{[(1-\alpha)(1-\theta^2)+2\mu(1-\alpha)\theta^2]C_2(0)}{(1-\alpha(\alpha+(1-\alpha)\theta^2))}) \\ & \{F_2(\omega_1)+F_2(\omega_2)+F_2(-\omega_1-\omega_2)\} \\ & +(\frac{[(1-\alpha)(1-\theta^2)+2\mu(1-\alpha)\theta^2]C_2(0)}{(1-\alpha(\alpha+(1-\alpha)\theta^2))})\{F_1(\omega_1) \\ & +F_1(\omega_2)+F_1(-\omega_1-\omega_2)\} \\ & +(C_3(0,0) \\ & -\frac{[(1-\alpha)(1-\theta^2)+2\mu(1-\alpha)\theta^2]C_2(0)}{(1-\alpha(\alpha+(1-\alpha)\theta^2))}) \\ & \{F_2(-\omega_1-\omega_2)F_1(-\omega_2)+F_2(-\omega_1-\omega_2)F_1(-\omega_1) \\ & +F_2(\omega_1)F_1(-\omega_2)+F_2(\omega_2)F_1(-\omega_1) \\ & +F_2(\omega_1)F_1(\omega_1+\omega_2)+F_2(\omega_2)F_1(\omega_1+\omega_2)\} \\ & +(\frac{[(1-\alpha)(1-\theta^2)+2\mu(1-\alpha)\theta^2]C_2(0)}{(1-\alpha(\alpha+(1-\alpha)\theta^2))}) \\ & \{F_1(-\omega_1-\omega_2)F_1(-\omega_2)+F_1(-\omega_1-\omega_2)F_1(-\omega_1) \\ & +F_1(\omega_1)F_1(-\omega_2)+F_1(\omega_2)F_1(-\omega_1) \\ & +F_1(\omega_1)F_1(\omega_1+\omega_2)+F_1(\omega_2)F_1(\omega_1+\omega_2)\}], \end{aligned} \quad (22)$$

where $F_1(\omega_k) = \frac{\alpha e^{i\omega_k}}{1-\alpha e^{i\omega_k}}$ *and*
$F_2(\omega_k) = \frac{\alpha(\alpha+(1-\alpha)\theta^2)e^{i\omega_k}}{1-(\alpha(\alpha+(1-\alpha)\theta^2))e^{i\omega_k}}$, *with* $k = 1, 2$.

PROOF. The proof is too long to be included here. ■
The normalized bispectral density function is calculated by **(6),** where $f_X(\omega_1, \omega_2)$ and $f(\omega_i)$ are given by (22) and (21).

NOTATION 7. *The higher order moments, cumulants , spectrum, bispectrum and normalized bispectrum for the GINAR(1) model that introduced by Al-Zaid and Al-Osh (1988) can be concluded by setting $\theta = 0$ in the higher order moments, cumulants, spectrum, bispectrum and normalized bispectrum of the DCGINAR(1).*

4.3 Estimation of the spectrum

Estimates of the spectrum, the bispectrum and normalized bispectrum using the smoothed periodogram method using the Parzen window and simulated series from the DCGINAR(1) model are calculated.
The theoretical spectrum $f_X(\omega)$, theoretical bispectral and normalized bispectral density of $f_X(\omega_1, \omega_2)$ and $g_X(\omega_1, \omega_2)$ are respectively obtained by setting $\alpha = .6$, $\theta = .7$ and $\mu = 1.8$ in (21), (22) and (6). Fig. 13, represents the simulated series of DCGINAR(1) with $\alpha = .6$, $\theta = .7$ and $\mu = 1.8$. Fig. 14 represents the theoretical spectrum and the estimate spectrum using Parzen lag window with M=7 from (7) and (9). Fig. 15 and Fig. 16 represent the theoretical bispectral and normalized bispectral modulus. Fig. 17 and Fig. 18 represent the estimate of the bispectrum and normalized bispectrum modulus with M=7 by Parzen window using (8), (10) and (9).
From Fig. 15 and Fig. 16, the normalized bispectrum modulus of the DCGINAR(1) is more flat than the non-normalized bispectrum modulus, since the the values of the normalized bispectrum modu-

