Least-Squares Collocation Bernstein Method for Solving System of Linear Fractional Integro-diffrential Equations

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ABSTRACT

This study gears towards finding a new simple numerical algorithm to solve system of linear fractional integro-differential equations. The technique involves the application of Caputo properties, the properties of Bernstein polynomials and least square collocation approach to reduce the problem to system of linear algebraic equations and then solved. To demonstrate the accuracy and applicability of the presented method some numerical examples are given. Numerical results show that the method is easy to implement and compares favorably with the exact results. The graphical solution of the method is displayed.

Keywords

System of linear fractional integro- differential equations; least squares collocation; Bernstein polynomials

1. INTRODUCTION

Fractional integro-differential equations has played a significant role in modelling of real world physical problems e.g the modeling of earthquake, reducing the spread of virus, control the memory behaviour of electric socket and many others. Fractional calculus is a field dealing with integral and derivatives of arbitrary orders, and their applications in science, engineering and other fields. The idea is from the ordinary calculus. According to [1- 3], It was discovered by Leibniz in the year 1695 few years after he discovered ordinary calculus but later forgotten due to the complexity of the formula. Since most Fractional Integro-diffrential Equations (FIDEs) cannot be solved analytically, much attention has been devoted to search for approximate and numerical techniques to the solution of FIDEs. Recently, many methods have been developed by researchers for providing approximate solutions of FIDEs. [4] employed Lagurre polynomials as basis functions for the solution of fractional Solving Fredholm integro-differential equations while [5] employed Bernstein polynomials as basis functions to approximate the solution of FIDEs. References [6 - 8] applied collocation techniques for solving FIDEs using different basis functions. [9] applied Sumudu transform method and Hermite Spectral collocation method for solving FIDEs. Author [10] introduced approximate solutions of Volterra-Fredholm integro-differential equations of fractional order. References [11 - 12] used Least - Squares method for the solution of FIDEs. [13 - 15] introduced numerical solution of fractional singular integro-differential equations by using Taylor series expansion and Galerkin method and a fast numerical algorithm based on the second kind of Chebyshev

polynomials. The author in [16] applied numerical solution of Fredholm-Volterra fractional integro-differential equation with nonlocal boundary conditions. Reference [17] employed Bernstein modified homotopy perturbation method for the Solution of Volterra fractional integro-differential equations. The objective of this work is to introduce a new technique called least squares collocation Bernstein method with application of Caputo properties that provide less rigorous works in terms of computational cost with improved accuracy for finding an approximate solution to system of Linear fractional integro- differential equations. The general form of the class of problem considered in this work is given as:

$$D^\alpha u_i(x) = p_i(x)u_i(x) + f_i(x) + \int_0^x k_i(x,t)(\sum_{r=1}^n u_r(t)) \, dt,$$
$$i = 1,2,\dots,n, \ o \le x, t \le 1, \tag{1}$$

With the following supplementary conditions:

$$u_i^{(j)}(x_0) = u_{ij} \ \ i = 0,1,2,\dots,m-1,$$
$$m - 1 < \alpha \le m, \ m \in, N \tag{2}$$

Where $D^\alpha u_i(x)$ indicates the $\propto th$ Caputo fractional derivative of $u_i(x)$; $p_i(x)$, $f_i(x)$, $k_i(x,t)$ are given smooth functions, x and t are real variables varying [0, 1] and $u_i(x)$ is the unknown function to be determined.

2. SOME RELEVANT BASIC DEFINITIONS

Definition 2.1: Riemann – Lowville fractional integral is defined as [18]:

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(x)}{(x-t)^{1-a}} \, dt, \ \alpha > 0, x > 0, \tag{3}$$

J^α denotes the fractional integral of order $\propto$

Definition 2.2: The Caputo Factional Derivative is defined as [18]:

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-s)^{n-\alpha-1} f^m(s) ds \tag{4}$$

Where m is a positive integer with the property that $n - 1 < \propto < n$

For example if $0 < \propto < 1$ the caputo fractional derivative is

$$D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} f^1(s) ds \tag{5}$$

Hence, we have the following properties:

(1) $J^\alpha J^v f = j^{\alpha+v} f, \ \alpha, v > 0, f \in C_\mu, \mu > 0$

(2) $J^\alpha x^\gamma = \frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \ \alpha > 0, \gamma > -1, x > 0$

(3) $J^\alpha \quad D^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} f^k(0) \frac{x^k}{k!}, \quad x > 0, n - 1 < \alpha \le n$

