Connected Perfect Domination of Interval-Valued Fuzzy Graphs

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ABSTRACT

In interval-valued fuzzy graphs, the connected perfect dominating set and connected perfect domination number are introduced in this paper. For standard theorems and results, as well as some examples, we compute the connected perfect domination number $\gamma_{cp}(G)$ and the upper connected perfect domination number $\Gamma_{cp}(G)$. The upper and lower bounds for the connected perfect domination number in interval-valued fuzzy graphs are derived using some other known parameters. Finally, some of the properties of connected perfect domination of interval-valued fuzzy graphs, as well as their various results are discussed.

General Terms

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Keywords

Interval-valued fuzzy graph, connected dominating set, perfect dominating set, connected perfect dominating set

1. INTRODUCTION

The interval-valued fuzzy set is a supplement to the fuzzy set that improves the accuracy of the uncertainty. Zadeh [16] was the first to introduce it (1975). Since a subset A of a vertex set V is called interval valued fuzzy set and it is denoted by $A = \{u, [\mu_1(u), \mu_2(u)] : u \in V\}$, where $\mu_1 : V \to [0,1]$ and $\mu_2 : V \to [0,1]$ are functions such that $\mu_1(u) \leq \mu_2(u)$ for $u \in V$. And a subset B of an edge set E is called interval valued fuzzy relation and it is denoted by $B = \{(u,v), [\rho_1(u,v), \rho_2(u,v)] : (u,v) \in E\}$, where $\rho_1 : E \to [0,1]$ and $\rho_2 : E \to [0,1]$ are functions such that $\rho_1(u,v) \leq \rho_2(u,v)$ for $(u,v) \in E$. If $G^* = (V,E)$ is a crisp graph, then $G = (A,B)$ is an interval-valued fuzzy graph. The definition of an interval-valued fuzzy graph, which is a generalization of fuzzy graph, was given by Akram and Dudek [1], they also defined interval-valued fuzzy graph operations and introduced several properties to them. The notion of fuzzy graph, interval-valued fuzzy complete graphs and Some features of self complementary graphs were addressed in previous studies, by [10], [7] and [6]. In a fuzzy graph, SomaSundaram [14, 15] explored the concepts of domination, independent domination, and connected domination. In fuzzy digraphs, sarala and Janaki [11] established the notion of connected domination number in (2019). Debnath (2013) investigates the ideas of domination in interval valued fuzzy graphs, [5]. Sarala and Kavitha proposed the ideas of strong (weak) dominance in interval-valued fuzzy graphs, as well as complete and complementary domination numbers in interval-valued fuzzy graphs, in their paper (2016) [12, 13]. Revathi et al [8, 9] looked at the concepts of perfect domination and connected perfect domination in fuzzy graphs. The concepts of a perfect domination, connected domination and total perfect domination in interval-valued fuzzy graphs were discussed in [2, 3, 4].
In this paper, the connected perfect dominating set and connected perfect domination number are introduced for interval-valued fuzzy graphs. We get a lot of information about these ideas. Finally, an interval-valued fuzzy graph is created to show the relationship between these concepts and others.

2. PRELIMINARIES

In this section, we go through some fundamental definitions for interval-valued fuzzy graphs, domination in interval-valued fuzzy graphs and perfect domination in interval-valued fuzzy graphs .
An interval-valued fuzzy graph (in short $IVFG$) of the graph $G^* = (V,E)$ is a pair $G = (A,B)$ where, $A = [\mu_1, \mu_2]$ is an interval-valued fuzzy set on V and $B = [\rho_1, \rho_2]$ is an interval-valued fuzzy relation on V, such that $\rho_1(wu) \leq \mu_1(w) \wedge \mu_1(u)$ and $\rho_2(wu) \leq \mu_2(w) \wedge \mu_2(u)$ for all $wu \in E$. Let $G = (A,B)$ be an $IVFG$. Then the order p and size q are

