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Stability of Conditional Invariant Sets of Control Systems

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ABSTRACT

In this paper we attempted to extend the work of study the stability properties of conditional invariant sets for a control system in R^n. Here necessary and sufficient conditions for relative to the given control system are determined.

Theorems are based on the work of Ladde, an optional control problem maximising a performance inertia is proved.

Keywords

Invariant Set, Asymptotical Control System, Perturbed System.

1. INTRODUCTION

Kayande and Lakshmikantham (26) introduced the notion of a conditional invariant set while Leela (38) has shown that the stability properties can be derived with the help of a single Lyapunov function. She also obtained some converse theorems showing the existence of such functions on the basis of stability properties.
In this paper it is attempted to extend the work of several authors to study the stability properties of conditional invariant sets for a control system in R^n.
Further it contains the definition of a conditional invariant set and some necessary and sufficient conditions for the stability behaviour of such sets in terms of suitable Liapunov functions. Also the definitions of strict stability of the conditional invariant set B relative to set A are given and necessary and sufficient conditions for these concepts to hold relative to the given control system are determined.
Then, comparison techniques based on differential inequalities are used to study the stability behaviour of conditional invariant set B relative to set A for the control system (2.1). Also converse theorems are proved.
Further it has some theorems on the asymptotic stability of conditional invariant set B relative to set A for the system (2.1).
Then after, we consider the system (2.1) as a perturbed system of the system (5.1). Using the existence of smooth Lyapunov functions (proved in this para), the stability behaviour of C.I. set B relative to A, for the system (2.1) is studied. The theorems are based on the work of Ladde, Lakshmikantam and Leela (33).
Lastly, an optimal control problem maximising a performance inertia is proved. This generalises the work of Rumiantsev (56).

2. SUFFICIENCY CONDITIONS FOR STABILITY OF A CONDITIONAL INVARIANT SET WITH RESPECT TO A CONTROL SYSTEM

Consider the control system:

$$\left.\begin{array}{ll} x' = f(t,x,u) & ('= d/dt) \\ \text{with} & \\ f(t,0,0) = 0 & \end{array}\right] \tag{2.1}$$

where $f \in c(IXDXE, R^n), I = [0, \infty)$ and D is a region in the real Euclidean space R^n and E, the set of controls, is a compact set in R^m. We assume D to be invariant for the system (2.1), so that the solutions starting in D for a fixed $u \in E$, remain in D for all $t \in I$. D may be R^n itself.
Whenever it is desired to ensure that through any point $(t_0, x_0) \in IXD$, there exists a unique solution $x_u(t, t_0, x_0)$ of (2.1) for each fixed $u \in E$, we assume the following Lipschitz condition:

$$|f(t,x,u) - f(t,y,v)| \leq k(t)|(x-y) + (u-v)| \tag{2.2}$$

where $| \cdot |$ denotes a convenient norm in R^n and $k(t) \in c(I, I)$. Then the solution $x_u = x_u(t, t_0, x_0)$ of (2.1) is also continuous with respect to initial conditions.
Let the sets $A, B \subset D$. We recall $d(x,a) = |x-a|$, $x, a \in D$ and $d(x,A) = \inf_{a \in A} d(x,a)$.
Clearly, $A \subset B$ implies $d(x,A) \geq d(x,B), x \in D$.
In the sequel A and B are assumed to be compact. Slight modifications will give similar results when B is closed.
We state the following definition for conditionally invariant set B relative to A for the system (2.1).

Definition 2.1. *The set B is said to be conditionally invariant relative to set A for the control system (2.1), if $u \in E$ and $x_0 \in A$ imply that $x_u(t, t_0, x_0) \in B$ for all $t \geq t_0 \geq 0$*

We write: C.I. set B to denote the conditionally invariant set B relative to A. In this notation, set A is not mentioned explicitely.
The following are due to the above definition.

Corollary 1. *If the set A is self-invariant i.e. $u \in E$ and $x_0 \in A$ imply that $x_u(t, t_0, x_0) \in A$ for all $t \geq t_0$), then any set $B \supset A$ is conditionally invariant relative to A for the system (2.1).*

Corollary 2. *If the set B is self invariant, then it is conditionally invariant relative to any set $A \subset B$ for the system (2.1).*

Thus our analysis will be useful even in the case where neither A nor B is self-invariant for the system (2.1).
Definitions: The C.I. set B is said to be

Definition 2.2. ***Stable***, *if for any $\in > 0$, however small and $t_0 \geq 0$, we can find a $\delta = \delta(t_0, \in) > 0$, $\delta \to 0$ as $\in \to 0$, such that for all $t \geq t_0$,*

$$d(x_0, A) \leq \delta \;\; implies \;\; d(x_u(t, t_0, x_0), B) < \in \tag{2.3}$$

where $x_u = x_u(t, t_0, x_0)$, is a solution of (2.1) as explained earlier.

Definition 2.3. ***Uniformly stable***, *if δ in (2.2) above is independent of t_0.*

Definition 2.4. ***asymptotically stable***, *if it is stable and for each $t_0 \geq 0$, there exists an $\alpha(t_0) > 0$ such that $d(x_0, A) \leq \alpha(t_0)$ implies*

$$\lim_{t\to\infty}[d(x_u(t, t_0, x_0), B)] = 0 \tag{2.4}$$

Definition 2.5. ***Uniformly asymptotically stable***, *if it is uniformly stable and α in (2.4) above is independent of t_0. Then for each $\in > 0$, there exists a $T(\in) > 0$, such that for each $t \geq t_0 + T$,*

$$d(x_0, A) \leq \alpha \;\; implies \;\; d(x_u(t, t_0, x_0), B) < \in \tag{2.5}$$

Definition 2.6. ***Exponentially asymptotically stable***, *if there exists constants $M > 0$ and $\alpha > 0$ such that*

$$d(x_u(t, t_0, x_0), B) \leq M d(x_0, A) \exp(-\alpha(t - t_0)), t \geq t_0 \geq 0. \tag{2.6}$$

We use Lyapunov functions to determine sufficient conditions for stability of C.I. set B.

$$\text{Let } V \in \gamma, \text{if } V \in c_{\text{lip}}(IXD, I), V(t, x) = 0 \; x \in A. \tag{2.7}$$

Definitions 2.7 & 2.8.
The function $V \in \gamma'$ if $V \in \gamma$ and possess continnous partial derivatives $\frac{\partial V}{\partial t}, \frac{\partial V}{\partial x_i} (i = 1, 2, \ldots, n)$ in IXD.
The function $V \in \gamma(\text{or} \in \gamma')$ is said to be positive definite with respect to the set B, if there exists a function $a \in K$ such that for $(t, x) \in IXD$,

$$a(d(x, B)) \leq V(t, x) \tag{2.8}$$

The function $V \in \gamma$ is said to be decrescent with respect to the set A, if there exists a function $b \in K$ such that for

$$(t, x) \in IXD, V(t, x) \leq b(d(x, A)) \tag{2.9}$$

Following are the theorems on stability of C.I. set B for the system (2.1).

