Metric Dimension of Graphs and its Application to Robotic Navigation

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ABSTRACT
Metric dimension of graphs has several applications to networking such as network navigation, network discovery and verification, wireless sensor network localization, and locating intruders in a network. This paper investigates the metric dimension in terms of contraction and bijection when a robot is navigating a network modeled by the$(2,1)C_4$-snake graph, $2\Delta_2$–snake graph and $3C_4$–snake graph.

General Terms
Graph Theory, Robotic Navigation

Keywords
Metric Dimension, Cardinal Number, Contraction, Adjacency Matrix

1. INTRODUCTION
For a robot that is moving it can send a signal to determine its distance to a set of fixed landmarks.The problem of finding out how many landmarks and where they should be placed so that the robot can always determine its location are crucial elements to know where it is right now.

A graph can be used to model the network navigation. A work place can be represented as a vertex in a graph, with edges indicating relationships between places. Then, robot navigation problem is reduced to find out the smallest locating subset of vertices β, $|\beta| = k$such that every vertex of the graph is uniquely determined by its coordinate of distances to this locating subset β, which is called metric basis of the graph and k is said to be metric dimension

The metric basis of the graph is the set of nodes where the landmarks are positioned and the number of landmarks is the metric dimension of the graph.

Considersimple connected graph$G = (V, E)$. The length of a shortest u-v path in$G$ is the distance between two vertices u, $v \in V$. Allow $\beta = (\beta_1, \beta_2,...,\beta_k)$ be an ordered subset of V. We can associate with v an ordered k-tuple that represents the distance between v and each of the vertices in β, indicated by $d(v, \beta) = (d(v, \beta_1),...,d(v,\beta_k))$. If we have $d(u,\beta) \neq d(v,\beta)$ for any two unique vertices u, $v \in V$, then the set β is termed a resolving set of G. The metric dimension of G corresponds to its cardinality and is represented by $\dim(G)$or $\beta(G)$. A basis of G is a resolving set of G with minimal cardinality.

P. J. Slater proposed the concept of metric dimension in [1], while Harary and Melter investigated it separately in [2]. The application of robot navigation in networks was explored in [3]. In [4] applications to pattern recognition and image processing problems involving hierarchical structures were made.

Gereyet al. [5] demonstrated that computing the metric dimension of an arbitrary graph is an NP-complete problem. The metric dimension problem for grid graph was investigated by Melter and Tomescu [6]. Caceres et al. [7] investigated the metric dimension of graphs formed by the cartesian product of two or more graphs. Chartrand et al. [8] introduced the graph with metric dimension 1, n -1 and n -2 , as well as the tight bound on the metric dimension of unicyclic graphs[9]. Shanmukhaet al. [10,11] calculated the parameters for wheels, graphs made by connecting wheels with paths, complete graphs, and so on. Susilowati et al. [12] determined the metric dimension of subdivision graph (G) for some $1 \leq k \leq |(G)|$ and some special graph G. Imran et al. [13] calculated the metric dimension of graphs derived from the rooted product graphs. Bailey et al. [14] discovered relationship between the base size of automorphism group of a graph and its metric dimension; which prompted researchers to investigate metric dimensions of distance regular graphs.Manjusha et al. [15] presented an algorithm to avoid the overlapping between the robots in a network. Beerliovaet al. [16] presented numerous upper and lower bounds for the competitive ratio (for the online network discovery problem) and the approximation ratio (for the off-line network verification problem) in both models.

This paper organized as follows: In Sect. 2 we introduce the basic concepts. Sect.3. Results and discussions are explained. Finally, Sect 4 presents the conclusion of this paper.

2. PRELIMINARIES
This section explains the basic concepts and results needed in the next sections.

2.1 Definition
The n-tuple $(m_{i1},m_{i2},\ldots\ldots m_{in})$, i $=1,2,\ldots.,m$ represents the coordinates of a vertex v_i .The coordinate for v_1 is $(m_{11},m_{12},\ldots.,m_{1n})$, where $m_{11} = d(v_1,v_1)$, $m_{12} = d(v_1,v_2)$.

2.2 Definition
Cardinal number of a basis element is indicated by$Ca(v_j)$, $j = 1,2,..., n$, $v_j \in W$ and is defines as the number of vertices of G identified by v_j with respect to $d(v_i ,v_j) = 1$, $i=1,2,\ldots.., m$.

2.3 Definition
An edge contraction is a graph operation that removes an edge from the graph while combining the two vertices it previously joined.