defined as follows: $p = \sum_{v_i \in V} \frac{1+\mu_2(v_i)-\mu_1(v_i)}{2}$ and $q = \sum_{(v_iv_j)\in E} \frac{1+\rho_2(v_iv_j)-\rho_1(v_iv_j)}{2}$. An $IVFG$ is said to be effective edge if
$\rho_1(uv) = \mu_1(u) \wedge \mu_1(v)$ and $\rho_2(uv) = \mu_2(u) \wedge \mu_2(v)$. Two vertices x and y are said to be neighbors in $IVFG$ if
$\rho_1(x, y) = \mu_1(x) \wedge \mu_1(y)$ and $\rho_2(x, y) = \mu_2(x) \wedge \mu_2(y)$. Let $G = (A, B)$ of $G^* = (V, E)$ where $A = [\mu_1, \mu_2]$ and $B = [\rho_1, \rho_2]$. Then an $IVFG$, G is called complete $IVFG$ if $\rho_1(xy) = \mu_1(x) \wedge \mu_1(y)$ and $\rho_2(xy) = \mu_2(x) \wedge \mu_2(y)$, for all $xy \in E$ and is denoted by K_μ. The complement of an $IVFG$, G is $IVFG$, $\overline{G} = (\overline{A}, \overline{B})$ where
$\overline{\mu_1} = \mu_1$; $\overline{\mu_2} = \mu_2$ for all vertices, also
$\overline{\rho_1}(xy) = \mu_1(x) \wedge \mu_1(y) - \rho_1(xy)$ and $\overline{\rho_2}(xy) = \mu_2(x) \wedge \mu_2(y) - \rho_2(xy)$ for all $xy \in E$.
An $IVFG$, G is called bipartite if the vertex set V of G can be partitioned into two non empty subsets V_1 and V_2 such that $V_1 \cap V_2 = \phi$.
A bipartite $IVFG$, G is called complete bipartite $IVFG$ if $\rho_1(v_iv_j) = min\{\mu_1(v_i), \mu_1(v_j)\}$ and $\rho_2(v_iv_j) = min\{\mu_2(v_i), \mu_2(v_j)\}$ for all $v_i \in V_1$ and $v_j \in V_2$, and it is denoted by K_{μ_1,μ_2}. Given a fuzzy graph G, choose $u \in V(G)$ and put $S = u$, for every u we have $N(S) = V - S$ denoted by $\acute{S}$? is the complete dominating set of an $IVFG$. The minimum cardinality of a complete dominating set of interval-valued fuzzy is called the complete domination number of G. Let $G = (A, B)$, be a Complementary $IVFG$ G on V and $u, v \in V$, We say u dominates v if $\overline{\rho_1}(xy) = min\{\mu_1(x), \mu_1(y)\}$ and $\overline{\rho_2}(xy) = min\{\mu_2(x), \mu_2(y)\}$. A subset S of V is called a dominating set in $\overline{G}$ if for every $v \notin S$, there exists $u \in S$ such that u dominates v. The minimum cardinality of a dominating set in G is called the domination number of G and is denoted by $\gamma(\overline{G})$. Let $G = (A, B)$ be $IVFG$ and let $S \in V(G)$, then a vertex sub set S of G is said to be independent set if $\rho_1(uv) < \mu_1(u) \wedge \mu_1(v)$, and $\rho_2(uv) < \mu_2(u) \wedge \mu_2(v)$ or $\rho_1(uv) = 0, \rho_2(uv) = 0$ for all $uv \in S$. Let $G = (A, B)$ be an $IVFG$ and let $u, v \in V(G)$. Then we say that u dominates v or v dominates u if (uv) is a effective edge, i.e $\rho_1(uv) = min(\mu_1(u), \mu_1(v))$ and $\rho_2(uv) = min(\mu_2(u), \mu_2(v))$. A vertex sub set $(D \subseteq V)$ of $V(G)$ is called dominating set in $IVFG$ G, if for every $v \in V - D$ there exists $u \in D$, such that (uv) is effective edge. A dominating set D of an $IVFG$ G, is called minimal dominating set if $D - \{u\}$ is not dominating set for every $u \in D$. A minimal dominating set D, with $|D| = \gamma(G)$ is denoted by $\gamma - set$. A dominating set D of an $IVFG$ G is said to be a total dominating set, if every vertex in V is dominated by a vertex in D. The minimum fuzzy cardinality of a total dominating set is called the total domination number of an $IVFG$ G and is denoted by $\gamma_t(G)$. Let $G = (A, B)$ be interval-valued fuzzy sub graph of graph $G^* = (V, E)$ and let $v; u \in V$. Then G is a connected $IVFG$, G if for every $v; u \in V(G)$. Thence is a path $u - v$ spanning u and v. A dominating set D in an $IVFG$, G is said to be connected dominating set if the induced interval-valued fuzzy sub graph $< D >$ is connected and is denoted by D_c. A connected dominating set D_c of an $IVFG$, G is said to be minimal connected dominating set if for every $u \in D_c$, $D_c - \{u\}$ is not connected dominating set of G. The minimum fuzzy cardinality of all minimal connected dominating set of $IVFG$, G is called the connected domination number of G and denoted by $\gamma_c(G)$. The maximum fuzzy cardinality of all minimal connected dominating set of an $IVFG$, G is called the upper connected domination number of G and denoted by $\Gamma_c(G)$. If each vertex $x \notin D$ is dominated by exactly one vertex $y \in D$, the dominating set D of an interval-valued fuzzy graph G is called the perfect dominating set of G. A perfect dominating set D_p of an interval-valued fuzzy graph G is called a minimal perfect dominating set if $D_p - u$ is not a perfect dominating set of G for every $u \in D_p$. The perfect domination number of interval-valued fuzzy graph G is denoted by $\gamma_p(G)$ and has the smallest fuzzy cardinality among all minimal perfect dominating sets. The minimum perfect dominating set, denoted by $\gamma_p - set$, is a perfect dominating set of interval-valued fuzzy graph G with the smallest fuzzy cardinality, i.e. $|D_p| = \gamma_p(G)$.

3. CONNECTED PERFECT DOMINATION OF INTERVAL-VALUED FUZZY GRAPHS

This section includes the definition of perfect connected dominating set and perfect connected domination number, theorems and results in an interval-valued fuzzy graph G.

DEFINITION 1. *Let $G = (A, B)$ of $G^* = (V, E)$ be an interval-valued fuzzy graph and D_p is a perfect dominating set of G with $D_p \subset V$. If the induced sub graph $< D_p >$ is connected, then D_p is said to be a connected perfect dominating set of G and it is denoted by D_{cp}.*

DEFINITION 2. *If $D_{cp} - \{x\}$ is not a connected perfect dominating set of an interval-valued fuzzy graph G for any $x \in D_{cp}$, the connected perfect dominating set D_{cp} of G is referred to as a minimal connected perfect dominating set of G.*

DEFINITION 3. *The minimum fuzzy cardinality of all minimal connected perfect dominating sets of an interval-valued fuzzy graph, G is called the lower connected perfect domination number of G and is denoted by $\gamma_{cp}(G)$.*

DEFINITION 4. *The maximum fuzzy cardinality of all minimal connected perfect dominating sets of an interval-valued fuzzy graph G, is called the upper connected perfect domination number of G and it is denoted by $\Gamma_{cp}(G)$.*

REMARK 3.1. *The smallest fuzzy cardinality of the connected perfect dominating set in interval-valued fuzzy graph G, is called minimum connected perfect dominating set and denoted by $\gamma_{cp} -$ set.*

EXAMPLE 1. *For the interval-valued fuzzy graph G shown in Figure 1, each edge of G is a strong edge.*

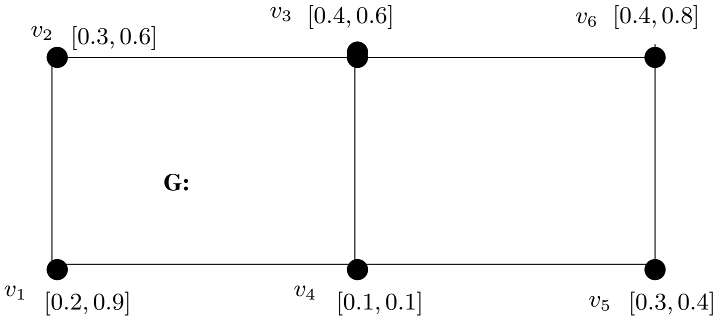

Fig. 1.