Theorem 2.1. *Let there exist a function $V = V(t, x) \in \gamma'$ such that for $(t, x) \in IXD$ and for each $u \in E$,*

$$(1) \qquad V'(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_i} f_i(t, x) \leq 0 \tag{2.10}$$

and

(2) V satisfies (2.8). Then the C.I. set B is stable.
If V also satisfies (2.9), then the stability is uniform.

Proof. From (2.10)

$$V(t, x_u(t, t_0, x_0)) \leq V(t_0, x_0) \tag{i}$$

Given $\in > 0$ however small, $t_0 \in I$, we can find a $\delta = \delta(t_0, \in) > 0$, $\delta \to 0$ as $\in \to 0$ such that

$$d(x_0, A) \leq \delta \text{ implies } V(t_0, x_0) < a(\in) \tag{ii}$$

due to the continuity of V and the condition that $V(t_0, x) = 0$ for $x \in A$ as $V \in \gamma'$.
Then from (2.8), (i) and (ii) above we have for $d(x_0, A) \leq \delta$,

$$\begin{aligned} a(d(x_u(t, t_0, x_0), B)) &\leq V(t, x_u(t, t_0, x_0)) \\ &\leq V(t_0, x_0) \\ &< a(\in) \end{aligned}$$

$$\text{implies } d(x_u(t, t_0, x_0), B) < \in, \text{ for } t \geq t_0. \tag{iii}$$

□

Hence the stability of C.I. set B.
If V satisfies (2.9), then can be chosen independent of t_0; with $\delta_1 = \delta b^{-1} a(\in)$, step (iii) above is verified.

Theorem 2.2. *If there exists a function $V \in \gamma$ such that the condition (2.10) in theorem (2.1) is replaced by*

$$D^+ V(t, x) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(t+h, x+hf(t, x, u)) - V(t, x)] \leq 0 \tag{2.11}$$

and other conditions remain unchanged, the conclusions of theorem 2.1 hold.

Proof. As $V \in \gamma, V$ Satisfies Lipschits condition in x so that $\lim_{h\to 0^+} \sup \frac{1}{h}[V(t + h, x + hf(t, x, u)) - V(t, x)] = V'(t, x)$. Thus (2.11) implies $V(t, x_u(t, t_0, x_0)) \leq V(t_0, x_0)$. The theorem follows at once, on the lines of the proof of theorem (2.1). □

Theorem 2.3. *Let there exist a function V satisfying the following properties -*

(1) $V(t, x) = 0$ for $x \in A$ and $V(t, x)$ is continuous in x on ∂A, the boundary of A.
(2) (2.8) holds (i.e., $V \in \gamma$ and V is positive definite and
(3) $V(t, x_u(t, t_0, x_0))$ is non-increasing along the trajectories for $t \in I$.
Then the C.I. set B is stable. If V also satisfies (2.9), then the stability is uniform.

Proof. From (3) $V(t, x_u(t, t_0, x_0)) \leq V(t_0, x_0)$. □

The rest of the argument follows on the same lines as in the proof of theorem (2.1).

Theorem 2.4. *If the C.I. set B is uniformly stable for the system (2.1) satisfying (2.2), then there exists a function satisfying the conditions of theorem (2.3).*

Proof. As the set B is uniformly stable with respect to A, given $\in > 0$, however small, there exists a $\delta = \delta(\in) > 0$ such that for $u \in E$, □
$d(x_0, A) \leq \delta$ implies $d(x_u(t, t_0, x_0), B) < \in$, for all $t \geq t_0$.
Let $\in(\delta)$ denote the inverse function of $\delta(\in)$. δ can be chosen to belong to K, the class of monotonic functions (Hahn (15)).
Define for $(t, x) \in IXD$,

$$V(t, x) = \sup_{T \geq 0} d(x_u(t + T, t, x), B).$$

For $x \in A, x_u(t + T, t, x) \in B$ so that for each $T \geq 0$,

$$d(x_u(t + T, t, x), B) = 0.$$

Hence $V(t, x) = 0$ for $x \in A$.
Also $V(t, x) \to 0$ as $d(x, A) \to 0$.
Thus $V(t, x)$ is continuous on A. This verifies the hypothesis (1).
Also $d(x, B) \leq V(t, x)$ follows from the definition of V. For each $T \geq 0$, $d(x_u(t+T, t, x), B) \leq \delta(d(x, A))$ by the hypothesis and choice of δ.
Therefore $V(t, x) \leq \delta(d(x, a)), \delta \in K$.
Thus we have verified (2.8) and (2.9).
Now,

$$\begin{aligned} &V(t_1, x_{u_1}(t_1, t_0, x_0)) \\ &= \sup_{T \geq 0} d(x_u(t_1 + T, t_1, x_{u_1}(t_1, t_0, x_0)), B) \\ &= \sup_{T \geq 0} d(x_u(t_1 + T, t_0, x_{u_0}), B) \end{aligned}$$

where $x_{u_0} = x_{u_1}(t_0, t, x_{u_1}(t, t_0, x_0))$
Similarly,

$$\begin{aligned} &V(t_2, x_{u_1}(t_2, t_0, x_0)) \\ &= \sup_{T \geq 0} d(x_u(t_2 + T, t_0, x_{u_0}), B). \end{aligned}$$

If $t_1 \geq t_2$ then $V(t_1, x_u(t_1, t_0, x_0)) \leq V(t_2, x_u(t_2, t_0, x_0))$ for any $u \in E$.
This verifies hypothesis (3). This completes the proof of the theorem.
Following the standard arguments (31), we can show that V satisfies Lipschitz condition in x and is also continuous on I. That proves the following theorem -

Theorem 2.5. *If the C.I. set B is uniformly stable relative to the set A for the control system (2.1) satisfying (2.2), then there exists a function V satisfying the hypotheses of theorem (2.2).*

We state the following definitions of strict stability properties of C.I. set B.
Definitions: Relative to A, the C.I. set B is said to be:

