International Journal of Computer Applications (0975 - 8887)
Volume 183 - No.13, July 2021

$\mathcal{G}_{\omega}^{\alpha}-$Open sets in Grill Topological Spaces

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ABSTRACT

In this paper, we investigate and introduce new class of open sets in grill topological spaces, class of $\mathcal{G}_{\omega}^{\alpha}-$open sets, which is weak form of the class of $\mathcal{G}\alpha$-open sets and strong form of class of $\mathcal{G}^{\omega}-$open sets. We give the relationship of this notion with the other known sets and introduce the notions of the closure operator and the interior operator of $\mathcal{G}_{\omega}^{\alpha}-$open sets.

AMS classification: Primary 54A05, 54A10.

Keywords

open set; Open set; $\alpha-$Open set; $\omega-$Open set; Grill topological space.

1. INTRODUCTION

Some classes of weak or strong forms of open sets in topological spaces are structured, investigated and introduced as important study in the general topology. In 1965, Njastad [5] introduced the class of $\alpha-$open sets in topological spaces as a weak form of class of open sets, a subset A of topological space X is called $\alpha-$open set if $A \subseteq Int(Cl(Int(A)))$, where $Cl(A)$ and $Int(A)$ are the closure set and the interior set of A, respectively. In 1982, Hdeib [2] introduced the class of $\omega-$open sets as a weak form of class of open sets in topological spaces, a subset A of a space X is called $\omega-$open set if for each $x \in A$, there is an open set U_x containing x such that $U_x - A$ is a countable set. In 2009, [4], they introduced in topological spaces class of $\alpha - \omega-$open sets, a weak form of class of $\alpha-$open sets. In 2016, [9], they introduced in topological spaces class of semi$-\omega-$open sets, a new generalized class of $\omega-$open sets and strong form of class of $\alpha - \omega-$open sets. In 2009, [8], they introduced the class of $\mathcal{G}\alpha$-open sets in grill topological spaces which lies between class of open sets and class of $\alpha-$open sets. In 2020, [1], they introduced the class of $\mathcal{G}^{\omega}-$open sets in grill topological spaces as a weak form of class of semi$-\omega-$open sets.

In this work, we investigate and introduce new class of sets in grill topological spaces, say class of $\mathcal{G}_{\omega}^{\alpha}-$open sets, which lies between the class of $\mathcal{G}\alpha$-open sets and class of $\mathcal{G}^{\omega}-$open sets. So in this paper, Section 2 contains some preliminaries which will be used in our work. Section 3 investigates and introduces the concept of $\mathcal{G}_{\omega}^{\alpha}-$open set in grill topological spaces. The relationship of this notion with the other known sets will be studied. Section 4 introduces the notions of the closure operator and the interior operator of $\mathcal{G}_{\omega}^{\alpha}-$open sets.

2. PRELIMINARIES

For a topological space (X, τ) and $A \subseteq X$, throughout this paper, we mean $Cl(A)$ and $Int(A)$ the closure set and the interior set of A, respectively.

THEOREM 2.1. [3] For a topological space (X, τ) and $A, B \subseteq X$, if B is an open set in X then $Cl(A) \cap B \subseteq Cl(A \cap B)$.

THEOREM 2.2. [3] For a topological space (X, τ),

(1) $Cl(X - A) = X - Int(A)$ for all $A \subseteq X$.
(2) $Int(X - A) = X - Cl(A)$ for all $A \subseteq X$.

DEFINITION 2.3. [2] A subset A of a space X is called $\omega-$*open set* if for each $x \in A$, there is an open set U_x containing x such that $U_x - A$ is a countable set. The complement of a $\omega-$open set is called a $\omega-$closed set. The set of all $\omega-$closed sets in X denoted by $\omega C(X, \tau)$ and the set of all $\omega-$open sets in X denoted by $\omega O(X, \tau)$.

THEOREM 2.4. [2] Let A be sub set of X in a topological space (X, τ). Then

(1) $Cl_{\omega}(X - A) = X - Int_{\omega}(A)$ for all $A \subseteq X$.
(2) $Int_{\omega}(X - A) = X - Cl_{\omega}(A)$ for all $A \subseteq X$.
(3) $Cl_{\omega}(A) \subseteq Cl(A)$ for all $A \subseteq X$.
(4) $Int(A) \subseteq Int_{\omega}(A)$ for all $A \subseteq X$.

THEOREM 2.5. [2] Every open set is $\omega-$open set.