(4) $D^\alpha J^\alpha f(x) = f(x), \ \ x > 0, n - 1 < \alpha \le n,$

(5) $D^\alpha C = 0, C$ is the constant,

$$(6) \begin{cases} 0, & \beta \in N_0, \beta < [\alpha], \\ D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, & \beta \in N_0, \beta \geq [\alpha], \end{cases}$$

Where $[\alpha]$ denoted the smallest integer greater than or equal to α and $N_0 = \{0, 1.2, \dots\}$

Definition 2.3: Bernstein basis polynomials: A Bernstein polynomial [19] of degree N is defined by

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i} \quad i = 0, 1 \dots n, \tag{6}$$

where,

$$\binom{m}{i} = \frac{m!}{i!(m-1)!} \tag{7}$$

Often, for mathematical convenience, we set $B_{i,m}(x) = 0$ if $< 0 \; or \; j > m$

Definition 2.4: Bernstein polynomials: A linear combination Bernstein [19] basis polynomials

$$u_i(x) = \sum_j^m a_j{}^i u_j(x) \tag{8}$$

the Bernstein polynomial of degree n where $a_j, \; j = 0, 1, 2, \dots \dots$ are constants.

3. DEMONSTRATION OF LEAST SQUARES COLLOCATION BERNSTEIN METHOD (LSCBM)

The Least Squares Collocation Bernstein Method is based on approximating the unknown function $u_i(x)$ in (1) by assuming an approximate solution of the form defined in (8). Consider equation (1) operating with J^α on both sides as follows:

$$J^\alpha[u_i(x)] = J^\alpha \left[p_i(x) u_i(x) + f_i(x) + \int_0^x k_i(x,t) (\sum_{v=1}^n u_v(t)) \, dt \right] \tag{9}$$

$$u_i(x) = \sum_{k=0}^{n-1} u_i{}^k(0) \frac{x^k}{k!} + J^\alpha \left[p_i(x) u_i(x) + f_i(x) + \int_0^x k_i(x,t) (\sum_{r=1}^n u_r(t)) \, dt \right] \tag{10}$$

Substituting (8) into (10)

$$\sum_{j=0}^m a_j{}^i u_j(x) = \sum_{k=0}^{m-1} u_i{}^k(0) \frac{x^k}{k!} + J^\alpha [p_i(x) \sum_j^m a_j{}^i u_j(x) + f_i(x) + \int_0^x k_i(x,t)(\sum_{r=1}^n [\sum_{j=0}^m a_j{}^i u_j(t)]) \, dt] \tag{11}$$

Hence, the residual equation is obtained as

$$R(a_0{}^i, a_1{}^i, \dots, a_m{}^i) =$$
$$\sum_{j=0}^m a_j{}^i u_j(x) - \{\sum_{k=0}^{m-1} u_i{}^k(0) \frac{x^k}{k!} + J^\alpha [p_i(x) \sum_{j=0}^m a_j{}^i u_j(x) + f_i(x) + \int_0^x k_i(x,t)(\sum_{r=1}^n [\sum_{j=0}^m a_j{}^i u_j(t)]) \, dt] \} \tag{12}$$

Let

$$S(a_0, a_1 \dots, a_m) = [R(a_0, a_1, \dots \dots, a_m)]^2 w(x) \tag{13}$$

Where $w(x)$ is the positive weight function defined in the interval, $[a, b]$. In this work, we take $w(x) = 1$ for simplicity. Thus,

$$S(a_0{}^i, a_1{}^i, \dots, a_m{}^i) = \left\{ \sum_{j=0}^m a_j{}^i u_j(x) - \{\sum_{k=0}^{m-1} u_i{}^k(0) \frac{x^k}{k!} + J^\alpha [p(x) \sum_{j=0}^m a_j{}^i u_j(x) + f_i(x) + \int_0^x k_i(x,t)(\sum_{r=1}^n [\sum_{j=0}^m a_j{}^i u_j(t)]) \, dt] \right\}^2 \tag{14}$$