Definition 2.7. ***Strict-stable***, *if there exists functions $a_1, a_2 \in K^*$ such that for some $\alpha > 0$, $d(x_0, A) \leq \alpha$ implies, for all $t \geq t_0$,*

$$a_2(t_0, d(x_0, B)) \leq d(x_u(t, t_0, x_0), B) \leq a_1(t_0, d(x_0, A)) \tag{2.12}$$

Definition 2.8. ***Uniform-strict-stable***, *if there exists $a_1, a_2 \in K$ satisfying (2.12) i.e., a_1, a_2 above are independent of t_0.*

Definition 2.9. ***Strict-asymptotically-stable***, *if there exist functions $a_1, a_2 \in K^*$ and $b_1, b_2 \in L^*$, such that*

$$\begin{aligned} a_2(t_0, d(x_0, B))b_2(t_0, t - t_0) &\leq d(x_u(t, t_0, x_0), B) \\ &\leq a_1(t_0, d(x_0, A))b_2(t_0, t - t_0) \end{aligned} \tag{2.13}$$

Definition 2.10. ***Uniform-strict-asymptotic-stable***, *if in (2.13) above $a_1, a_2 \in K$ and $b_1, b_2 \in L$ i.e., a_1, a_2 are independent of t_0 while b_1, b_2 belong to class L.*

Remark 1. *(2.7) and (2.8) rule out the possibility of asymptotic and uniform asymptotic stability. Moreover, if $\delta_2 = d(x_0, B)$ and $\delta_1 = d(x_0, A)$, $\delta_1, \delta_2 > 0$. Then (2.12) becomes $a_2(t_0, \delta_2) \leq d(x_u(t, t_0, x_0), B) \leq a_1(t_0, \delta_1)$. Thus the solution remains in some tube-like domain in the neighbourhood of B. The definition thus corresponds to stability in tube-like domain (19,31). We can write the definitions (2.7) to (2.10) in equivalent forms in terms of $\in$ and δ.*

Remark 2. *It is important to note that $d(x_0, B)$ on the left sides of inequalities in the definitions (2.7) to (2.10) cannot be replaced by $d(x_0, A)$, unless $B = A$, for if $x_0 \in B - A$, then $d(x_0, A) \geq 0$ but $d(x_0, B) = 0$. Thus the inequalities cannot be satisfied even at $t = t_0$.*

The following are theorems on strict-stability:

Theorem 2.6. *Let there exist functions V_1, V_2 such that*

$$(1) \quad D^+V_1 = \lim_{h \to 0^+} \sup \frac{1}{h}[V(t+h, x+hf(t, x, u)) - V(t, x)] \leq 0 \tag{2.14}$$

$$(2) \quad D^-V_2 = \lim_{h \to 0^+} \inf \frac{1}{h}[V(t+h, x+hf(t, x, u)) - V(t, x)] \geq 0 \tag{2.15}$$

and
(3) there exist functions $a_1, a_2 \in K^*$ and $a_3, a_4 \in K$ such that

$$a_3(d(x, B)) \leq V_1(t, x) \leq a_1(t, d(x, A)) \tag{2.16}$$

and

$$a_2(t, d(x, B)) \leq V_2(t, x) \leq a_4(d(x, 3)) \tag{2.17}$$

Then the C.I. set B is strict-stable relative to the set A.

Proof. From (1) and (2) we get

$$V_1(t, x_u(t, t_0, x_0)) \leq V_1(t_0, x_0) \tag{i}$$

and

$$V_2(t, x_u(t, t_0, x_0)) \geq V_2(t_0, x_0) \tag{ii}$$

From (3) - (2.16) and (i)

$$a_3(d(x_u, B)) \leq V_1(t, x_u) \leq V_1(t_0, x_0) \leq a_1(t_0, d(x_0, A))$$

implies

$$d(x_u, B) \leq a_3^{-1}(a_1(t_0, d(x_0, A))) \tag{iii}$$

Again from (3)-(2.17) and (ii),

$$a_4(d(x_u, B)) \geq V_2(t, x_u) \geq V_2(t_0, x_0) \geq a_2(t_0, d(x_0, B))$$

implies

$$d(x_u, B) \geq a_4^{-1}(a_2(t_0, d(x_0, B))) \tag{iv}$$

(iii) and (iv) imply

$$a(t_0, d(x_0, B)) \leq d(x_u, B) \leq b(t_0, d(x_0, A))$$

where $a = a_4^{-1}a_2 \in K^*$ and $b = a_3^{-1}a_1 \in K^*$. □

Hence the C.I. set B is strict stable relative to A.

Theorem 2.7. *If, in theorem (2.6), $a_1, a_2 \in K$ aswell, then the C.I. set B is uniform strict stable relative to A.*

Proof. On the same line as the proof of theorem (2.6). □

Theorem 2.8. *Let V_1 be non-increasing and V_2 be non-decreasing along the trajectories of the equation (2.1), satisfying the hypothesis (3) theorem (2.6). Then the C.I. set B is strict stable relative to A. If $a_1, a_2 \in K$, then the strict stability is uniform.*

Proof. V_1, V_2 being respectively non-increasing and non-decreasing imply the steps (i) and (ii) in the proof of the theorem (2.6). The rest of the argument is the same as in the proof of the theorem (2.6). □

Assuming that (2.2) is satisfied, following are the theorems on the existence of Lyapunov functions for strict stability.

Theorem 2.9. *Let the C.I. set B be uniformly strict stable relative to set A for the system (2.1). Then there exist functions V_1 and V_2 satisfying the hypotheses of theorem (2.7).*

Proof. For $(t, x) \in IXD$, define, with any fixed $u \in B$,

$$V_1(t,x) = \inf_{0 \le T \le t} d(x_u(T,t,x), A) \quad \text{(i)}$$

and

$$V_2(t,x) = \sup_{0 \le T \le t} d(x_u(T,t,x), B) \quad \text{(ii)}$$

The C.I. set B is uniform strict stable; hence from definition 2.8, there exist $b_1, b_2 \in K$ such that

$$b_2(d(x_0, B)) \le d(x_u(t, t_0, x_0), B) \le b_1(d(x_0, A)) \text{ for } t \ge t_0. \quad \text{(iii)}$$