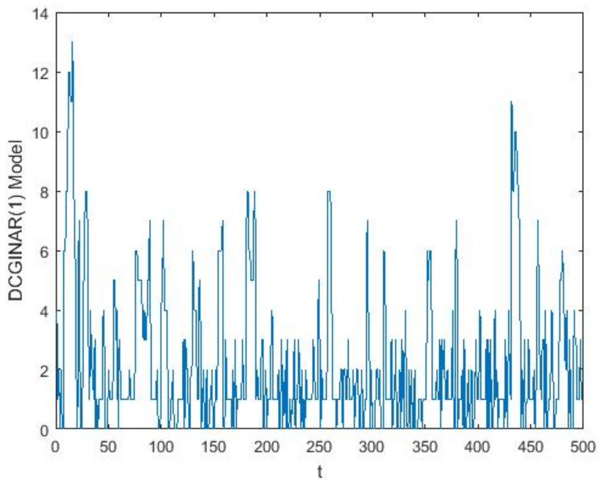

Fig. 13. Simulated series of DCGINAR(1) with $\alpha = .6$, $\theta = .7$ and $\mu = 1.8$.

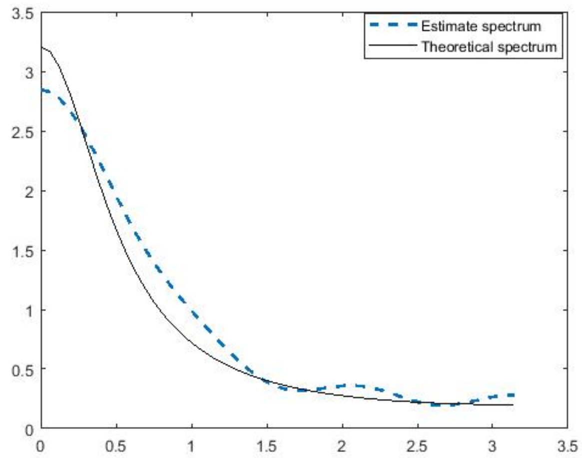

Fig. 14. Theoretical spectrum and estimated spectrum at M=7.

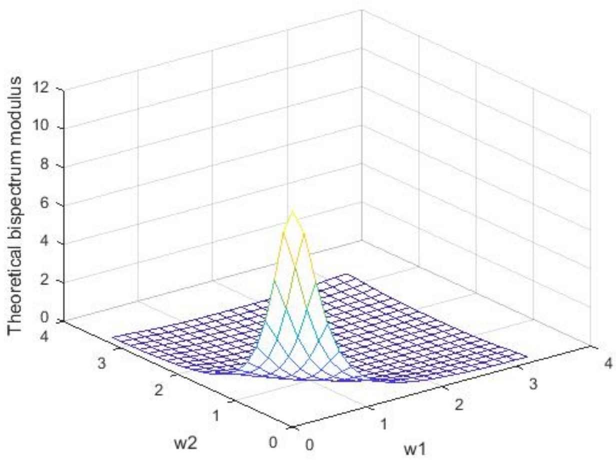

Fig. 15. Theoretical bispectral modulus of DCGINAR(1) with $\alpha = .6$, $\theta = .7$ and $\mu = 1.8$.

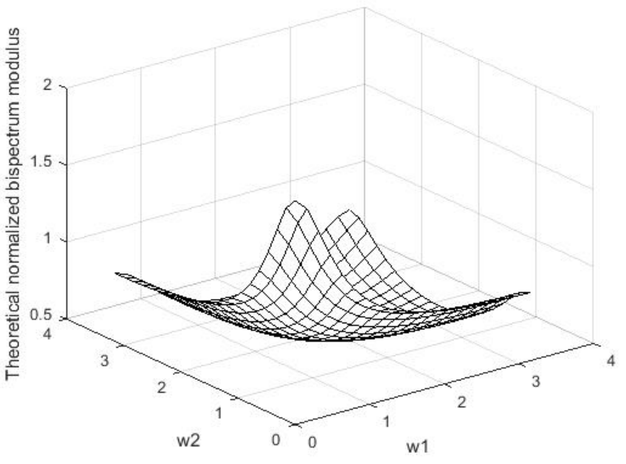

Fig. 16. Theoretical normalized bispectral modulus with $\alpha = .6$, $\theta = .7$ and $\mu = 1.8$ of DCGINAR(1).