THEOREM 2.6. [2] For a topological space (X, τ), the pair $[X, \omega O(X, \tau)]$ forms a topological space.

In this work, $Int_\omega(A)$ denotes the $\omega-$interior operator of A defined as the union of all $\omega-$open sets which contained in A and $Cl_\omega(A)$ denotes the $\omega-$closure operator of A defined as the intersection of all $\omega-$closed sets which contain A .

DEFINITION 2.7. A subset A of a topological space (X, T) is called:

(1) $\alpha-$*open set* [5] if $A \subseteq Int(Cl(Int(A)))$. The complement of $\alpha-$open set is called $\alpha-$closed set.
(2) $\alpha - \omega-$*open set* [4] if $A \subseteq Int_\omega(Cl(Int_\omega(A)))$. The complement of $\alpha - \omega-$open set is called $\alpha - \omega-$closed set.
(3) *Semi*$-\omega-$*open set* [9] if $A \subseteq Cl(Int_\omega(A))$. The complement of semi$-\omega-$open set is called semi$-\omega-$closed set.

It is clear that every $\alpha-$open set is $\alpha - \omega-$open set.

THEOREM 2.8. [9] Every $\alpha - \omega-$open set is semi$-\omega-$open set.

DEFINITION 2.9. [6] A non-null collection $\mathcal{G}$ of subsets of a topological spaces (X, τ) is said to be a *grill* on X if $\mathcal{G}$ satisfies the following conditions:
(i) $A \in \mathcal{G}$ and $A \subseteq B$ implies that $B \in \mathcal{G}$
(ii) $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

For a grill topological space $(X, \tau, \mathcal{G})$, the operator $\Phi : P(X) \to P(X)$ from the power set $P(X)$ of X to $P(X)$ was defined in [7] in the following manner : For any $A \in P(X)$,

$$\Phi(A) = \{x \in X : U \cap A \in \mathcal{G}, \text{ for each open neighborhood } U \text{ of } x\}.$$

This operator is called the operator associated with the grill $\mathcal{G}$ and the topology τ.

Then the operator $\Psi : P(X) \to P(X)$, given by $\Psi(A) = A \cup \Phi(A)$, for $A \in P(X)$, was also shown in [7] to be a Kuratowski closure operator. So for a grill topological space $(X, \tau, \mathcal{G})$ there exists an unique topology $\tau_\mathcal{G}$ on X defined by

$$\tau_\mathcal{G} = \{U \subseteq X : \Psi(X - U) = X - U\},$$

where $\tau \subseteq \tau_\mathcal{G}$ and for any $A \subseteq X$, $\Psi(A) = Cl_\mathcal{G}(A)$ such that $Cl_\mathcal{G}(A)$ denotes the set of all $\mathcal{G}-$closure points of A. A point $x \in X$ is called a $\mathcal{G}-$*closure point* of A if for every open set U in $(X, \tau_\mathcal{G})$ containing x, $U \cap A \neq \emptyset$. A point $x \in A$ is called a $\mathcal{G}-$*interior point* of A if there is open set U in $(X, \tau_\mathcal{G})$ such that $x \in U \subseteq A$. The set of all $\mathcal{G}-$interior points of A denoted by $int_\mathcal{G}(A)$.

THEOREM 2.10. [7] Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then for $A, B \subseteq X$, the following properties hold:

(1) $A \subseteq B$ implies that $\Phi(A) \subseteq \Phi(B)$;
(2) $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$;
(3) $\Phi(\Phi(A)) \subseteq \Phi(A) = Cl(\Phi(A)) \subseteq Cl(A)$;
(4) If $U \in \tau$ then $U \cap \Phi(A) \subseteq \Phi(U \cap A)$.

DEFINITION 2.11. [1] A subset A of grill topological space $(X, \tau, \mathcal{G})$ is called a $\mathcal{G}^\omega-$*open set* if $A \subseteq Cl(Int_\omega(\Psi(A)))$. The complement of $\mathcal{G}^\omega-$open set is called $\mathcal{G}^\omega-$*closed set.*

DEFINITION 2.12. [8] A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is called a $\mathcal{G}\alpha$-*open set* if $A \subseteq Int(\Psi(Int(A))$. The complement of $\mathcal{G}\alpha$-open set is called $\mathcal{G}\alpha$-*closed set.*

3. $\mathcal{G}^\alpha_\omega-$OPEN SETS.