Thus from the step (iii), for each $T \in [0, t]$, we have

$$d(x_u(T,t,x), A) \ge b_1^{-1}(d(x,B)) \quad \text{(iv)}$$

and

$$d(x_u(T,t,x), B) \le b_2^{-1}(d(x,B)) \quad \text{(v)}$$

Writing $x = x_u(T, t, x)$, from the uniqueness of solutions (2.1), we get

$$V_1(t,x) \le d(x,A) \text{ and } V_1(t,x) \ge b_1^{-1}(d(x,B))$$

and

$$V_2(t,x) \ge d(x,B) \text{ and } V_2(t,x) \le b_2^{-1}(d(x,B))$$

Thus we get

$$b_1^{-1}(d(x,B)) \le V_1(t,x) \le d(x,A)$$

and

$$d(x,B) \le V_2(t,x) \le b_2^{-1}(d(x,B))$$

These verify the hypothesis (3) of theorem(2.6). The continuity of V_1 and V_2 follows from the continuous dependence of solutions of (2.1) on initial conditions.
For Lipschitz property, we have $V_1(t.x) = d(x_u(T_1, t, x), A)$ for some $T_1 \in [0, t]$, and $V_2(t, x) = d(x_u(T_2, t, x), B)$ for some $T_2 \in [0, t]$, due to the compactness of $[0, t]$. Hence from the fact that $|d(x, A) - d(y, A)| \le K_1(t)|x - y|$ and the standard inequalities on the trajectories of (2.1) satisfying (2.2), we are lead to

$$|V_i(t,x) - V_i(t,y)| \le |x-y| \exp \int_0^t K_1(s)ds, (i = 1,2) \quad \text{(vi)}$$

Thus $V_i (i = 1, 2)$
Now

$$\begin{aligned} V_1(t, x_{u_1}(t,t_0,x_0)) &= \inf_{0 \le T \le t} d(x_u(T,t,x_{u_1}(t,t_0,x_0)), A) \\ &= \inf_{0 \le T \le t} d(x_u(T,t_0,x_{u_0}), A) \end{aligned}$$

due to the uniqueness of solutions of (2.1).
Also for

$$\begin{aligned} h \ne 0, \quad & V_1(t+h,\ x_{u_1}(t+h,t_0,u_0)) \\ &= \inf_{0 \le T \le t} d(x_u(T,t_0,x_{u_0}), A) \\ &\le V_1(t, x_{u_1}(t,t_0,x_0)) \end{aligned} \quad \text{(vii)}$$

Similarly,

$$V_2(t+h, x_{u_1}(t+h,t_0,x_0)) \ge V_2(t, x_{u_1}(t,t_0,x_0)) \quad \text{(viii)}$$

(vii) and (viii) show that V_1 and V_2 are non-increasing and non-decreasing respectively along the trajectories of (2.1). Hence V_1 and V_2 satisfy the hypotheses of theorem (2.8) and this implies the hypothesis of theorem (2.6). □

This proves the converse of theorem (2.6).

3. STABILITY BEHAVIOUR OF CONDITIONAL INVARIANT SET B WITH RESPECT TO A FOR THE CONTROL SYSTEMS

Let $V \in \gamma'$ and for all $(t, x, u) \in IXDIE$

$$V'(t,x) \le g(t, V(t,x)) \quad (3.1)$$

where $g \in c(IXI, R)$, $g(t, 0) \ge 0$ and the maximal solution

$$\left.\begin{aligned} r(t,t_0,r_0) \text{ of: } r' = g(t,r) \\ \text{with } r(t_0) = r_0 \end{aligned}\right] \quad (3.2)$$

exists for all $t \in [t_0, \infty]$.

Theorem 3.1. *Let there exist a function $V \in \gamma'$ such that (3.1) and (2.8) hold, with $V(t, x) = 0$ for $x \in A$. If (1)*
If (1) the trivial solution of (3.2) is stable, then the C.I. set B is stable, and if (2) V satisfies (2.9) and the trivial solution of (3.2) is uniformly stable, then the C.I. set B is uniformly stable.

Proof. From (3.1), by standard comparison theorems (see 31) it follows $V(t_0, x_0) \le r_0$
implies

$$V(t, x_u(t,t_0,x_0)) \le r(t,t_0,r_0), t \ge t_0 \quad \text{(i)}$$

For any $\in > 0$, $t_0 \in I$, there exists, due to the stability of the trivial solution of (3.2), a $\delta_1 = \delta(t_0, \in) > 0 \delta_1 \to 0$ as $\in \to 0$ such that for all $t \ge t_0, r_0 \le \delta$, implies

$$r(t,t_0,r_0) < a_1(\in), a_1 \in K \quad \text{(ii)}$$

Due to the continuity of V and the fact that $V(t, x) = 0$ for $x \in A$, there exists $\delta_2 = (t_0, \delta_1) > 0$ such that $d(x_0, A) \le \delta_2$ and $V(t_0, x_0) \le \delta_1$ hold simultaneously.

With this δ_2, the choice of $r_0 \leq V(t_0, x_0)$, $d(x_0, A) \leq \delta_2$ implies

$$\begin{aligned} a_1(d(x,B)) &\leq V(t,x) && \text{from (2.8)} \\ &\leq r(t,t_0,x_0) && \text{from step (i)} \\ &< a_1(\in) && \text{from step (ii)} \end{aligned}$$

implies $d(x, B) <\in$, $t \geq t_0$, which means that the C.I. set B is stable.
(2) In this case δ_1 is independent of t_0 and as V satisfies (2.9), δ_2 can also be chosen independent of t_0. □

A weaker version of the above theorem, which follows from the assumption due to $V \in \gamma$, (3.3) implies step (i) of the proof of the theorem (3.1) is the following theorem -

Theorem 3.2. *In theorem (3.1), the condition (3.1) can be replaced by $V \in \gamma$ and*

$$D^+V(t,x) \leq g(t, V(t,x)) \tag{3.3}$$

Following is a converse theorem.

Theorem 3.3. *Suppose that*
(1) f satisfies (2.2)
(2) there exists a function $a \in K$ such that

$$d(x_u(t,t_0,x_0), B) \leq a(d(x,A)),\ t \geq t_0 \geq 0.$$

(3)

$$g \in c(IXI, R),\ g(t,0) \geq 0$$

and

$$|g(t,r_1) - g(t,r_2)| \leq k_2(t)|r_1 - r_2|$$

for

$$t \in I, r_1, r_2 \in I, k_2 \in c(I, I).$$

(4) the solution $r(t, 0, r_0)$ of

$$r' = g(t,r) \tag{3.4}$$

has the property

$$c_1(r_0) \leq r(t,0,r_0) \leq c_2(r_0) \tag{3.5}$$

where $c_1, c_2 \in K$.
Then there exists a function $V = V(t,x), (t,x) \in IXD$ with the properties–
(i) $V \in \gamma$
(ii) (2.8) and (2.9) are satisfied by V, and
(iii) $D^+V(t,x) \leq g(t, V(t,x))$.