In order to minimize equation (15), we obtained the values of $a_j \; (j \geq 0)$ by finding the minimum value of S as:

$$\frac{\partial S_i}{\partial a_j{}^i} = 0, j = 0, 1, 2 \dots, m \tag{15}$$

Applying (15) on (14), we have

$$\left\{ \sum_{j=0}^m a_j{}^i u_j(x) - \{\sum_{k=0}^{m-1} u_i{}^k(0) \frac{x^k}{k!} + J^\alpha [p(x) \sum_{j=0}^m a_j{}^i u_j(x) + f_i(x) + J^\alpha [\int_0^x k_i(x,t)(\sum_{r=1}^n [\sum_{j=0}^m a_j{}^i u_j(t)]) \, dt] \right\} \times \{u_j(x) - J^\alpha[p(x)u_j(x)] - J^\alpha[\int_0^x k_i(x,t)(\sum_{r=1}^n [\sum_{j=0}^m u_j(t)]) \, dt] \} \tag{16}$$

Thus, (16) is then simplified for $j = 0, 1, \dots n$ and collocated at equally spaced point $x_i = a + \frac{(b-a)i}{m}$, $(i = 1(1)m)$ to obtain $(m+1)$ algebraic system of equations in $(m+1)$ unknown $a^i{}_j$ which are then put in matrix form as follow:

$$A = \begin{pmatrix} R_i(x, a_0{}^i) h_0{}^i & R_i(x, a_1{}^i) h_0{}^i & \cdots & R_i(x, a_m{}^i) h_0{}^i \\ R_i(x, a_0{}^i) h_1{}^i & R_i(x, a_1{}^i) h_1{}^i & \cdots & R_i(x, a_m{}^i) h_1{}^i \\ \vdots & \vdots & \ddots & \vdots \\ R_i(x, a_m{}^i) h_m{}^i & R_i(x, a_m{}^i) h_m{}^i & \dots & R_i(x, a_j{}^i) h_m{}^i \end{pmatrix}, \tag{17}$$

$$B = \begin{pmatrix} \left[J^\alpha f_i(x) + \sum_{k=0}^{n-1} u_i{}^k(0) \frac{x^k}{k!} \right] h_0{}^i \\ \left[J^\alpha f_i(x) + \sum_{k=0}^{n-1} u_i{}^k(0) \frac{x^k}{k!} \right] h_1{}^i \\ \vdots \\ \left[J^\alpha f_i(x) + \sum_{k=0}^{n-1} u_i{}^k(0) \frac{x^k}{k!} \right] h_m{}^i \end{pmatrix} \tag{18}$$

Where

$$h_j{}^i = u_j(x) - J^\alpha[p(x)u_j(x)] - J^\alpha[\int_0^x k_i(x,t)(\sum_{r=1}^n [\sum_{j=0}^m u_j(t)]) \, dt], \; j = 0, 1, \dots, m \tag{19}$$

$$R_i(x, a_j{}^i) = \sum_{j=0}^m a_j{}^i u_j(x) - \{ J^\alpha [p(x) \sum_{j=0}^m a_j{}^i u_j(x) + J^\alpha[\int_0^x k_i(x,t)(\sum_{r=1}^n [\sum_{j=0}^m a_j{}^i u_j(t)]) \, dt], \tag{20}$$
$$j = 0, 1, \dots, m, \; I = 1, 2, \dots, n$$

The $(m+1)$ linear equations are then solved to obtain the unknown constants $a_j (j = 0(1)m)$, which are then substituted back into the assumed approximate solution to give the required approximation solution.

5. NUMERICAL EXAMPLES

In this section, the above technique is implemented on some problems. The problems are then solved via the Bernstein polynomials as basis functions. The problems are then solved to illustrate the computational cost accuracy and efficiency of the proposed method using Maple 18.

Example 5.1: Consider the following fractional Integro-differential [20]

$$D^\alpha u_1(x) = -\frac{x}{6} + \frac{3x^{\frac{1}{3}}}{\Gamma(\frac{1}{3})} + \int_0^1 2xt[u_1(t) + u_2(t)]dt, \tag{21}$$

$$D^{\frac{2}{3}} u_2(x) = -\frac{5x^3}{6} + \frac{9x^{\frac{4}{3}}}{2\Gamma(\frac{1}{3})} + \int_0^1 x^3[u_1(t) - u_2(t)]dt \tag{22}$$

Subject to initial condition $u_1(0) = -1, u_2(0) = 0$ with the exact solution $u_1(x) = x - 1, u_2(x) = x^2$