DEFINITION 3.1. A subset A of grill topological space $(X, \tau, \mathcal{G})$ is called a $\mathcal{G}^\alpha_\omega-$*open set* if $A \subseteq Int(\Psi(Int_\omega(A)))$. The complement of $\mathcal{G}^\alpha_\omega-$open set is called $\mathcal{G}^\alpha_\omega-$*closed set.* The set of all $\mathcal{G}^\alpha_\omega-$open sets in X is denoted by $\mathcal{G}^\alpha_\omega O(X, \tau)$ and the set of all $\mathcal{G}^\alpha_\omega-$closed sets in X is denoted by $\mathcal{G}^\alpha_\omega C(X, \tau)$.

EXAMPLE 3.2. Let $(X, \tau, \mathcal{G})$ be a grill topological space on the set $X = \{1, 2, 3\}$ with $\tau = \{\emptyset, X, \{2, 3\}\}$ and $\mathcal{G} = P(X) - \emptyset$. Then we observe that

$$\mathcal{G}^\alpha_\omega O(X, \tau) = \{\emptyset, X, \{2, 3\}\}$$

and

$$\mathcal{G}^\alpha_\omega C(X, \tau) = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1, 3\}, \{2, 1\}\}.$$

EXAMPLE 3.3. Let $(R, \tau, \mathcal{G})$ be a grill topological space on the set of real numbers R with $\tau = \{\emptyset, R, \{1, 2\}\}$ and $\mathcal{G} = P(R) - \{\emptyset\}$, where $P(R)$ is the power set of R, that is, the collection of all subsets of R. It is clear that $A = R - \{1, 2\}$ is $\omega-$open set and

$$Int(\Psi(Int_\omega(A))) = Int(\Psi(A)) = Int(A) = \emptyset,$$

that is, a set A is not $\mathcal{G}^\alpha_\omega-$open set in a grill topological space.

For a grill topological space $(R, \tau, \mathcal{G})$, there are not relationships between class of $\omega-$open sets and class of $\mathcal{G}^\alpha_\omega-$open sets. See in the example above that a set A is $\omega-$open set but not $\mathcal{G}^\alpha_\omega-$open set and B in the following example is $\mathcal{G}^\alpha_\omega-$open set but not $\omega-$open set:

EXAMPLE 3.4. Let $(R, \tau, \mathcal{G})$ be a grill topological space on the set of real numbers R with $\tau = \{\emptyset, R, [1, 2]\}$ and $\mathcal{G} = P(R) - \{\emptyset\}$. Let $B = (0, 3)$. Since

$$Int(\Psi(Int_\omega(B))) = Int(\Psi([1, 2])) = Int(R) = R,$$

then a set B is $\mathcal{G}^\alpha_\omega-$open set in a grill topological space.

The following theorem clears that there are relationships between class of $\omega-$open sets and class of $\mathcal{G}^\alpha_\omega-$open sets for grill topological spaces $(X, \tau, \mathcal{G})$ with countable set X.

THEOREM 3.5. For any grill topological space$(X, \tau, \mathcal{G})$ with a countable set X, every $\mathcal{G}^\alpha_\omega-$open set in a grill topological space $(X, \tau, \mathcal{G})$ is $\omega-$open set.

PROOF. In this case if X is a countable sets then all subsets of X are both $\omega-$open and $\omega-$closed set. Then every $\mathcal{G}^\alpha_\omega-$open set in a grill topological space $(X, \tau, \mathcal{G})$ is $\omega-$open set. □

THEOREM 3.6. Every $\mathcal{G}^\alpha_\omega-$open set in a grill topological space $(X, \tau, \mathcal{G})$ is $\alpha - \omega-$open set in topological space (X, τ).

PROOF. Let A be any $\mathcal{G}^\alpha_\omega-$open set. Then

$$A \subseteq Int(\Psi(Int_\omega(A))).$$

Hence

$$\begin{aligned} A &\subseteq Int(\Psi(Int_\omega(A))) \subseteq Int_\omega(\Psi(Int_\omega(A))) \\ &\subseteq Int_\omega(Cl(Int_\omega(A))). \end{aligned}$$

That is, A is $\alpha - \omega-$open set in topological space (X, τ). □

The converse of theorem above no need to be true.

EXAMPLE 3.7. Let $(R, \tau, \mathcal{G})$ be a grill topological space on the set of real numbers R with $\tau = \{\emptyset, R, \{2\}\}$ and $\mathcal{G} = P(R) - \{\emptyset\}$ then a set $A = R - \{2\}$ is $\alpha - \omega-$open set in topological space (X, τ), but not $\mathcal{G}^\alpha_\omega-$open set in a grill topological space.