2.4 Example
Consider the graph in Figure 1, which has two metric dimensions with regard to the basis $W=\{v_1,v_3\}$.

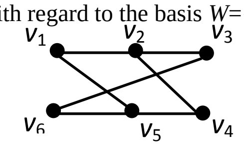

Fig 1:Ca(v_1) = 2 and Ca(v_3) = 2 with respect to d (v_i,v_j) =1

2.5 Robotic Assignment

Let $\beta= v_1,v_2,......,v_j,...,v_n$, j=1,2,....,$n$ be the basis and $V(G) = v_1,v_2,...,v_i,...,v_m,i = 1,2,....,m$be the vertex set of G.The coordinates for a vertex $v_i \in V(G)$ are $(m_{i1},m_{i2},...,m_{in}),i=1,2,.....,m$. If Min $(m_{i1},m_{i2},...,m_{in})= m_{ij}$for a given j, the basic element can be assigned (Robot) v_j.Robot has ability to begin a wide range of tasks, move around in a cluttered environment, recognize real-world thingsand interpret regular speech.

2.6 Example

In Figure 2 the coordinate of vertex is(1, 3) with respect to the basisβ={ v_1, v_3}. Here, *Min* (1,3) = 1, therefore basis element $v_2$is assigned to v_3.

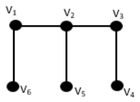

Fig 2: Comb graph $P_3 \odot 1K_1$

In Figure 2 the coordinate of $v_2$is (1,1). Therefore $d(v_1,v_2)=d(v_2,v_3)=1$.

3. MAIN RESULTS

3.1 Theorem

If the graph is $(2,1)C_4$-snake with n vertices$v_1,v_2,....,v_6$, then the contracted graph $(2,1)C_4$-snake.$e$with no loops and parallel edgeshas metric basis $\beta((2,1)C_4$-snake. e) = 4.

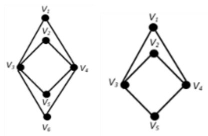

Fig 3: represents(2,1)C_4-snake and contracted graph of ((2,1)C_4-snake. e)

Proof: We have $\beta(2,1)C_4$-snake = 5. Let $e = v_4v_6$ be an edge in β ((2,1)C_4-snake) where v_4 and $v_6$are neighboring to each other by n-2 vertices. Consider the contracted graph ((2,1)C_4-snake. e). Every vertex in ((2,1)C_4-snake. e) is linked to the other n-2 vertices. Here the edge $e\notin E$((2,1)C_4-snake.e) and the vertices v_3,v_4 and $v_6$are substituted by $v_3$and v_4 .Clearly, the remaining n -2 vertices must be next to each other $v_3$and $v_4$since vertices v_1, v_2 and v_5 are all adjacent to those vertices. As a result, the simple graph ((2,1)C_4-snake.e) should contain exactly n-1vertices.

3.2 Theorem

If the graph is($2\Delta_2$ –snake)with 7 vertices$v_1,v_2,....,v_7$ thenthe contracted graph $2\Delta_2$ –snake. e. with no loops and parallel edges has metric basis β(($2\Delta_2$ –snake. e)) =2is denoted as

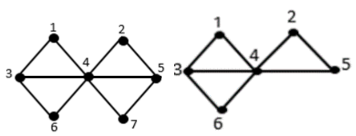

Fig4:represents $2\Delta_2$ –snake and $2\Delta_2$ –snake.e

Proof: Let $e = v_5\, v_7$beanedge in $2\Delta_2$ –snakewhere v_5 and v_7 are adjacent to every other $n-2$ vertices. Consider the contracted graph ($2\Delta_2$ –snake. e). Every vertex in ($2\Delta_2$ –snake. e) are adjacent to the other $n-2$ vertices. Here the edge $e\notin E$($2\Delta_2$ –snake. e) and the vertices $v_4,v_5$and v_7 are replaced by v_4,v_5. Then the simple graph $2\Delta_2$–snake.e. contains exactly $n-1$ vertices.As a result, the proof.

3.3 Theorem

If the graphis ($3C_4$ –snake)with 10 vertices$v_1,v_2,....,v_{10}$, thenthe contracted graph$3C_4$–snake.$e$with no loops and parallel edgeshas metric basis β(($3C_4$–snake. e)) =3.

Proof: We haveβ($3C_4$–snake) =4.Let $e = v_9v_{10}$beanedge in$3C_4$ –snakewhere v_9 and v_{10} are adjacent to every other $n-2$ vertices. Consider the contracted graph ($3C_4$ –snake. e). Every vertex in ($3C_4$ –snake. e) are adjacent to the other $n-2$ vertices. Here the edge $e\notin E$($3C_4$ –snake. e) and the vertices v_8,$v_9$and v_{10} are replaced by v_8,v_9. Then the simple graph $3C_4$ –snake.e contains exactly $n-1$ vertices.As a result, the proof. Figure 5 represents $3C_4$ –snakeand its contraction (simple graph) with the edge e.