Proof. Choose any $u \in E$ and define

$$V(t,x) = r(t, 0, \inf_{0 \leq T \leq t} d(x_u(T,t,x), A)).$$

From (2) for each

$$T \in [0,t], d(x_u(T,t,x), A) \leq a^{-1}(d(x,B)).$$

Therefore

$$\inf_{0 \leq T \leq t} d(x_u(T,t,x), A) \geq a^{-1}d(x,B).$$

Thus from (3.5)

$$V(t,x) \geq c_1(\inf_{0 \leq T \leq t} d(x_u(T,t,x), A)) \geq c_1(a^{-1}(d(x,B)))$$

and

$$V(t,x) \leq c_2(\inf_{0 \leq T \leq t} d(x_u(T,t,x), A)) \leq c_2(d(x,A)).$$

Taking $a_1 = c_1 a^{-1}$, $a_2 = c_2$, $a_1, a_2 \in K$ and hence V satisfies (2.8) and (2.9). Using standard arguments, it can be shown that $V \in \gamma$ and satisfies (iii). □

Note: (2) verifies the uniform stability of C.I. set B relative to A. Result (i) is equivalent to strict stability of the trivial solution of (3.4) at $t_0 = 0$.
The following theorem is on the lower estimates on the distance of the trajectories from C.I. set B.

Theorem 3.4. *Let there exist a function $V_2 \in \gamma'$, such that for $(t,x) \in IXD$,*

(1)

$$V_2'(t,x) \geq p(t, V_2(t,x)), p \in c(IXI, R) \tag{3.6}$$

and (2) there exist functions $b_3 \in K$ and $b_4 \in K^*$ such that

$$b_4(t, d(x,B)) \leq V_2(t,x) \leq b_3(d(x,B)) \tag{3.7}$$

Let $r(t, t_0, r_0)$ denote the minimal solution of

$$r' = p(t,r), r(t_0) = r_0 \tag{3.8}$$

existing to the right of t_0 (i.e. $t \geq t_0$).
Then for $t \geq t_0 \geq 0$, and $r_0 \geq \alpha > 0$,
(i) $r(t, t_0, r_0) \geq a_3(t_0, r_0)$, $a_3 \in K^*$ implies

$$d(x_u(t,t_0,x_0), B) \geq b_3^{-1} a_3(t_0, b_4(t_0, d(x_0,B))),$$

and
(ii) $r(t, t_0, r_0) \geq a_3(t_0, r_0) c_3(t_0, t - t_0)$, $a_3 \in K^*$, $c_3 \in L^*$ implies

$$d(x_u(t,t_0,x_0), B) \geq a_3^{-1}[a_3(t_0, b_4(t_0, d(x_0,B))) c_3(t_0, t - t_0)]$$

Also if $b_4(t, d(x,B)) = b_5(d(x,B))$, $b_5 \in K$, and
$c_3(t_0, t - t_0) = c_4(t - t_0)$, $c_4 \in L$, then
(iii) $r(t, t_0 r_0) \geq a_4(r_0)$, $a_4 \in K$, implies
$d(x_u(t,t_0,x_0), B) \geq (b_3^{-1} a_4 b_5)(d(x_0,B))$, and
(iv) $r(t,t_0,r_0) \geq a_5(r_0) c_5(t - t_0)$, $a_5 \in K$, $c_5 \in L$, implies

$$d(x_u(t,t_0,x_0), B) \geq b_3^{-1}[a_5(b_5(d(x_0,B))) c_5(t - t_0)]$$

The results (i) to (iv) hold for all $x_0 \in D$ such that $d(x_0, B) \geq b_3^{-1}(\alpha)$.

Proof. From (3.6) and standard comparison theorem

$$\left.\begin{aligned} V_2(t, x_u(t,t_0,x_0)) \geq r(t,t_0,r_0) \\ \text{whenever } V_2(t_0,x_0) \geq r_0 \end{aligned}\right] \tag{i}$$

Choose $r_0 = V_2(t_0, x_0)$. Let $x_0 \in D$ so that $d(x_0, B) \geq b_3^{-1}(\alpha)$. Then from (3.7) $r_0 = b_3(d(x_0,B)) = V_2(t_0,x_0) > \alpha$ (since the equality holds at $t = t_0$) or $r_0 > \alpha$.
Consider

$$\begin{aligned} b_3(d(x_u(t,t_0,x_0), B)) &\geq V_2(t, x_u(t,t_0,x_0)) \\ &\geq r(t,t_0,r_0) \\ &\geq a_3(t_0, r_0) \\ &\geq a_3(t_0, V_2(t_0,x_0)) \\ &\geq a_3(t_0, b_4(t_0, d(x_0,B))),\ \text{as } a_3 \in K^*. \end{aligned}$$

□

Thus

$$d(x_u(t,t_0,x_0).B) \geq b_3^{-1} a_3(t_0, b_4(t_0, d(x_0,B))).$$

Hence

$$d(x_u(t,t_0,x_0),B) \geq b_3^{-1} a_3(t_0, b_4(t_0, d(x_0,B)))$$

verifies (i).
Other results (ii) to (iv) follow on the same lines.
A weaker version of theorem (3.4), which follows from assumption that $V \in \gamma$ implies (3.6) which in turn implies step (i) of the proof of theorem (3.4) is the following.

Theorem 3.5. *The condition (3.6) in theorem (3.4) can be replaced by*

$$D^- V_2(t,x) \geq p(t, V_2(t,x)) \tag{3.9}$$

Converse theorem on the existence of V_2 satisfying the assumptions of theorem (3.5).

Theorem 3.6. *Suppose that -*

(1) f satisfies (2.2)
(2) there exists a funcion $a \in K$ such that for

$$t \geq t_0 \geq 0, a(d(x_0,B)) \leq d(x_u(t,t_0,x_0),B) \tag{3.10}$$

(3) $p \in C(IXI,R)$, $p(t,0) \geq 0$, and

$$|p(t,r_2) - p(t,r_1)| \leq k_3(t)|r_1 - r_2| \tag{3.11}$$

and
(4) the solution $r(t,0,r_0)$ of

$$r' = p(t,r), \quad (p \text{ defined in (3)}) \tag{3.12}$$

has the property -

$$c_3(r_0) \leq r(t,0,r_0) \leq c_4(r_0) \tag{3.13}$$

where $c_3, c_4 \in K$.
Then there exists a function $V_2 = V_2(t,x)$ such that
(i) $V_2 \in \gamma$
(ii) V_2 satisfies (2.8) and (2.9) and
(iii) $D^-(V_2(t,x)) \geq p(t, V_2(t,x))$.