From the above Figure, a subset $S_1 = \{v_3, v_4\}, S_2 = \{v_1, v_4, v_5\}, S_3 = \{v_2, v_3, v_6\}$ are minimal connected perfect dominating sets of G. Then $S_1 = \{v_3, v_4\}$ is a minimum connected

perfect dominating set of G, and $\gamma_{cp}(G) = 1.1$.
While $S_3 = \{v_2, v_3, v_6\}$ is a maximum connected perfect dominating set of G, and $\Gamma_{cp}(G) = 1.95$. We note that $\gamma_{cp} - set = S_1$.

The following includes some of the properties of connected perfect domination of interval-valued fuzzy graphs, as well as their various results.

THEOREM 5. *Let $G = (A, B)$ be any connected interval-valued fuzzy graph. If D_{cp} is a connected perfect dominating set of G. Then*
(i) D_{cp} is a dominating set of G, but the contrary is not true.
(ii) D_{cp} is a connected dominating set of G, but the contrary is not true.
(iii) D_{cp} is a perfect dominating set of G, but the contrary is not true.

PROOF. Let $G = (A, B)$ be any connected interval-valued fuzzy graph. Suppose that D_{cp} be a connected perfect dominating set of G.
(i) We will prove that D_{cp} is a dominating set of G. Since D_{cp} is a connected perfect dominating set of G, then by Definition of D_{cp}, for each vertex $w \notin D_{cp}$, w is adjacent to exactly one vertex of D_{cp} and the induced interval-valued fuzzy subgraph $< D_{cp} >$ is connected. Since $w \notin D_{cp}$. Hence $w \in V - D_{cp}$ and w dominated by exactly one vertex in D_{cp}, which is a dominating set of G, but the converse need not is true.
(ii) We will prove that D_{cp} is a connected dominating set of G. Since D_{cp} is a connected perfect dominating set of G, hence clear that D_{cp} is also connected dominating set of G, but the converse need not is true.
(iii) We will prove that D_{cp} is a perfect dominating set of G. We know that, a connected perfect dominating set if for each vertex $u \notin D_{cp}$, u dominated by exactly one vertex of D_{cp}, and also the induced interval-valued fuzzy sub graph $< D_{cp} >$ is connected. It is obvious that D_{cp} is a perfect dominating set of G, but the converse is not the case. □

The following Example shows that the above Theorem is true.

EXAMPLE 2. *For the interval-valued fuzzy graph G given in the following Figure, where every edge of G is a strong edge.*

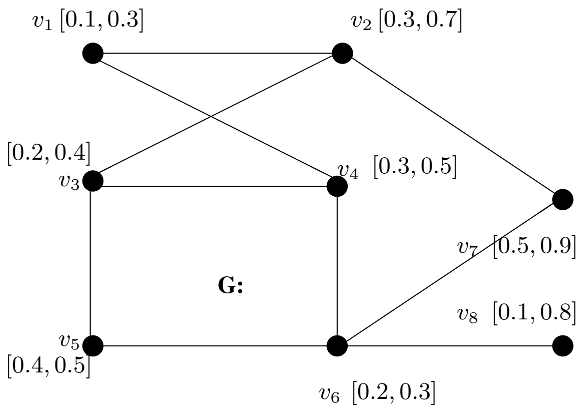

Fig. 2.

According to the above Figure, a vertex subset $S_1 = \{v_1, v_4, v_6\}, S_2 = \{v_4, v_6, v_7\}$ and $S_3 = \{v_2, v_6, v_7\}$ are connected perfect dominating sets of an interval-valued fuzzy graph G. As a result, S_1, S_2 and S_3 are also dominating sets, connected dominating sets and perfect dominating sets of G.
A vertex subset $D_1 = \{v_2, v_6\}$ is a dominating set of an interval-valued fuzzy graph G, but it is not a connected perfect dominating set of G.
And $D_2 = \{v_3, v_4, v_6\}$ is a connected dominating set of G, but it is not a connected perfect dominating set of G.
And $D_3 = \{v_2, v_3, v_8\}$ is a perfect dominating set of G, but it is not a connected perfect dominating set of G.

COROLLARY 3.1. *Let $G = (A, B)$ be any connected interval-valued fuzzy graph. Then*

$$\gamma(G) \leq \gamma_p(G) \leq \gamma_{cp}(G).$$

PROOF. Let $G = (A, B)$ be any connected interval-valued fuzzy graph. By Theorem 5, every connected perfect dominating set of an interval-valued fuzzy graph G is a dominating set and it is a perfect dominating set of G. Hence

$$\gamma(G) \leq \gamma_{cp}(G)\text{.........}(1),$$

and

$$\gamma_p(G) \leq \gamma_{cp}(G)\text{.........}(2).$$

Since every perfect dominating set of an interval-valued fuzzy graph G is a dominating set(Theorem 3.2, [2]). Hence

$$\gamma(G) \leq \gamma_p(G)\text{.........}(3).$$

From (1), (2) and (3), we get

$$\gamma(G) \leq \gamma_p(G) \leq \gamma_{cp}(G).$$

□

REMARK 3.2. *Let G be a connected interval-valued fuzzy graph. Then every connected dominating set of an interval-valued* *fuzzy graph G is not a perfect dominating set of G and the converse is true.*