COROLLARY 3.8. Every $\mathcal{G}_\omega^\alpha-$open set in a grill topological space $(X, \tau, \mathcal{G})$ is a semi$-\omega-$open set in topological space (X, τ).

PROOF. Form Theorems(2.8) and (3.6). □

Example(3.7) and Theorem(2.8) show that the converse of theorem above no need to be true.

THEOREM 3.9. Every $\mathcal{G}_\omega^\alpha-$open set in a grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}^\omega-$open set in a grill topological space $(X, \tau, \mathcal{G})$.

PROOF. Let A be any $\mathcal{G}_\omega^\alpha-$open set. Then

$$A \subseteq Int(\Psi(Int_\omega(A))).$$

Hence

$$\begin{aligned} A &\subseteq Int(\Psi(Int_\omega(A))) \subseteq \Psi(Int_\omega(A))) \\ &\subseteq Cl(Int_\omega(A))) \subseteq Cl(Int_\omega(\Psi(A))). \end{aligned}$$

That is, A is $\mathcal{G}^\omega-$open set in topological space (X, τ). □

The converse of theorem above no need to be true.

EXAMPLE 3.10. Let $(R, \tau, \mathcal{G})$ be a grill topological space on the set of real numbers R with $\tau = \{\emptyset, R, R - \{2\}\}$ and $\mathcal{G} = P(R) - \{\emptyset\}$ then a set $A = \{1\}$ is $\mathcal{G}^\omega-$open set, but not $\mathcal{G}_\omega^\alpha-$open set in a grill topological space $(R, \tau, \mathcal{G})$.

THEOREM 3.11. Every $\mathcal{G}\alpha$-open set in a grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}_\omega^\alpha-$open set in a grill topological space $(X, \tau, \mathcal{G})$.

PROOF. Let A be any $\mathcal{G}\alpha$-open set. Then

$$A \subseteq Int(\Psi(Int(A))).$$

Hence

$$A \subseteq Int(\Psi(Int(A))) \subseteq Int(\Psi(Int_\omega(A))).$$

That is, A is $\mathcal{G}_\omega^\alpha-$open set in topological space (X, τ). □

The converse of theorem above no need to be true.

EXAMPLE 3.12. Let $(R, \tau, \mathcal{G})$ be a grill topological space on the set of real numbers R with $\tau = \{\emptyset, R, R - \{2\}$ and $\mathcal{G} = P(R) - \{\emptyset\}$ then a set $A = R - \{1\}$ is $\mathcal{G}_\omega^\alpha-$open se but not $\mathcal{G}\alpha-$open set in a grill topological space $(X, \tau, \mathcal{G})$.

From all the previous relationships in our work and the preliminaries, we have the following figure.

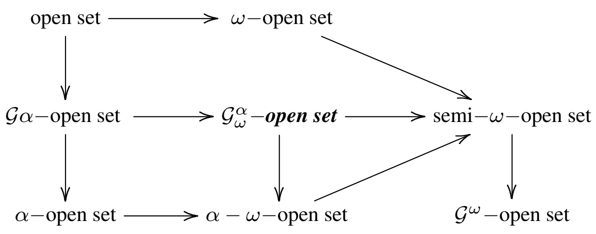

Fig. 1. Relation for sets

THEOREM 3.13. A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}_\omega^\alpha-$closed set if and only if $Cl(int_\mathcal{G}(Cl_\omega(A))) \subseteq A$.

PROOF. Let A be any $\mathcal{G}_\omega^\alpha-$closed set in grill topological space $(X, \tau, \mathcal{G})$. Then $(X - A)$ is $\mathcal{G}_\omega^\alpha-$open set in grill topological space. Then we have

$$(X - A) \subseteq Int(\Psi(Int_\omega(X - A))).$$

By using Theorem(2.2), theorem(2.4) and theorem(2.10) we get

$$\begin{aligned} (X - A) &\subseteq Int(\Psi(Int_\omega(X - A))) \\ &= Int(\Psi(X - Cl_\omega A))) \\ &= Int(Cl_\mathcal{G}(X - Cl_\omega A))) \\ &= Int(X - Int_\mathcal{G}(Cl_\omega A))) \\ &= X - Cl(int_\mathcal{G}(Cl_\omega(A))). \end{aligned}$$

Hence $Cl(int_\mathcal{G}(Cl_\omega(A))) \subseteq A$. The conversely is similar. □

THEOREM 3.14. Let A_μ be any $\mathcal{G}_\omega^\alpha-$open set in grill topological space $(X, \tau, \mathcal{G})$, for each $\mu \in I$ then $\cup_{\mu \in I} A_\mu$ is $\mathcal{G}_\omega^\alpha-$open set in grill topological space $(X, \tau, \mathcal{G})$, where I is an index set.