Proof. As (1) is satisfied for each fixed $u \in E$, the solution $x_u(t,t_0,x_0)$ of (2.1) through the point $(t_0,x_0) \in IXD$, is unique and depends continuously on initial conditions.
Let for a fixed $u \in E$, $x = x_u(t,t_0,x_0)$. Then $x_0 = x_u(t_0,t,x)$ due to uniqueness of the solution of (2.1).
Similarly the solutions of (3.12) are unique and depend continuously on initial conditions.
For $(t,x) \in IXD$, define

$$V_2(t,x) = r(t, 0, \sup_{0 \leq T \leq t} d(x_u(T,t,x),B)).$$

From (3.13) and the properties of supremum, we get

$$V_2(t,x) \geq c_3(d(x,B)) \tag{i}$$

and from (3.10), (3.13)

$$V_2(t,x) \leq (c_4 a^{-1}) d(x,B) \tag{ii}$$

Steps (i) and (ii) above verify the conclusion (ii) of the theorem.
By standard arguments, properties (i) and (iii) are proved. □

The strict stability properties are now verified combining some results of theorems (3.1), (3.2) with (3.4) and (3.5) respectively. We state below the theorem corresponding to uniform strict stability of the C.I. set B relative to A, without proof.

Theorem 3.7. *Let there exist functions V_1 and V_2 such that*

(1) V_1 and V_2 satisfy the conditions (2.8) and (2.9) (with the monotonic functions of class K)
(2) V_1 satisfies (3.3) and condition (2) of theorem (3.1), and
(3) V_2 satisfies (3.9) and condition (3) of theorem (3.4).
Then the C.I. set B is uniform strict stable relative to A.
Theorems (3.3) and (3.6) show the existence of functions V_1 and V_2 in case the C.I. set B is uniform strict stable relative to A.

4. THEOREMS ON ASYMPTOTIC STABILITY OF CONDITIONAL INVARIANT SET B RELATIVE TO A FOR THE SYSTEM (2.1)

Theorem 4.1. *Let there exist a function $V = V(t,x)$ such that*

(1) $V(t,x) = 0$ for $x \in A$ and $V(t,x)$ is continuous in x on the boundary ∂A of A.
(2) (2.8) holds, and
(3) $V(t,x)$ decreases monotonically in t along the trajectories of (2.1) for each $u \in E$, and $\lim_{t\to\infty} V(t, x_u(t,t_0,x_0)) = 0$, for some $x_0 \in \bar{S}(A,\delta)$, $\delta = \delta(t_0)$.
Then the C.I. set B is asymptotically stable relative to A. If is independent of t_0 then the asymptotic stability is uniform.

Proof. Hypotheses (1), (2) and (3) show, by theorem (2.3) that the C.I. set B is stable relative to A. Now from (2.8) and the hypothesis (3) we have for

$$x_0 \in \bar{S}(A,\delta) \lim_{t\to\infty} a(d(x_u(t,t_0,x_0),B)) \leq \lim_{t\to\infty} V(t, x_u(t,t_0,x_0)) = 0.$$

Hence

$$\lim_{t\to\infty} d(x_u(t,t_0,x_0),B) = 0.$$

Hence the C.I. set B is asymptotically stable relative to A. □

Theorem 4.2. *If there exists a function $V \in \gamma'$ such that for $(t,x) \in IXD$,*

(1) $\eta(t) a(d(x,B)) \leq V(t,x)$, where η increases monotonically to infinity, $\eta(0) = 1$, $a \in K$.
(2) $V(t,x) = 0$, $x \in A$, and
(3) $V'(t,x) \leq 0$.
Then the C.I. set B is asymptotically stable relative to A.

Proof. If $\eta(t) = 1$, hypotheses (1), (2) and (3) show, by theorem (3.1), that the C.I. set B is stable. From (3) and (1) follows that for $x_0 \in \bar{S}(A,\delta)$,

$$\eta(t) a(d(x_u(t,t_0,x_0),B)) \leq V(t, x_u(t,t_0,x_0)) \leq V(t_0,x_0).$$

The R.H.S. is constant while $\eta(t) \to \infty$ as $t \to \infty$.
Therefore

$$\lim_{t\to\infty} a(d(x_u(t,t_0,x_0),B)) = 0.$$

Since $a \in K$,

$$\lim_{t\to\infty} d(x_u(t,t_0,x_0),B) = 0.$$

Hence the C.I. set B is asymptotically stable relative to A. □

Theorem 4.3. *If there exists a function $V \in \gamma'$ such that*
(1) $V'(t,x) \le -c(d(x,A))$, $c \in K$, and
(2) (2.8) and (2.9) hold.
Then the C.I. set B is asymptotically stable.

Proof. From hypotheses (1) and (2), it follows that the C.I. set B is uniformly stable. Let $\in > 0$ be given and $\delta = \delta(\in)$ be the function determined from uniform stability. Let $\alpha > 0$, $t_0 \ge 0$ be given. Choose $T = \frac{b(\alpha)}{c(\delta)}$.
We claim that
$d(x_0, A) \le \alpha$ implies, there exists $t_1 \in [t_0, t_0 + T]$ such that

$$d(x_u(t_1, t_0, x_0), A) \le \delta.$$

If not, for all

$$t \in [t_0, t_0 + T], d(x_u(t, t_0 x_0), A) > \delta.$$

This shows that

$$\begin{aligned} 0 < a(\delta) &\le V(t_0 + T, x_u(t_0 + T, t_0, x_0)) \\ &\le V(t_0, x_0) - c(d(x_u(s, t_0, x_0), A))ds \qquad \text{(i)} \\ &\le b(d(x_0, A)) - c(\delta)(T + t_0 - t_0) \\ &= b(\alpha) - c(\delta)T = 0 \end{aligned}$$

i.e., $0 < 0$ which is absurd. Hence the claim.
Thus there exists a $t_1 \in [t_0, t_0 + T]$ such that

$$d(x_u(t_1, t_0, x_0), A) \le \delta.$$

Thus from uniform stability of C.I. set B relative to A

$$d(x_u(t, t_0, x_0), B) < \in \quad \text{for all } t \ge t_1.$$

Hence for $t \ge t_0 + T, d(x_0, A) \le \alpha$ implies that

$$d(x_u(t, t_0, x_0), B) < \in .$$

As T is independent of t_0, the asymptotic stability is uniform. □

Theorem 4.4. *If a function $V \in \gamma'$ exists such that*
(1) (2.8) adn (2.9) are satisfied,
(2) $V(t,x) \ge 0$, and
(3) $V(t,x) \to 0$ uniformly as $V'(t,x) \to 0$
i.e., for any $\in > 0$, we can find a $\delta(\in) > 0$ such that $V(t,x) < \in$ implies $V'(t,x) < \delta$.
Then the C.I. set B is uniformly asymptotically stable relative to A.