THEOREM 6. *Let G be any connected interval-valued fuzzy graph, with $\gamma_{cp} - set$ of G. Then a connected dominating set of an interval-valued fuzzy graph is a perfect dominating set of G and conversely if only if $\gamma_{cp} - set$ is a connected perfect dominating set of G.*

PROOF. Let G be any connected interval-valued fuzzy graph, with $\gamma_{cp} - set$ of G. Suppose that S_1 is the minimal connected dominating set of an interval-valued fuzzy graph G and S_2 is the minimal perfect dominating set of an interval-valued fuzzy graph G. Then we have two cases:
Case1: If $S_1 \neq S_2$, which is not connected perfect dominating set of G, it is contradicting to the Theorem.
Case2: If $S_1 = S_2$, which $\gamma_{cp} - set$ both connected dominating set and perfect dominating set of G. Therefore $\gamma_{cp} - set$ is connected perfect dominating set of G.
Conversely: Suppose that $\gamma_{cp} - set$ be the connected perfect dominating set of an interval-valued fuzzy graph G. By Theorem 5, this obvious. □

EXAMPLE 3. *Consider the interval-valued fuzzy graph G, given in the following Figure, where all edges of G are strong.*

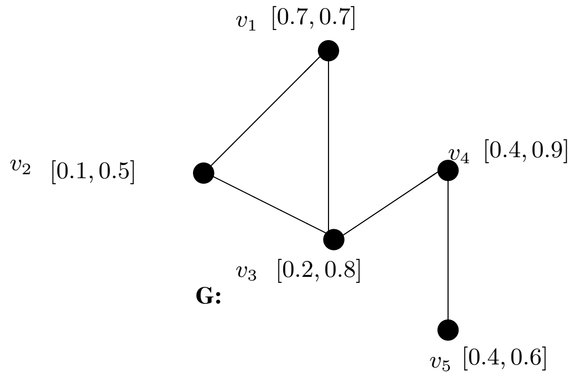

Fig. 3.

From the above Figure, a vertex subset $D_1 = \{v_1, v_3, v_4\}$ is connected dominating set of an interval-valued fuzzy graph G, but is not perfect dominating set of G, and $D_2 = \{(v_1, v_5)\}$ is perfect dominating set of an interval-valued fuzzy graph G, but is not connected dominating set of G. So this $D_3 = \{v_3, v_4\}$ is connected perfect dominating set of G, thus D_3 be together a connected dominating set and a perfect dominating set of G.

REMARK 3.3. *For any interval-valued fuzzy graph, G if $V(G)$ is an independent. Then*

$$\gamma_{cp}(G) = 0 \quad and \quad \Gamma_{cp}(G) = 0.$$

THEOREM 7. *Let G be connected interval-valued fuzzy graph, with $n \geq 5$ and D_{cp} be a $\gamma_{cp} - set$ of G. Then $V - D_{cp}$ need not be a connected perfect dominating set of G.*

PROOF. Let G be a connected interval-valued fuzzy graph, $n \geq 5$ and D_{cp} be a minimum connected perfect dominating set of G. Assume that $V - D_{cp}$ is a connected perfect dominating set of G. Then by the Definition of Perfect dominating set, for each vertex $x \notin V - D_{cp}$, x is dominates by exactly one vertex of $V - D_{cp}$, and the perfect dominating set is a connected perfect dominating set if the induced sub graph $< D_{cp} >$ is connected. Hence there is a vertex $y \in V - D_{cp}$, such that y does not adjacent to any vertex in D_{cp} or y dominates at least one vertex in D_{cp}. Therefor, D_{cp} is not the minimum connected perfect dominating set of G, which is a contradiction. Thus $V - D_p$ is not a connected perfect dominating set of G. □

EXAMPLE 4. *For the interval-valued fuzzy graph G given in the following Figure, where every edge of G is a strong edge.*

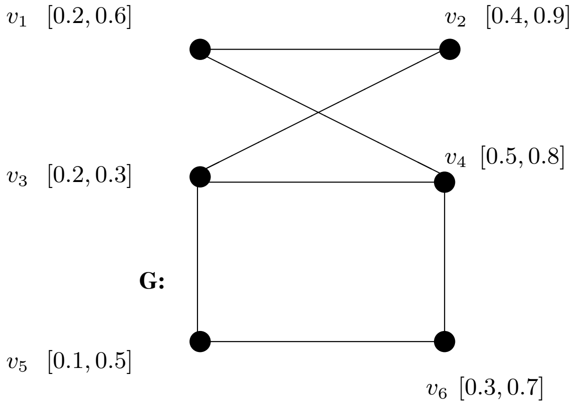

Fig. 4.

From the above Figure, we see that a vertex subset $D_1 = \{v_1, v_4, v_6\}$, $D_2 = \{v_2, v_3, v_5\}$ and $D_3 = \{v_3, v_4\}$ are minimal connected perfect dominating sets of an interval-valued fuzzy graph G. Then $D_3 = \{v_3, v_4\}$ is minimum connected perfect dominating set of G. Hence $V - D_3 = \{v_1, v_2, v_5, v_6\}$ is not a connected perfect dominating set of an interval-valued fuzzy graph G.