PROOF. Since A_μ is $\mathcal{G}_\omega^\alpha-$open set for each $\mu \in I$ then $A_\mu \subseteq Int(\Psi(Int_\omega(A_\mu)))$.
From the previous relationships in our work and the preliminaries, we have the following

$$\begin{aligned} \cup_{\mu \in I} A_\mu &\subseteq \cup_{\mu \in I} Int(\Psi(Int_\omega(A_\mu))) \\ &\subseteq Int(\cup_{\mu \in I}(\Psi(Int_\omega(A_\mu)))) \\ &= Int(\cup_{\mu \in I}(\Phi(Int_\omega(A_\mu)) \cup Int_\omega(A_\mu))) \\ &\subseteq Int(\Phi(\cup_{\mu \in I}(Int_\omega(A_\mu))) \cup (\cup_{\mu \in I} Int_\omega(A_\mu))) \\ &\subseteq Int(\Phi(Int_\omega(\cup_{\mu \in I} A_\mu)) \cup (Int_\omega(\cup_{\mu \in I} A_\mu))) \\ &= Int(\Psi(Int_\omega(\cup_{\mu \in I} A_\mu))). \end{aligned}$$

Hens $\cup_{\mu \in I} A_\mu$ is $\mathcal{G}_\omega^\alpha-$open set in a grill topological space $(X, \tau, \mathcal{G})$. □

The intersection of two $\mathcal{G}_\omega^\alpha-$open set no need to be $\mathcal{G}_\omega^\alpha-$open set.

EXAMPLE 3.15. Let $(X, \tau, \mathcal{G})$ be a grill topological space on the set $X = \{1, 2, 3\}$ with $\tau = \{\emptyset, X, \{2, 3\}\}$ and $\mathcal{G} = P(X) - \{\emptyset\}$ then $A = \{1, 2\}$ and $B = \{1, 3\}$ are $\mathcal{G}_\omega^\alpha-$open sets in grill topological space $(X, \tau, \mathcal{G})$, but $A \cap B = \{1\}$ is not a $\mathcal{G}_\omega^\alpha-$open set in a grill topological space $(X, \tau, \mathcal{G})$.

THEOREM 3.16. Let A be any open set in topological space (X, τ) and B be any $\mathcal{G}_\omega^\alpha-$open set in a grill topological space $(X, \tau, \mathcal{G})$ then $A \cap B$ is $\mathcal{G}_\omega^\alpha-$open set in a grill topological space $(X, \tau, \mathcal{G})$.

PROOF. Since B is $\mathcal{G}_\omega^\alpha-$open set then $B \subseteq Int(\Psi(Int_\omega(B)))$.
By Theorem(2.1)

$$\begin{aligned} A \cap B &\subseteq A \cap Int(\Psi(Int_\omega(B))) = Int(A) \cap Int(\Psi(Int_\omega(B))) \\ &= Int(A \cap \Psi(Int_\omega(B))) \subseteq Int(\Psi(A \cap Int_\omega(B))) \\ &= Int(\Psi(Int_\omega(A) \cap Int_\omega(B))) = Int(\Psi(Int_\omega(A \cap B))). \end{aligned}$$

Hens $A \cap B$ is $\mathcal{G}_\omega^\alpha-$open set in a grill topological space $(X, \tau, \mathcal{G})$. □

4. $\mathcal{G}_\omega^\alpha-$OPERATORS

DEFINITION 4.1. Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \subseteq X$ The $\mathcal{G}_\omega^\alpha-$interior operator of A denoted by $_{\mathcal{G}_\omega^\alpha} Int(A)$ and defined by

$$_{\mathcal{G}_\omega^\alpha} Int(A) = \cup\{B \subseteq X : B \subseteq A \text{ and } B \in \mathcal{G}_\omega^\alpha O(X, \tau)\}.$$

That is, ${}_{\mathcal{G}^\alpha_\omega}Int(A)$ is the union of all $\mathcal{G}^\alpha_\omega-$open sets contained in A. The $\mathcal{G}^\alpha_\omega-$closure operator of A denoted by ${}_{\mathcal{G}^\alpha_\omega}Cl(A)$ and defined by

$${}_{\mathcal{G}^\alpha_\omega}Cl(A) = \cap\{B \subseteq X : A \subseteq B \text{ and } B \in \mathcal{G}^\alpha_\omega C(X, \tau)\}.$$

That is, ${}_{\mathcal{G}^\alpha_\omega}Cl(A)$ is the intersection of all $\mathcal{G}^\alpha_\omega-$closed sets containing A.