Proof. If $t_0 \ge 0$ and $d(x_0, A) \le \delta, \delta = b^{-1}a(\alpha)$, then $d(x_u(t, t_0, x_0), B) < \alpha$ for $t \ge t_0$. For any $\in > 0, \in \le \alpha$, we can determine a $\delta_1(\in)$ such that $V(t,x) < b(\delta)$ implies $V'(t,x) < \delta_1(\in)$.
Let $T(\in) = \frac{2a(\alpha)}{\delta_1(\in)}$. Suppose that for $t \in [t_0, t_0 + T]$ $V(t,x) \ge \delta_1$, $d(x_0, A) < \delta$, then we have

$$\begin{aligned} 0 &\le V(t_0 + T, x_u(t_0 + T, t_0 x_0)) \\ &= V(t_0, x_0) + V'(s, x_u(s, t_0, x_0))\, ds \\ &\le b(d(x_0, A)) - \delta_1 T \\ &= b(b^{-1}a(\alpha)) - \delta_1 T = a(\alpha) - 2a(\alpha) = -a(\alpha) < 0. \end{aligned}$$

Thus there exists a $t_1 \in [t_0, t_0 + T]$ such that $V(t,x) < \delta_1(\in)$. Hence $V'(t_1, x_u(t_1, t_0, x_0)) < a(\in)$. But by (2) and (2.8),

$$\begin{aligned} a(d(x_u(t, t_0, x_0), B)) &\le V(t, x_u(t, t_0, x_0)) \\ &= V(t_1, x_u(t_1, t_0, x_0)) < a(\in), \text{ for all } t \ge t_1. \end{aligned}$$

Thus for all $t \ge t_1$, hence $t \ge t_0 + T$, $d(x_u(t, t_0, x_0), B) < \in$, provided that $d(x_0, A) < \alpha$.
This completes the proof of the theorem. □

Theorem 4.5. *If there exists a function V such that*
(1) $V(t,x) = 0$ for $x \in A$, V continuous on ∂A, the boundary of A,
(2) V satisfies (2.8), and
(3)

$$\lim_{h \to 0^+} \sup \frac{1}{h}[V(t+h, x_u(t+h, t_0, x_0)) - V(t, x_u(t, t_0, x_0))]$$

$\le -c(V(t, x_u(t, t_0, x_0)))$ for each $u \in E$ and $t \ge t_0$, where $c \in K$.
Then the C.I. set B is asymptotically stable relative to A.

Proof. Due to the hypotheses (1), (2) and (3) we have stability of the C.I. set B relative to A from theorem (2.3).
Let $d(x_0, A) < \delta$, δ being chosen from the stability criterion, given $\in > 0$.
Now (3) shows that $\lim_{t \to \infty} V(t, x_u(t, t_0, x_0)) = V$, exists for $x_0 \in S(A, \delta)$. Now suppose that $V < 0$; then $V(t, x_u(t, t_0, x_0)) < V$ for all $t \ge t_0$. However this shows from (3) that

$$0 < a(V) \le V(t, x_u(t, t_0, x_0)) \le V(t_0, x_0) - c(V)(t - t_0)$$

which is impossible for sufficiently large t.
Thus $V \ge 0$. Then from (2.8) we have

$$\lim_{t \to \infty} d(x_u(t, t_0, x_0), B) = 0 \text{ for } x_0 \in S(A, \delta).$$

Hence the proof. □

Theorem 4.6. *Let a function V satisfying the hypotheses (1) and (2) of theorem (4.5) and (2.9) exist.*

Moreover, let (2)

$$\begin{aligned} \lim_{h \to 0^+} \sup \frac{1}{h}&[V(t+h, x_u(t+h, t_0, x_0)) - V(t_0, x_u(t, t_0, x_0))] \\ &\le -c(d(x_u(t, t_0, x_0), A)), \end{aligned}$$

for each $u \in E$ and $t \ge t_0$, where $c \in K$.
Then the C.I. set B is uniformly asymptotically stable relative to A.

Proof. Parallel to the proof of theorem (4.3). The hypothesis (3) guarantees that step (i) in the proof of theorem (4.3) is satisfied. □

Theorem 4.7. *The theorems (4.5) and (4.6) hold if there exists a function $V \in \gamma$ and the L.H.S. of (3) in the theorems are replaced by $D^+V(t,x)$, other conditions remaining the same.*

Proof. As $V \in \gamma$, V satisfies Lipschitz condition in x; the inequalities (3) follow from the above replacements. Hence the result. □

Note: The function V' in theorems (4.3), (4.4) can also be replaced by $D^+V(t,x)$, if $V \in \gamma$.
We prove the following converse theorem.

Theorem 4.8. *If the C.I. set B is uniformly asymptotically stable relative to A, then there exists a function V satisfying the hypotheses of theorem (4.1).*

Proof. For $(t, x) \in IXD$, define

$$V(t,x) = \inf_{u\in E} \sup_{T\geq 0} d(x_u(t+T,t,x),B)$$

Then from theorem (2.4) hypotheses of theorem (4.1) are verified except the part $\lim_{t\to\infty} V(t, x_u(t, t_0, x_0)) = 0$.
From the definition,

$$\begin{aligned} V(t,x_u(t,t_0,x_0)) &\leq \inf_{u\in E} \sup_{T\geq 0} d(x_u(t+T,t_0,x_0(t,t_0,x_0)),B) \\ &\leq \inf_{u\in E} \sup_{T\geq 0} d(x_u(t+T,t_0,x_0),B). \end{aligned}$$

As $t \to \infty$, $d(x_u(t+T, t_0, x_0), B) \to 0$.
Hence $V(t, x_u(t, t_0, x_0)) \to 0$, This completes the proof. □

Assume that the system is autonomous, so that f is independent of t and (2.2) is also satisfied.

Theorem 4.9. *Let there exist a function $V = V(x)$ such that*

(1) $a(d(x, B)) \leq V(x), V(x) = 0$ if $x \in A$, and
(2) $V'(x) \leq 0$ for $x \in M$, $V'(x) > 0$ for $x \notin M$
where M is the set of points not containing the entire trajectories of (2.1).
Then the C.I. set B is asymptotically stable relative to A.