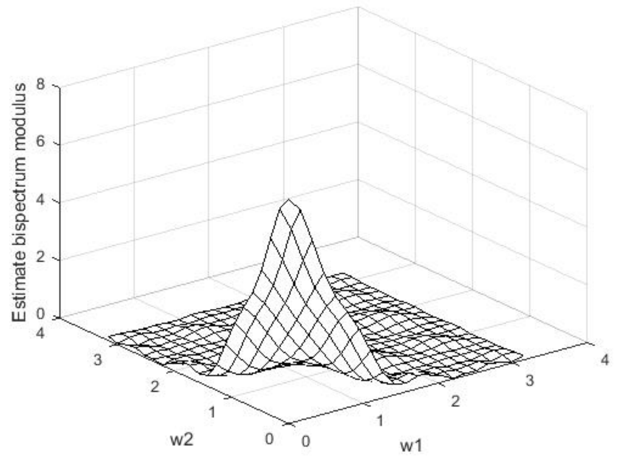

Fig. 17. Estimate bispectral modulus of DCGINAR(1) at M=7.

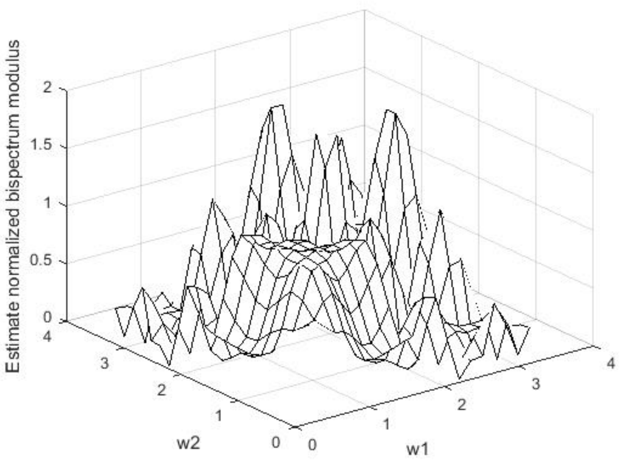

Fig. 18. Estimate normalized bispectral modulus of DCGINAR(1) at M=7.

lus lies between (0.5,2) and the values of the non-normalized bispectrum modulus lies between (0,12).

5. CONCLUSIONS

Bispectrum and normalized bispectrum are used for checking the linearity of the models. The higher order moments, spectrum, bispectrum and normalized bispectrum of the NSINAR(1), SGINAR(1)-II and DCGINAR(1) models are computed. The spectrum, bispectrum and normalized bispectrum are estimated using a smoothed periodogram based on the Parzen lag window and using a simulated series from each model. Moreover, the higher order moments, spectrum and bispectrum for the GINAR(1) are concluded as a special case of the DCGINAR(1) model. The normalized bispectrum modulus of these models are more flat than the non-normalized bispectrum modulus, so the test of linearity that given by Subba Rao and Gabr (1980) and its modification given by Hinich (1982) can be used for integer valued time series models.