THEOREM 8. *For any connected interval-valued fuzzy graph G, with D_{cp} be a connected perfect dominating set of G. If each $u \in D_{cp}$ is dominates by exactly one vertex of $V - D_{cp}$. Then $V - D_{cp}$ is perfect dominating set of G. But $V - D_{cp}$ is not connected perfect dominating set of G.*

PROOF. Let G be any connected interval-valued fuzzy graph, with D_{cp} be a connected perfect dominating set of G. Suppose that u is any vertex in D_{cp}. Then, if each $u \in D_{cp}$ is dominates by exactly one vertex of $V - D_{cp}$, because D_{cp} is a connected perfect dominating set of G. Hence the number of vertices in V is twice of the number of vertices in D_{cp}, i.e the number of vertices in D_{cp} equal the number of vertices in D_{cp}. Therefor, $V - D_{cp}$ is a perfect dominating set of G. But $V - D_{cp}$ need not is a connected perfect dominating set of G. □

COROLLARY 3.2. *Let G be a connected interval-valued fuzzy graph, such that G is not a tree or path, and D_{cp} be a connected perfect dominating set of G. If V has twice the number of vertices as D_{cp}, then $V - D_{cp}$ is a connected perfect dominating set of G.*

PROOF. Let G be any connected interval-valued fuzzy graph, such that G is not a tree or path, and D_{cp} be a connected perfect dominating set of G. Suppose that the number of vertices in V is twice that of the number of vertices in D_{cp}. Then by Theorem 8, if each $u \in D_{cp}$ is dominates by exactly one vertex of $V - D_{cp}$. Since D_{cp} be a connected perfect dominating set of G and G is not tree or path. Thus $V - D_{cp}$ is also connected. Therefore, $V - D_{cp}$ is a connected perfect dominating set of G. □

EXAMPLE 5. *In Figure 4, we obtain* $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$*, then* $n = 6$*. Since* $D_{cp} = \{v_1, v_4, v_6\}$ *is a connected perfect dominating set of an interval-valued fuzzy graph G and* $n_1 = 3$*. Hence* $V - D_{cp} = \{v_2, v_3, v_5\}$ *is a connected perfect dominating set of G.*

The following gives γ_{cp} and Γ_{cp} for the complete interval-valued fuzzy graph K_μ and its complementary.

THEOREM 9. *For any complete interval-valued fuzzy graph* $G = k_\mu$,

$$\gamma_{cp}(k_\mu) = min\{|v|; \forall v \in V(G)\},$$

and

$$\Gamma_{cp}(k_\mu) = max\{|u|; \forall u \in V(G)\}.$$

PROOF. Let G be any complete interval-valued fuzzy graph. Then all edges in K_μ are strong edges and each vertex in k_μ dominates all the other vertices in G. Hence a connected perfect dominating set has exactly one vertex say v in k_μ, where v has minimum membership value in k_μ or has maximum say u. Thus,

$$\gamma_{cp}(k_\mu) = min\{|v|; \forall v \in V(G)\},$$

and

$$\Gamma_{cp}(k_\mu) = max\{|u|; \forall u \in V(G)\}.$$

□

COROLLARY 3.3. *For any interval-valued fuzzy graph* $G = k_\mu$,

$$\gamma(k_\mu) = \gamma_p(k_\mu) = \gamma_{cp}(k_\mu), \quad and \quad \Gamma_(k_\mu) = \Gamma_p(k_\mu) = \Gamma_{cp}(k_\mu).$$

The proof for this corollary is simple.

REMARK 3.4. *for any complete interval-valued fuzzy graph G with* D_{cp} *be a connected perfect dominating set of G. Then a connected dominating set of G exists in* $V - D_{cp}$*. But a perfect dominating set of G does not exist in* $V - D_{cp}$*.*

THEOREM 10. *Let G be any complete interval-valued fuzzy graph and let* $\overline{G}$ *be the complementary of G. Then a connected perfect dominating set in* $\overline{G}$ *does not exist.*

PROOF. Let G be any complete interval-valued fuzzy graph and let $\overline{G}$ be a complement of G. Then all edges of G are strong edges and every vertex in G dominates all the other vertices of G. So a connected perfect dominating set D_{cp} of G is exists and $|D_{cp}| = \gamma_{cp}(G) = min\{|v|; \forall v \in V(G)\}$. Since $\overline{G}$ is the complementary of G. Then $\overline{G}$ is null interval-valued fuzzy graph. Therefore the connected perfect dominating set of $\overline{G}$ does not exist. □

COROLLARY 3.4. *For any complete interval-valued fuzzy graph* G,

$$\gamma(\overline{k_\mu}) > \gamma_{cp}(\overline{k_\mu}).$$

PROOF. Let $G = k_\mu$ be a complete interval-valued fuzzy graph, then $\overline{k_\mu}$ is null interval-valued fuzzy graph, i.e. each vertex is isolated vertex. So $\gamma(\overline{k_\mu}) = p$ and $\gamma_{cp}(\overline{k_\mu} = 0$. Hence, we get the corollary. □

The following gives γ_{cp} and Γ_{cp} for The complete bipartite interval-valued fuzzy graph $K_{m,n}$ and its complement.

THEOREM 11. *For any complete bipartite interval-valued fuzzy graph,* $G = k_{m,n}$*. Then*

$$\Gamma_{cp}(k_{m,n}) = max\{|w| : w \in A_1\} + max\{|u| : u \in A_2\},$$

and

$$\gamma_{cp}(k_{m,n}) = min\{|w| : w \in A_1\} + min\{|u| : u \in A_2\}.$$

PROOF. Suppose that $k_{m,n}$ be any complete bipartite interval-valued fuzzy graph, such that m and n are the number of vertices A_1 and A_2 respectively. Since each edge in $K_{m,n}$ is a strong edge and each vertex in A_1 dominates all vertices in A_2 and each vertex in A_2 dominates all vertices in A_1. Then a connected perfect dominating set of $k_{m,n}$ has exactly two vertices w, u where $w \in A_1$ and $u \in A_2$. Thus, w has the maximum membership value in A_1 and u has the maximum membership value of A_2. Similarly w has the minimum membership value in A_1 and u has the minimum membership value of A_2. Therefore,

$$\Gamma_{cp}(K_{m,n}) = max\{|w| : w \in A_1\} + max\{|u| : u \in A_2\},$$

and

$$\gamma_{cp}(K_{m,n}) = min\{|w| : w \in A_1\} + min\{|u| : u \in A_2\}.$$

□

THEOREM 12. *For any complete bipartite interval-valued fuzzy graph* $G = k_{m,n}$ *and* D_{cp} *be a minimal connected perfect dominating set of* $k_{m,n}$*. Then* $V - D_{cp}$ *is a connected dominating set of* $k_{m,n}$*.*