Applying above mothed on example 1, taking $\alpha = \frac{3}{4}$ and $m = 2$. The following constants were obtained as: $a_0 = -1.000000014, a_1 = -0.4999999007, a_2 = 0$ for equation (21) and $a_0 = 9.696724573 \times 10^{-10}, a_1 = -4.132001054 \times 10^{-9} 0, a_2 = 1.000000011$ for equation (22). Substituting the values back into the assumed approximate solution, we obtain the approximate solution as:
$u_1(x) = -2.126 \times 10^{-7} x^2 + 1.0000000227 x - 1.000000014,$
$u_2(x) = 1.000000020 x^2 - 1.020334702 \times 10^{-8} x + 9.696724573 \times 10^{-10}$. Comparing the result obtained by [20] with the new method, it tends to be said that the proposed method performed more accurately since the table of error found is smaller than [20] and the graph of the approximate solution is the same as the graph of exact solution.
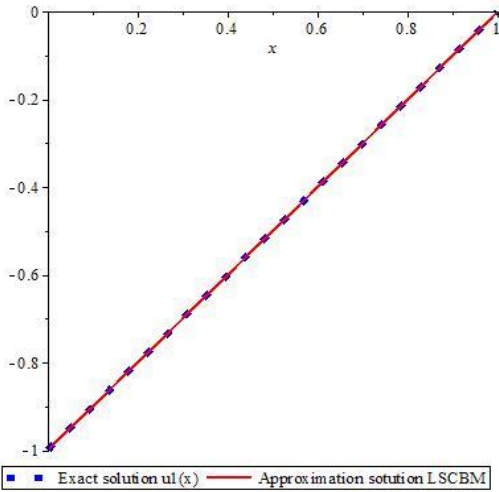
Figure 1: Showing the graph of approximation solution $u_1(x)$ and exact of example 1
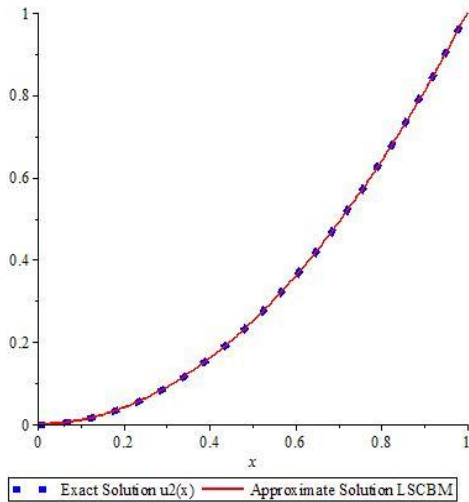

Figure 2: Showing the graph of approximation solution $u_2(x)$ and exact of example 1

Example 5.2: Consider the following fractional Integro-differential [20]

$$D^\propto u_1(x) = -\frac{1}{20} - \frac{x}{12} + \frac{4x^{\frac{1}{4}}(15-23x^2)}{15\Gamma\left(\frac{1}{3}\right)} + \int_0^1 (x+t)[u_1(t) +$$

$$u_2(t)]dt, \tag{23}$$

$$D^\propto u_2(x) = -\frac{5x^3}{6} + \frac{9x^{\frac{4}{3}}}{2\Gamma\left(\frac{1}{3}\right)} + \int_0^1 x^{\frac{1}{2}}t^2[u_1(t) - u_2(t)]dt \tag{24}$$

Subject to initial condition $u_1(0) = 0, u_2(0) = 0$ with the exact solution $u_1(x) = x - x^3, u_2(x) = x^2 - x$.
Solving Example 2, following the same procedure above, we take $\propto = \frac{3}{4}$ and $m = 3$. The following constants were obtained as: $a_0 = 9.030799779 \times 10^{-9}, a_1 = 0.3333335125$, $a_2 = 0.6666672478, a_3 = 0$ for equation (23) and $a_0 = 0, a_1 = -0.3333333432, a_2 = -0.3333333621, a_3 = 0$ for equation (24). Substituting the values back into the assumed approximate solution, the approximate solution is obtained as:

$u_1(x) = -1.000001214x^3 + 6.95 \times 10^{-7}x^2 + 1.000000511x + 9.0307997$,

$u_2(x) = 5.6 \times 10^{-10}x^3 + 0.9999999x^2 - 1.000000030t$.