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π	18.95													
$.95\pi$	18.75	18.37	17.67											
$.90\pi$	18.19	17.67	16.87	15.89	14.81									
$.85\pi$	17.34	16.72	15.89	14.91	13.88	12.86	11.89							
$.80\pi$	16.30	15.64	14.81	13.88	12.92	11.98	11.10	10.30	9.58					
$.75\pi$	15.16	14.50	13.72	12.86	11.98	11.14	10.34	9.62	8.98	8.42	7.94			
$.70\pi$	14.02	13.39	12.67	11.89	11.10	10.34	9.63	8.99	8.42	7.93	7.50	7.15	6.86	
$.65\pi$	12.93	12.35	11.69	11.00	10.30	9.62	8.99	8.42	7.92	7.48	7.11	6.80	6.55	6.35
$.60\pi$	11.92	11.40	10.82	10.20	9.58	8.98	8.42	7.92	7.47	7.09	6.76	6.49	6.27	
$.55\pi$	11.02	10.56	10.04	9.50	8.95	8.42	8.93	7.48	7.09	6.75	6.46	6.23		
$.50\pi$	10.22	9.82	9.37	8.89	8.41	7.94	7.50	7.11	6.76	6.46	6.21			
$.45\pi$	9.54	9.19	8.80	8.38	7.95	7.54	7.15	6.80	6.49	6.23				
$.40\pi$	8.95	8.65	8.31	7.94	7.54	7.20	6.86	6.55	6.27					
$.35\pi$	8.45	8.20	7.91	7.59	7.26	6.93	6.63	6.35						
$.30\pi$	8.04	7.83	7.58	7.30	7.01	6.72	6.45							
$.25\pi$	7.71	7.54	7.32	7.08	6.82	6.57								
$.20\pi$	7.45	7.31	7.13	6.92	6.69									
$.15\pi$	7.25	7.14	6.99	6.81										
$.10\pi$	7.11	7.03	6.91											
$.05\pi$	7.02	6.97												
0	7.00													
ω_2														
ω_1	0	$.05\pi$	$.10\pi$	$.15\pi$	$.20\pi$	$.25\pi$	$.30\pi$	$.35\pi$	$.40\pi$	$.45\pi$	$.50\pi$	$.55\pi$	$.60\pi$	65π

Table 1.

Theoretical bispectral modulus of NSINAR(1) with $\alpha = .25, \lambda = 3$ and $\mu = 5$.

π	.7319													
$.95\pi$	.7322	.7328	.7340											
$.90\pi$	.7331	.7340	.7353	.7369	.7387									
$.85\pi$	.7345	.7355	.7369	.7385	.7402	.7420	.7436							
$.80\pi$	.7362	.7373	.7387	.7402	.7419	.7435	.7451	.7465	.7478					
$.75\pi$	.7380	.7391	.7405	.7420	.7435	.7451	.7466	.7479	.7491	.7501	.7510			
$.70\pi$	.7399	.7409	.7422	.7436	.7451	.7466	.7479	.7492	.7503	.7513	.7522	.7529	.7534	
$.65\pi$	.7416	.7426	.7438	.7451	.7465	.7479	.7492	.7504	.7515	.7524	.7532	.7538	.7544	.7548
$.60\pi$	.7432	.7441	.7452	.7465	.7478	.7491	.7503	.7515	.7525	.7534	.7541	.7547	.7552	
$.55\pi$	.7447	.7455	.7465	.7476	.7489	.7501	.7513	.7524	.7534	.7542	.7549	.7555		
$.50\pi$	.7459	.7466	.7476	.7487	.7498	.7510	.7522	.7532	.7541	.7549	.7555			
$.45\pi$	.7470	.7476	.7485	.7495	.7506	.7518	.7529	.7538	.7547	.7555				
$.40\pi$	.7479	.7485	.7493	.7502	.7513	.7524	.7534	.7544	.7552					
$.35\pi$	.7487	.7492	.7499	.7508	.7519	.7529	.7539	.7548						
$.30\pi$	.7494	.7498	.7505	.7513	.7523	.7533	.7542							
$.25\pi$	.7499	.7502	.7509	.7517	.7526	.7536								
$.20\pi$	.7503	.7506	.7512	.7520	.7528									
$.15\pi$	.7506	.7509	.7514	.7521										
$.10\pi$	.7508	.7510	.7515											
$.05\pi$	.751	.7511												
0	.75104													
ω_2														
ω_1	0	$.05\pi$	$.10\pi$	$.15\pi$	$.20\pi$	$.25\pi$	$.30\pi$	$.35\pi$	$.40\pi$	$.45\pi$	$.50\pi$	$.55\pi$	$.60\pi$	65π

Table 2.

Theoretical normalized bispectral modulus of NSINAR(1) with $\alpha = .25, \lambda = 3$ and $\mu = 5$.