THEOREM 4.2. Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \subseteq X$. Then ${}_{\mathcal{G}^\alpha_\omega}Int(A) = A$ if and only if A is $\mathcal{G}^\alpha_\omega-$open set.

PROOF. Let ${}_{\mathcal{G}^\alpha_\omega}Int(A) = A$. Then from definition of ${}_{\mathcal{G}^\alpha_\omega}Int(A)$ and by Theorem(3.14), ${}_{\mathcal{G}^\alpha_\omega}Int(A)$ is $\mathcal{G}^\alpha_\omega-$open set. Hence A is $\mathcal{G}^\alpha_\omega-$open set.
Conversely, let A be $\mathcal{G}^\alpha_\omega-$open set. Then from definition of $\mathcal{G}^\alpha_\omega-$open set we have ${}_{\mathcal{G}^\alpha_\omega}Int(A) \subseteq A$. Since A is $\mathcal{G}^\alpha_\omega-$open set, then from definition of ${}_{\mathcal{G}^\alpha_\omega}Int(A)$, we get that $A \subseteq {}_{\mathcal{G}^\alpha_\omega}Int(A)$. Hence ${}_{\mathcal{G}^\alpha_\omega}Int(A) = A$. □

THEOREM 4.3. Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \subseteq X$. Then $x \in {}_{\mathcal{G}^\alpha_\omega}Int(A)$ if and only if there $\mathcal{G}^\alpha_\omega-$open set U such that $x \in U \subseteq A$.

PROOF. Let $x \in {}_{\mathcal{G}^\alpha_\omega}Int(A)$ take $U = {}_{\mathcal{G}^\alpha_\omega}Int(A)$. Then by definition ${}_{\mathcal{G}^\alpha_\omega}Int(A)$ and Theorem(3.14), we get that U is $\mathcal{G}^\alpha_\omega-$open set and $x \in U \subseteq {}_{\mathcal{G}^\alpha_\omega}Int(A)$.
Conversely, let there is $\mathcal{G}^\alpha_\omega-$open set U such that $x \in U \subseteq A$. Then by definition of ${}_{\mathcal{G}^\alpha_\omega}Int(A)$ we get that $x \in {}_{\mathcal{G}^\alpha_\omega}Int(A)$. □

THEOREM 4.4. Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \subseteq X$. Then $x \in {}_{\mathcal{G}^\alpha_\omega}Cl(A)$ if and only if for all $\mathcal{G}^\alpha_\omega-$open set U containing x such that $U \cap A \neq \emptyset$.

PROOF. Let $x \in {}_{\mathcal{G}^\alpha_\omega}Cl(A)$ and U be $\mathcal{G}^\alpha_\omega-$open set containing x. If $U \cap A = \emptyset$ then $A \subseteq X - A$. Since A is $\mathcal{G}^\alpha_\omega-$open set then $X - A$ is $\mathcal{G}^\alpha_\omega-$closed set. So ${}_{\mathcal{G}^\alpha_\omega}Cl(A) \subseteq X - A$, and

$$x \in {}_{\mathcal{G}^\alpha_\omega}Cl(A) \subseteq X - A.$$

This is contradiction, because $x \in U$. Hence $U \cap A \neq \emptyset$.
Conversely, let $x \notin {}_{\mathcal{G}^\alpha_\omega}Cl(A)$. Then $X - {}_{\mathcal{G}^\alpha_\omega}Cl(A)$ is $\mathcal{G}^\alpha_\omega-$open set containing x. Hence by hypothesis $X - {}_{\mathcal{G}^\alpha_\omega}Cl(A) \neq \emptyset$. his is contradiction, because $X - {}_{\mathcal{G}^\alpha_\omega}Cl(A) \subseteq X - A$. □

THEOREM 4.5. Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \subseteq X$ then ${}_{\mathcal{G}^\alpha_\omega}Cl(A) = A$ if and only if A is $\mathcal{G}^\alpha_\omega-$closed set.