PROOF. Let $G = k_{m,n}$ be complete bipartite interval-valued fuzzy graph and $V = (A_1, A_2)$. By Theorem 11, a connected perfect dominating set of G contains exactly two vertices w, u, where $w \in A_1$ and $u \in A_2$. So $V - D_{cp} = A_1 - \{w\} + A_2 - \{u\}$ is a connected dominating set of $G = k_{m,n}$. □

COROLLARY 3.5. *Let* $G = k_{m,n}$ *be a complete bipartite interval-valued fuzzy graph. Then*

$$\gamma_p(G) = \gamma_{cp}(G) = \gamma_p(\overline{G}).$$

PROOF. Since, we know that $\gamma_p(G) = \gamma_{cp}(G)$. and $\gamma_p(G) = \gamma_p(\overline{G})$. for $G = k_{m,n}$. Then, we get prove. □

The following gives γ_{cp} and Γ_{cp} for the star interval-valued fuzzy graph $S_n = S_{u,v_i}$ and its complementary.

THEOREM 13. *Let* $G = S_n$ *be a strong star interval-valued fuzzy graph, such that* n *is the number of vertices of* S_n*. Then* $\gamma_{cp}(G) = |u|$*, such that u is a root vertex.*

PROOF. Let G be a strong star interval-valued fuzzy graph, then the vertex set of G are $\{u, v_i\}$, where u the root vertex of G and $v_i = \{v_1, v_2, ..., v_{n-1}\}$, such that u dominates to all vertices v_i, for $i = \{1, 2, ..., n-1\}$. Hence a connected perfect dominating set of G contains $\{u\}$. Therefor, the connected perfect domination number $\gamma_{cp}(G) = |u| :$ u is a root vertex. □

COROLLARY 3.6. *For any strong star interval-valued fuzzy graph* S_n,

$$\gamma_p(S_n) = \gamma_{cp}(S_n) = \Gamma_{cp}(S_n).$$

PROOF. This proof for this corollary is simple. □

EXAMPLE 6. *For the interval-valued fuzzy graph* S_7*, shown in the following Figure, such that all edges of* S_7 *are strong.*

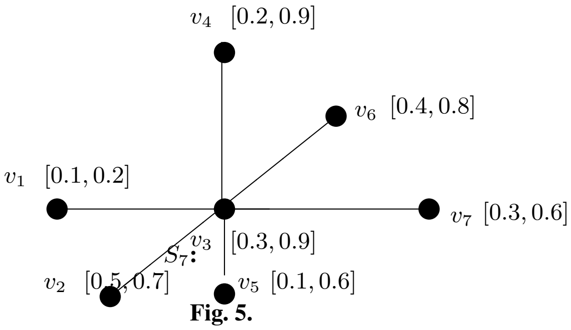

Fig. 5.

According to Figure 5, a vertex subset $D = \{v_3\}$ is the minimal connected perfect dominating set of the star interval-valued fuzzy graph S_7. Then a connected perfect domination number $= \gamma_{cp}(S_7) = 0.6$.

REMARK 3.5. *For any a strong star interval-valued fuzzy graph $G = S_n$ with D_{cp} be a minimal connected perfect dominating set of S_n. Then*
(i) $V - D_{cp}$ is a dominating set of G.
(ii) $V - D_{cp}$ is not connected dominating set of S_n.
(iii) $\overline{S_n} - \{u\}$ is a complete, where $\overline{S_n}$ is a complementary of S_n and u is a root vertex of S_n.

The following gives γ_{cp} and Γ_{cp} for the wheel interval-valued fuzzy graph $W_n = W_{u,v_i}$.

THEOREM 14. *Let $G = W_n$ be a strong wheel interval-valued fuzzy graph. Then $\gamma_{cp}(W_n) = |u|$: u is a root vertex, and n the number of vertices of W_n.*

PROOF. This proof has the same result as Theorem 13. □

COROLLARY 3.7. *For any strong Wheel interval-valued fuzzy graph W_n,*

$$\gamma_{cp}(W_n) = \Gamma_{cp}(W_n).$$

PROOF. This proof for this corollary is trivial. □

EXAMPLE 7. *Consider the interval-valued fuzzy graph W_7 given in the following Figure, where all edges of W_7 are effective.*

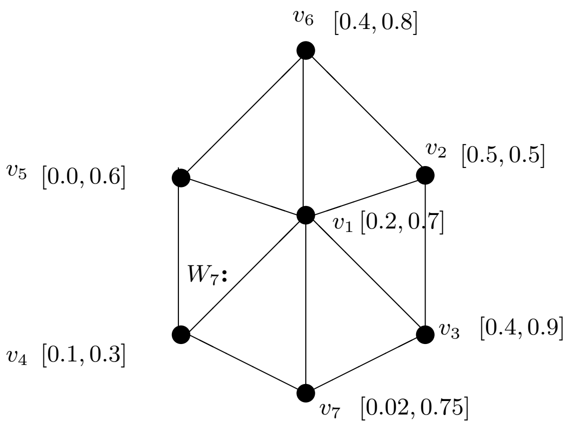

Fig. 6.

From the above Figure, we get only $D_{cp} = \{v_1\}$ is a connected perfect dominating set of an interval-valued fuzzy graph W_7, and v_1 is the root vertex of W_7. Then $\gamma_{cp} = \Gamma_{cp} = |v_1| = 0.75$.

The following gives γ_{cp} and Γ_{cp} for The cycle interval-valued fuzzy graph C_n.