.Comparing the result obtained by [20] with the new method, it tends to be said that the proposed method performed more

accurately since the table of error found is smaller than [20] and the graph of the approximate solution is the same as the graph of exact solution
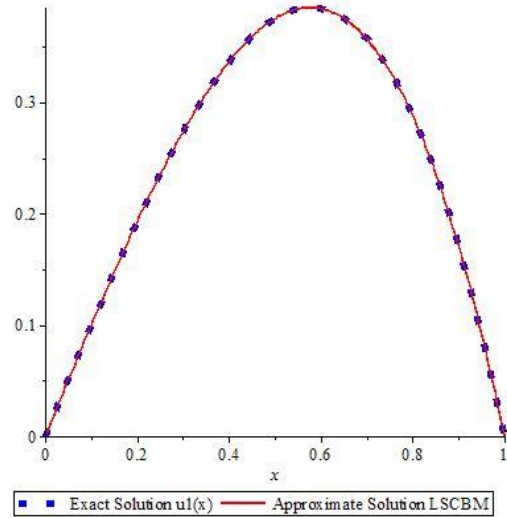

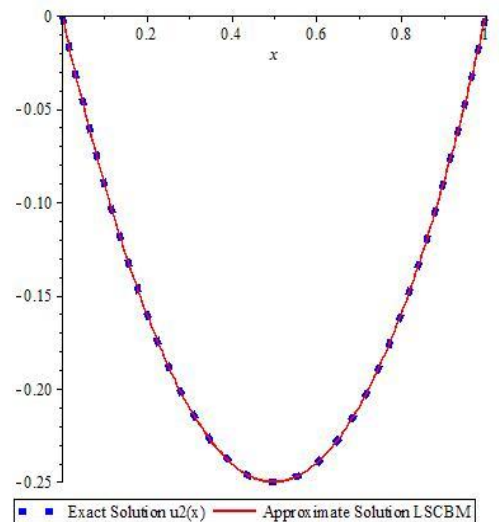**Figure 3: Showing the graph of approximation solution $u_1(x)$ and exact of example 2**

Figure 4: Showing the graph of approximation solution $u_2(x)$ and exact of example 2

Example 5.3: Consider the following fractional Integro-differential [20]

$$D^{\frac{4}{5}}u_1(x) = \frac{83x}{80} + \frac{x}{12} + \frac{25x^{\frac{6}{5}}(11+15x)}{33\Gamma\left(\frac{1}{5}\right)} + \int_0^1 2xt[u_1(t) +$$

$$u_2(t)]dt, \tag{25}$$

$$D^{\frac{3}{4}}u_2(x) = -\frac{5x^3}{6} + \frac{9x^{\frac{4}{3}}}{2\Gamma\left(\frac{1}{3}\right)} + \int_0^1 (x+t)[u_1(t) - u_2(t)]dt. \tag{26}$$

Subject to initial condition $u_1(0) = 0, u_2(0) = 0$ with the exact solution $u_1(x) = x^3 - x^2, u_2(x) = \frac{15}{8}x^2$. Similarly solving Example 3, following the same procedure above, we take $\propto = \frac{3}{4}$ and $m = 2$. The following constants were obtained as: $a_0 = 0, a_1 = 0, a_2 = 0.3333333306, a_3 = 0$ for equation (25) and $a_0 = 0, a_1 = 0, a_2 = 0.6249996843, a_3 = 1.8749993560$ for equation (26). Substituting the values back

into the assumed approximate solution, the approximate solution is obtained as:

$u_1(x) = 0.9999999918x^3 - 0.9999999918x^3$,

$u_2(x) = 3.03 \times 10^{-7}x^3 + 1.87499053x^2$. Comparing the result obtained by [20] with the new method, it tends to be said that the proposed method performed more accurately since the table of error found is smaller than [20] and the graph of the approximate solution is the same as the graph of exact solution
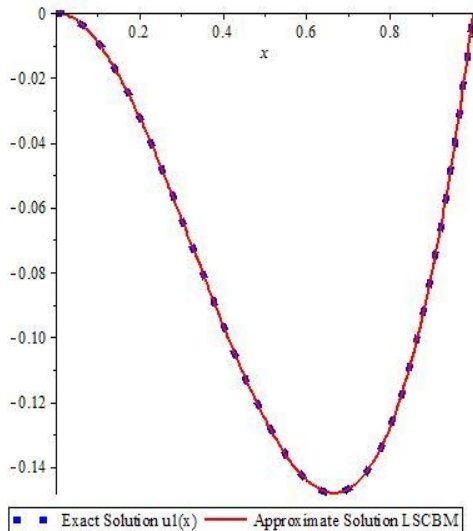