PROOF. Let ${}_{\mathcal{G}^\alpha_\omega}Cl(A) = A$. Then from definition of ${}_{\mathcal{G}^\alpha_\omega}Cl(A)$ and by Theorem(3.14), ${}_{\mathcal{G}^\alpha_\omega}Cl(A)$ is $\mathcal{G}^\alpha_\omega-$closed set. Hence A is $\mathcal{G}^\alpha_\omega-$closed set.
Conversely, let A be $\mathcal{G}^\alpha_\omega-$closed set. Then from definition of $\mathcal{G}^\alpha_\omega-$closed set, we have $A \subseteq {}_{\mathcal{G}^\alpha_\omega}Int(A)$. Since A is $\mathcal{G}^\alpha_\omega-$open set, then from definition of ${}_{\mathcal{G}^\alpha_\omega}Int(A)$, we get that ${}_{\mathcal{G}^\alpha_\omega}Cl(A) \subseteq A$. Hence ${}_{\mathcal{G}^\alpha_\omega}Cl(A) = A$. □

THEOREM 4.6. Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A, B \subseteq X$. Then the following hold:

(1) If $A \subseteq B$ then ${}_{\mathcal{G}^\alpha_\omega}Cl(A) \subseteq {}_{\mathcal{G}^\alpha_\omega}Cl(B)$.
(2) ${}_{\mathcal{G}^\alpha_\omega}Cl(A) \cup {}_{\mathcal{G}^\alpha_\omega}Cl(B) \subseteq {}_{\mathcal{G}^\alpha_\omega}Cl(A \cup B)$.
(3) ${}_{\mathcal{G}^\alpha_\omega}Cl(A \cap B) \subseteq {}_{\mathcal{G}^\alpha_\omega}Cl(A) \cap {}_{\mathcal{G}^\alpha_\omega}Cl(B)$.
(4) ${}_{\mathcal{G}^\alpha_\omega}Cl(A) \subseteq Cl(A)$.

PROOF. 1. Let $x \in {}_{\mathcal{G}^\alpha_\omega}Cl(A)$. Then by Theorem(4.4), for all $\mathcal{G}^\alpha_\omega-$open set U containing x, $U \cap A \neq \emptyset$. Since $A \subseteq B$, then $U \cap B \neq \emptyset$. Hence $x \in {}_{\mathcal{G}^\alpha_\omega}Cl(B)$. That is, ${}_{\mathcal{G}^\alpha_\omega}Cl(A) \subseteq {}_{\mathcal{G}^\alpha_\omega}Cl(B)$.
2. It is clear from the Part (1).
3. It is clear from the Part (1).
4. By the fact, every open set U is $\mathcal{G}^\alpha_\omega$-open set. □

In the last theorem ${}_{\mathcal{G}^\alpha_\omega}Cl(A \cup B) \neq {}_{\mathcal{G}^\alpha_\omega}Cl(A) \cup {}_{\mathcal{G}^\alpha_\omega}Cl(B)$.

EXAMPLE 4.7. In Example(3.15), take $A = \{2\}$ and $B = \{3\}$. Then

$${}_{\mathcal{G}^\alpha_\omega}Cl(A) \cup {}_{\mathcal{G}^\alpha_\omega}Cl(B) = A \cup B = \{2, 3\}$$

and

$${}_{\mathcal{G}^\alpha_\omega}Cl(A \cup B) = {}_{\mathcal{G}^\alpha_\omega}Cl(\{2, 3\}) = X.$$

THEOREM 4.8. Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A, B \subseteq X$. Then the following hold:

(1) If $A \subseteq B$ then ${}_{\mathcal{G}^\alpha_\omega}Int(A) \subseteq {}_{\mathcal{G}^\alpha_\omega}Int(B)$.
(2) ${}_{\mathcal{G}^\alpha_\omega}Int(A) \cup {}_{\mathcal{G}^\alpha_\omega}Int(B) \subseteq {}_{\mathcal{G}^\alpha_\omega}Int(A \cup B)$.
(3) ${}_{\mathcal{G}^\alpha_\omega}Int(A \cap B) \subseteq {}_{\mathcal{G}^\alpha_\omega}Int(A) \cap {}_{\mathcal{G}^\alpha_\omega}Int(B)$.
(4) $Int(A) \subseteq {}_{\mathcal{G}^\alpha_\omega}Int(A)$.