THEOREM 15. *Let G be any a strong cycle (C_n) interval-valued fuzzy graph. Then a connected perfect dominating set of C_n is exists, such that n the number of vertices of (C_n).*

PROOF. Let G be a strong cycle interval-valued fuzzy graph. If $G = C_3$, this indicates that the cycle has three vertices, or $n = 3$. Hence, a connected perfect dominating set has one vertex $u \in C_3$, or $D_{cp} = n - 2 = 3 - 2 = 1$. If $G = C_4$, the cycle contains four vertices, $n = 4$. As a result, a connected perfect dominating set has two vertex $x$and $y$of C_4, resulting in $D_{cp} = n - 2 = 4 - 2 = 2$. If $G = C_5$ is a cycle containing five vertices, hence $C_{cp} = n - 2 = 5 - 2 = 3$ etc. Therefore, a connected perfect dominating set exists for any cycle interval-valued fuzzy graph $G = C_n$ and $D_{cp} = n - 2$. □

The following example shows that the connected perfect dominating set of an interval-valued fuzzy graph in the above Theorem exists for any strong cycle of G.

EXAMPLE 8. *For the interval-valued fuzzy graph $G = C_6$ in the following Figure , where all the edges of C_6 are effective.*

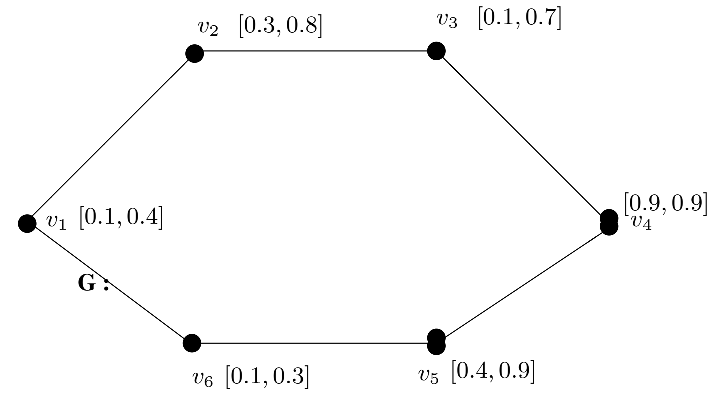

Fig. 7.

From the above Figure, we see that a vertex subset $D_1 = \{v_1, v_2, v_3, v_4\}$, $D_2 = \{v_2, v_3, v_4, v_5\}$, $D_3 = \{v_3, v_4, v_5, v_6\}$, $D_4 = \{v_4, v_5, v_6, v_1\}$, $D_5 = \{v_5, v_6, v_1, v_2\}$ and $D_5 = \{v_6, v_1, v_2, v_3\}$ are minimal connected perfect dominating sets of an interval-valued fuzzy graph $G = C_6$. Therefore, a connected perfect dominating set exists. $D_{cp} = n-2 = 6-2 = 4$, such that n is the number of vertices of the connected perfect dominating set in C_n.

COROLLARY 3.8. *For any cycle interval-valued fuzzy graph G of a cycle G^*,*

$$\gamma_{cp}(G) = min\left(\sum_{i=1}^{n-2} |u_i|\right),$$

and

$$\Gamma_{cp}(G) = max \left(\sum_{i=1}^{n-2} |u_i| \right).$$

PROOF. Let $G = C_n$ be any cycle interval-valued fuzzy graph, where n the number of vertices in G, then by above Theorem 15, $D_{cp} = n - 2$. Therefor,

$$\gamma_{cp}(G) = min \left(\sum_{i=1}^{n-2} |u_i| \right),$$

or

$$\Gamma_{cp}(G) = max \left(\sum_{i=1}^{n-2} |u_i| \right).$$

□

The following gives γ_{cp} and Γ_{cp} for the path interval-valued fuzzy graph P_n and the bipartite $K_{(\mu_1,\mu_2)}$. Additionally, other results.

THEOREM 16. *For any P_n strong path or $K_{(\mu_1,\mu_2)}$ bipartite interval-valued fuzzy graph with $n \geq 3$. Then,*

$$|D_{cp}| = \sum_{i=1}^{n-2} |v_i| = \sum_{i=1}^{n-2} \frac{1 + \mu_2(v_i) - \mu_1(v_i)}{2},$$

for $v_i \in V$.

PROOF. By Theorem 15 and Corollary 3.8, the proof of this Theorem is trivial. □

COROLLARY 3.9. *If G be a strong path or bipartite, then γ_{cp}= Γ_{cp}.*

THEOREM 17. *Let G be any strong cycle, strong path, or bipartite interval-valued fuzzy graph with $n \geq 5$. Then $\gamma_{cp}(G) \geq \frac{p}{2}$.*

PROOF. Let G be an interval-valued fuzzy graph. If G is a strong path or bipartite, hence G has one connected perfect dominating set of G. It is $D_{cp} = n - 2$. Also, if G is a strong cycle, then the connected perfect dominating set of G is the minimum of all minimal connected perfect dominating sets. It is $D_{cp} = min\{n - 2\}$, where n is the number of vertices in G. Since $\gamma_{cp}(G) \leq p$ and $n \geq 5$. Thus, $2(\gamma_{cp}(G)) \geq p$.
Hence, $\gamma_{cp}(G) \geq \frac{p}{2}$. □

EXAMPLE 9. *Consider the interval-valued fuzzy graph given in the following Figure, such that all edges of G are strong.*

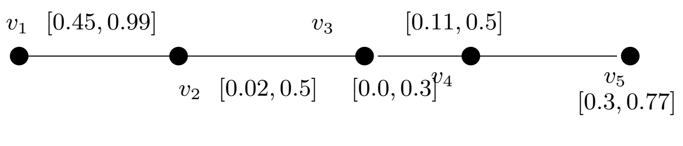

Fig. 8

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As shown in the above Figure, $D_{cp} = \{v_2, v_3, v_4\}$ is a connected perfect dominating set of interval-valued fuzzy graph G. Then, $\gamma_{cp}(G) = 2.124$, $p = |V| = 3.544$, and $\frac{p}{2} = \frac{3.544}{2} = 1.772$. Hence, $\gamma_{cp}(G) \geq \frac{p}{2}$.