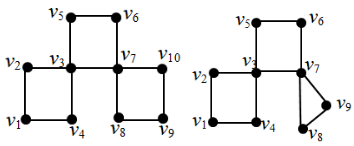

Fig5:$3C_4$ –snake and $3C_4$ –snake.e

3.4 Adjacency matrix of *S*

The relationships between graph vertices and basis elements are interesting to study.We know that each element in the adjacency matrix $[S]_{nxn}$indicates the length one path between any two vertices in the graph, entry in $[S]^2$ gives the length two path between any two matrices and so on.The differentpaths that flow from basis elements to non-basis vertices in the graph are the focus of our attention. Each item in the sub matrix $[S]_{nxn}$of order $m\times n$, $m \leq n$ represents a path between robots and network nodes.The diagonals of the matrices $[S]^2$and MM^T are simply equivalent, where M is the incidence matrix of S.

3.5 Example

Suppose the following graph represents the robotic spanning tree S of some graph G and its adjacency matrix and incidence matrix are given in Figure 6.

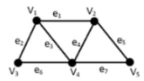

Fig6 Graph *G*

The diagonals of $[S]^2$and $MM^T$are proven to be the same.In other words, we can simply find a two-length route from a robot to itself.

$$[S]=\begin{bmatrix}0&1&1&1&0\\1&0&0&1&1\\1&0&0&1&0\\1&1&1&0&1\\0&1&0&1&0\end{bmatrix} \quad M=\begin{bmatrix}1&1&1&0&0&0&0\\1&0&0&1&1&0&0\\0&1&0&0&0&1&0\\0&0&1&1&0&1&1\\0&0&0&0&1&0&1\end{bmatrix}$$

$$[S]^2=\begin{bmatrix}0&1&1&1&0\\1&0&0&1&1\\1&0&0&1&0\\1&1&1&0&1\\0&1&0&1&0\end{bmatrix}\begin{bmatrix}0&1&1&1&0\\1&0&0&1&1\\1&0&0&1&0\\1&1&1&0&1\\0&1&0&1&0\end{bmatrix}=\begin{bmatrix}3&1&2&2&2\\1&3&2&2&1\\2&2&3&2&1\\2&2&2&4&1\\2&1&1&1&2\end{bmatrix}$$

$$MM^T=\begin{bmatrix}1&1&1&0&0&0&0\\1&0&0&1&1&0&0\\0&1&0&0&0&1&0\\0&0&1&1&0&1&1\\0&0&0&0&1&0&1\end{bmatrix}\begin{bmatrix}1&1&0&0&0\\1&0&1&0&0\\1&0&0&1&0\\0&1&0&1&0\\0&1&0&0&1\\0&0&1&1&0\\0&0&0&1&0\end{bmatrix}=\begin{bmatrix}3&1&1&1&0\\1&3&0&1&1\\1&0&2&1&0\\1&1&1&4&0\\0&1&0&1&1\end{bmatrix}$$

Since S is a spanning tree,τ (S) = 1 is the number of various spanning trees of S. with respect to a contraction,the following theorem proves that there is a one to one correspondence between $E(S)$ and τ ($S.e$). We actually reduce the cardinal number of each basis element by using contraction. As a result, contraction is critical in the routing of complicated networks.

3.6 Theorem [15]

There is bijection between the edge set of S and $S.e$ if S is the robotic spanning tree of a graph G and $S.e$ is the contraction with regard to nonloop$e$.

3.7 Example

Consider the Robotic spanning tree in the Figure 7.

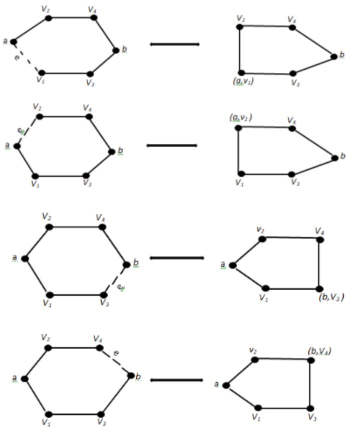

Fig7.Correspondence stated in *G*.

4. CONCLUSION

We determined the metric dimension of (2,*1*)C_4-snake graph, $2\Delta_2$–snake graph and $3C_4$–snake graph with regard to contraction and bijection between them. Also, we applied in robotic assignment spanning subgraph in a complex network.

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