PROOF. Similar for the proof of Theorem(4.6) □

In the last theorem ${}_{\mathcal{G}^\alpha_\omega}Int(A \cap B) \neq {}_{\mathcal{G}^\alpha_\omega}Int(A) \cap {}_{\mathcal{G}^\alpha_\omega}Int(B)$.

EXAMPLE 4.9. In Example(3.15), take $A = \{1, 2\}$ and $B = \{1, 3\}$. Then

$${}_{\mathcal{G}^\alpha_\omega}Int(A) \cap {}_{\mathcal{G}^\alpha_\omega}Int(B) = A \cap B = \{1\}$$

and

$${}_{\mathcal{G}^\alpha_\omega}Int(A \cap B) = {}_{\mathcal{G}^\alpha_\omega}Int(\{1\}) = \emptyset.$$

THEOREM 4.10. Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \subseteq X$.Then the following hold:

(1) ${}_{\mathcal{G}^\alpha_\omega}Int(X - A) = X - \mathcal{G}^\alpha_\omega Cl(A)$.
(2) ${}_{\mathcal{G}^\alpha_\omega}Cl(X - A) = X - {}_{\mathcal{G}^\alpha_\omega}Int(A)$.

PROOF. 1. Since $A \subseteq {}_{\mathcal{G}^\alpha_\omega}Cl(A)$, then $X - {}_{\mathcal{G}^\alpha_\omega}Cl(A) \subseteq X - A$. Since $X - {}_{\mathcal{G}^\alpha_\omega}Cl(A)$ is $\mathcal{G}^\alpha_\omega-$open set in a grill topological space $(X, \tau, \mathcal{G})$ then

$${}_{\mathcal{G}^\alpha_\omega}Int[X - ({}_{\mathcal{G}^\alpha_\omega}Cl(A))] = X - {}_{\mathcal{G}^\alpha_\omega}Cl(A) \subseteq {}_{\mathcal{G}^\alpha_\omega}Int(X - A) \quad (1)$$

Now let $x \in {}_{\mathcal{G}^\alpha_\omega}Int(X - A)$. Then there is $\mathcal{G}^\alpha_\omega-$open set U such that $x \in U \subseteq X - A$. Since $X - U$ is $\mathcal{G}^\alpha_\omega-$closed set containing A and $x \notin (X - U)$, then $x \notin {}_{\mathcal{G}^\alpha_\omega}Cl(A)$, that is $x \in X - {}_{\mathcal{G}^\alpha_\omega}Cl(A)$. Then

$${}_{\mathcal{G}^\alpha_\omega}Int(X - A) \subseteq X - {}_{\mathcal{G}^\alpha_\omega}Cl(A). \quad (2)$$

From (1) and (2), we get ${}_{\mathcal{G}^\alpha_\omega}Int(X - A) = X - {}_{\mathcal{G}^\alpha_\omega}Cl(A)$.

2. Since ${}_{\mathcal{G}^\alpha_\omega}Int(A) \subseteq A$, then $X - A \subseteq X - {}_{\mathcal{G}^\alpha_\omega}Int(A)$. Since $X - {}_{\mathcal{G}^\alpha_\omega}Int(A)$ is $\mathcal{G}^\alpha_\omega-$closed set in a grill topological space $(X, \tau, \mathcal{G})$ then

$${}_{\mathcal{G}^\alpha_\omega}Cl(X - A) \subseteq {}_{\mathcal{G}^\alpha_\omega}Cl[X - {}_{\mathcal{G}^\alpha_\omega}Int(A)] = X - {}_{\mathcal{G}^\alpha_\omega}Int(A) \quad (3)$$

For the other side, let $x \in X - {}_{\mathcal{G}^\alpha_\omega}Int(A) = X - {}_{\mathcal{G}^\alpha_\omega}Int(X - (X - A))$. This implies

$$X - [X - {}_{\mathcal{G}^\alpha_\omega}Cl(X - A)] = {}_{\mathcal{G}^\alpha_\omega}Cl(X - A).$$

Then $x \in {}_{\mathcal{G}^\alpha_\omega}Cl(X - A)$. Hence

$$X - {}_{\mathcal{G}^\alpha_\omega}Int(A) \subseteq {}_{\mathcal{G}^\alpha_\omega}Cl(X - A). \quad (4)$$

From (3) and (4) we get that ${}_{\mathcal{G}^\alpha_\omega}Cl(X - A) = X - {}_{\mathcal{G}^\alpha_\omega}Int(A)$. □

5. REFERENCES

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