THEOREM 18. *For any connected interval-valued fuzzy graph G.*

$$\gamma_{cp}(G) \leq p - \Delta_N(G) + 1.$$

PROOF. By Theorem 3.27 [2], the proof of this Theorem is self-evident. □

COROLLARY 3.10. *For any connected interval-valued fuzzy graph G.*

$$\gamma_{cp}(G) \leq p - \Delta_E(G) + 1.$$

THEOREM 19. *For any connected interval-valued fuzzy graph G.*

$$\frac{p}{2(\Delta_N + 1)} \leq \gamma_{cp}(G) \leq p - \Delta_N(G) + 1.$$

PROOF. Let G be any connected interval-valued fuzzy graph, with $\gamma_{cp} - set$ and let u be a vertex of G with $\Delta_N(u) = d_N(u)$. Firstly we prove the lower bound. (By Remark 3.98, [4]), $\frac{p}{\Delta_N(G)+1} \leq \gamma_{cp}(G)$ or $\frac{p}{\Delta_N(G)+1} \geq \gamma_{cp}(G)$. Then if $\frac{p}{\Delta_N(G)+1} \leq \gamma_{cp}(G)$, it is trivial, thus $\frac{p}{2(\Delta_N+1)} \leq \gamma_{cp}(G)$ or if $\frac{p}{\Delta_N(G)+1} \geq \gamma_{cp}(G) \Rightarrow \frac{p}{\Delta_N(G)+1} \leq 2(\gamma_{cp}(G)) \Rightarrow$

$$\frac{p}{2(\Delta_N + 1)} \leq \gamma_{cp}(G)...........(1).$$

Secondly we prove the upper bound. By Theorem 18

$$\gamma_{cp}(G) \leq p - \Delta_N(G) + 1...........(2).$$

Therefore, from (1) and (2) we get

$$\frac{p}{2(\Delta_N + 1)} \leq \gamma_{cp}(G) \leq p - \Delta_N(G) + 1.$$

□

COROLLARY 3.11. *For any connected interval-valued fuzzy graph G.*

$$\frac{q}{2(\Delta_E + 1)} \leq \gamma_{cp}(G) \leq q - \Delta_E(G) + 1.$$

4. CONCLUSION

This paper, the concepts of connected perfect dominating set and connected perfect dominance number in interval-valued fuzzy graphs were introduced. Also found $\gamma_{cp}(G)$ and $\Gamma_{cp}(G)$ for some known parameters in interval-valued fuzzy graphs G, such as path, star, cycle, wheel, complete, and bipartite etc. Numerous conclusions were drawn from the instances, and many aspects of these notions were discussed with suitable examples.

5. REFERENCES

[1] M. Akram and W. A. Dudek, "Interval-valued fuzzy graphs", Cmput. Math. Appl. 61, (2011), 289-299.
[2] F. M. Al-Ahmadi and Mahiuob M. Q. Shubatah. Perfect dominating set of an interval-valued fuzzy graphs, Asian Journal of Probability and Statistics, 11(4), (2021), 35-46.
[3] F. M. Al-Ahmadi and Mahiuob M. Q. Shubatah. Connected dominating set in interval-valued fuzzy graphs, IJRDO Journal of Mathematics, 8(2), (2022), 1-9.

[4] Faisal M. Al-Ahmadi, Mahiuob M. Q. Shubatah and Yahya Q. Hasan. Total perfect domination of interval-valued fuzzy graphs, IJRDO Journal of Mathematics, 8(2), (2022), 1-32.

[5] P. Debnath, Domination in interval-valued fuzzy graphs, Annals of Fuzzy Mathematics and Informatics. 6(2), (2013), 363-370.

[6] M. Naga Maruthi Kumari and R. Chandrasekhar, "Operations on interval-valued fuzzy graphs", IJARSE; 4(4), (2015), 618-627.

[7] H. Rashmanlou and Y. B. Jun, Complete interval valued Fuzzy graphs, Annals of Fuzzy Mathematics and Informatics, 3(1), (2013), 107-130.

[8] S.Revathi, P.J.Jayalakshmi, C.V.R.Harinarayanan Perfect Dominating Sets in fuzzy graphs IOSR Journal of Mathematics, 8(3), (2013), 43-47.

[9] S.Revathi, R. Muthuraj and C.V.R.Harinarayanan Connected Perfect Dominating Sets in fuzzy graph Golden Research thoughts, Vol-5. (2015), 1-5.

[10] A. Rosenfeld, Fuzzy graphs, In. L. A. Zadeh, K. S. Fu and M. Shimura, (Eds). "Fuzzy Sets and their Applications", Academic Press, New York, (1975), 77-95.

[11] N. Sarala, P. Janaki, Connected domination number of fuzzy digraphs, research article IJSRR, 8(2), (2019), 4096-4100.

[12] N. Sarala, T. Kavitha, Complete and Complementary Domination Number in Interval Valued Fuzzy Graphs. IJSR, 5 (2016), 2046-2050.

[13] N. Sarala, T. Kavitha, Strong (Weak) Domination in Interval Valued Fuzzy Graph. Discovery, 52(248), (2016), 1626-1634.

[14] A. Somasundaram, S. Somasundaran, "Domination in fuzzy graphs-I", Pattern Recognition Letters, 19, (1998), 787-791.

[15] A. Somasundaram, "Domination in fuzzy graphs-II", Journal of Fuzzy Mathematics, 20, (2000), 281-289.

[16] L. A. Zadeh, The concept of a linguistic and application to approximate reasoning I, Inform. Sci. 8, (1975), 149-249.