Figure 5: Showing the graph of approximation solution
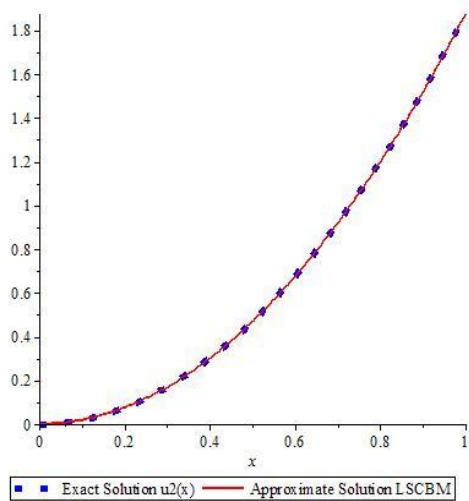$u_1(x)$ **and exact of example 3**

Figure 6: Showing the graph of approximation solution
$u_2(x)$ **and exact of example 3**

7. TABLE OF RESULTS

Table 1. Comparison of the absolute errors for Example 1

x	LSCBM $u_1(x)$	ADM [20]
0.0	1.400×10^{-8}	7.776×10^{-5}
0.2	2.290×10^{-8}	2.468×10^{-4}
0.4	4.278×10^{-8}	7.838×10^{-4}
0.6	4.566×10^{-8}	1.540×10^{-3}
0.8	3.154×10^{-8}	2.488×10^{-3}
1.0	4.000×10^{-8}	3.609×10^{-3}

Table 2. Comparison of the absolute errors for Example 1

X	LSCBM $u_2(x)$	ADM [20]
0.0	9.697×10^{-10}	7.854×10^{-8}
0.2	2.710×10^{-10}	4.413×10^{-5}
0.4	8.833×10^{-11}	1267×10^{-5}
0.6	2.048×10^{-9}	6.604×10^{-5}
0.8	5.607×10^{-9}	1.609×10^{-4}
1.0	1.077×10^{-8}	3.647×10^{-4}

Table 3. Comparison of the absolute errors for Example 2

x	LSCBM $u_1(x)$	ADM [20]
0.0	9.031×10^{-9}	$0.000 \times 10^{+10}$
0.2	1.293×10^{-7}	3.381×10^{-3}
0.4	2.469×10^{-7}	6.510×10^{-3}
0.6	3.036×10^{-7}	9.942×10^{-3}
0.8	2.411×10^{-7}	1.372×10^{-2}
1.0	1.031×10^{-9}	1.786×10^{-2}

Table 4. Comparison of the absolute errors for Example 2

x	LSCBM $u_2(x)$	ADM [20]
0.0	$0.000 \times 10^{+10}$	$0.000 \times 10^{+10}$
0.2	6.632×10^{-9}	4.753×10^{-4}
0.4	1.274×10^{-8}	1.130×10^{-3}
0.6	1.562×10^{-8}	1.876×10^{-3}
0.8	1.261×10^{-8}	2.689×10^{-3}
1.0	1.000×10^{-9}	3.534×10^{-3}

Table 5. Comparison of the absolute errors for Example 3

x	LSCBM $u_1(x)$	ADM [20]
0.0	$0.000 \times 10^{+10}$	$0.000 \times 10^{+10}$
0.2	2.624×10^{-10}	6.852×10^{-4}
0.4	7.872×10^{-10}	2.386×10^{-3}
0.6	1.181×10^{-9}	4.950×10^{-3}
0.8	1.050×10^{-9}	8.309×10^{-3}
1.0	$0.000 \times 10^{+10}$	1.241×10^{-2}

Table 6. Comparison of the absolute errors for Example 3

x	LSCBM $u_2(x)$	ADM [20]
0.0	$0.000 \times 10^{+10}$	$0.000 \times 10^{+10}$
0.2	3.546×10^{-8}	2.400×10^{-3}
0.4	1.321×10^{-7}	3.307×10^{-3}
0.6	2.755×10^{-7}	5.331×10^{-3}
0.8	4.509×10^{-7}	7.662×10^{-3}
1.0	6.440×10^{-7}	1.029×10^{-2}

6. CONCLUSION

In this study, Bernstein polynomials, least square collocation together with Caputo properties are used to find the solution of system of linear fractional integro- differential equations. There is a high rate of convergence of the approximate solutions to the exact solutions. Specifically, the performance of the proposed method was compared with existing results in the literature and are found to be more efficient in the terms of accuracy. It is also observed that the number of iterations needed in solving the problems using the proposed method is few and with lower values of M (the degree of the approximant).

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