Proof. Hypotheses (1) and (2) show that the set B is uniformly stable. Thus given $\in > 0$, there exists $\delta(\in) > 0$ such that $d(x_0, A) < \delta$ implies $d(x_u(t, t_0, x_0), B) < \in$, for $t \geq t_0$.
We show that

$$d(x_0, A) < \delta \text{ implies } \lim_{t\to\infty} V(x_u(t,t_0,x_0)) = 0.$$

Assume that this is <u>not</u> true. Then due to (2)

$$V(x_u(t,t_0,x_0)) \to V > 0.$$

As $x_u(t, t_0, x_0)$ is bounded, (B is compact), $x_u(t, t_0, x_0)$ has a limit point x. By continuity of V, $V(x) = V$.
Since the solution does not lie entirely in the set M, for some $T > t_0$ we have

$$V(x(T,t_0,x)) > V \tag{i}$$

As $x_u(t, t_0, x_0) \to x$, there exists a sequence: $t_k = t_0 + k\tau$, τ fixed, such that $x_u(t_k, t_0, x_0) \to x$.
The continuous dependence on initial conditions and continuity of V shows that there exists a number $T > t_0 \geq 0$ and $N > 0$ such that

$$V(x_u(T,t_0,x_u(t_0+k_i\tau,t_0,x_0))) > V \tag{ii}$$

From the uniqueness property and the group property of autonomous differential equations, we have,

$$\begin{aligned} &x_u(T,t_0,x_u(t_0+k_i\tau,t_0,x_0)) \\ &= x(T+k_i\tau,t_0+k_i\tau,x_u(t_0+k_i\tau,t_0,x_0)) \\ &= x_u(T+k_i\tau,t_0,x_0) \end{aligned} \tag{iii}$$

Substituting from (iii) in (i) we get $V(x(T + k_i\tau, t_0, x_0)) > V$ for $k_i > N$, which contradicts the hypothesis $V'(x) \leq 0$, unless $V = 0$.
Therefore

$$\lim_{t\to\infty} V(x_u(t,t_0,x_0)) = 0.$$

Hence the proof. □

We state the following comparison theorem for asymptotic stability of C.I. set B, without proof.

Theorem 4.10. *If there exists a function $V \in \gamma$, such that (3.3) holds with g having properties as in (3.1), then the trivial solution of (3.2) is asymptotically stable, implies the C.I. set B is asymptotically stable relative to A provided that (2.8) holds and $V(t,x) = 0$ for $x \in A$.*

Theorem 4.11. *Let there exist a function $V \in \gamma$, such that (1) (3.3) holds with g having the properties as in (3.1), (2) (2.8) and (2.9) holds, and (3) the trivial solution of the equation (3.2) is uniformly asymptotically stable. Then the C.I. set B is uniformly asymptotically stable.*

The following theorem gives necessary and sufficient conditions for exponential asymptotic stability of the C.I. set B relative to A for the system (2.1) satisfying (2.2).

Theorem 4.12. *The C.I. set B relative to A is exponentially asymptotically stable if and only if there exists a function $V \in \gamma$ such that*

(1) (2.8) and (2.9) hold, with $a(r) = r$ and

$$|b(r) - b(r_1)| < L|r - r_1|, L > 0$$

(2)

$$D^+V(t,x) \leq -\alpha V(t,x).$$

Proof. **(Sufficiency):** From hypothesis (2) we have

$$V(t,x_u(t,t_0,x_0)) \leq V(t_0,x_0)\exp[-\alpha(t-t_0)]$$

From (2.8) and (2.9), we know that

$$a(d(x_u(t,t_0,x_0),B)) \leq b(d(x_0,A))\exp[-\alpha(t-t_0)]$$

Therefore

$$d(x_u(t,t_0,x_0),B) \leq b(d(x_0,A))\exp[-\alpha(t-t_0)] \tag{i}$$

since $a(r) = r$ itself.
(Necessary): Define for $(t, x) \in IXD$,

$$V(t,x) = \inf_{0\leq T\leq t} b(d(x_u(T,t.x),A))\exp[-\alpha(t-T)]$$

From step (i) above, we have $V(t, x) \geq d(x, B)$ and $V(t, x) \leq b(d(x, A))$ verifying (1). Standard arguments show that V is continuous and as b satisfies Lipschitz condition in r, it can be shown that

$$|V(t,x) - V(t,y)| \leq L|x-y|\exp\int_0^t k(s)ds.\exp[-\alpha(t-t_1)] \tag{ii}$$

where $t_1 \in [0, t]$, so that

$$|V(t,x) - V(t,y)| < L|x-y|\exp\int_0^t k(s)ds.$$

Thus $V \in \gamma$. Now using usual arguments, we can prove the hypothesis (2). □

Note: If, in (2.2), k is constant and such that $k \geq \alpha$ then the Lipschitz constant in step (ii) is independent of t.

5. CONTROL SYSTEM AS A PERTURBED SYSTEM

Here the system (2.1) is considered as a perturbation of the system defined below, by letting $f(t,x,u) = g(t,x) + R(t,x,u)$. Consider the system –

$$x' = g(t,x) \tag{5.1}$$

and

$$x' = g(t,x) + R(t,x,u) = f(t,x,u) \tag{2.1}$$

Let $g \in C(IXD, R^n)$, $g(t,0) = 0$, and g satisfies the Lipschitz condition:

$$|g(t,x) - g(t,y)| \leq k(t)|x-y| \tag{5.2}$$

$R(t,o,u) = 0, u \in E, t \in I; R \in C(IXDXE, R^n)$ and for

$$(t,x,u) \in IXDXE, R(t,x,u) = \eta(t)\beta(d(x,B)) \tag{5.3}$$

where $\eta \in L'[0,\infty]$ and $\beta \in K$.
In order that we study the preservation of stability properties of the C.I. set B relative to A for the system (2.1), assuming these for the system (5.1), it is necessary that the Lyapunov function V should be determined, satisfying, for $x, y \in \bar{S}(B,\alpha)$

$$|V(t,x) - V(t,y)| \leq L|x-y|, L > 0 \tag{5.4}$$

where L depends on α only. In the converse results proved in sections 2.3 and 2.4 it is not possible to do so. Assuming some more conditions, we obtain, in the next two theorems a function that satisfies (5.4). We then use this function to derive some stability results for the system (2.1). The theorem (5.2) is a special case of theorem in (33).

Theorem 5.1. *Assume that trajectories $x(t,t_0,x_0)$ of (5.1) satisfy the estimate below –*

$$|d(x(t,t_0,x_0),B) - d(x(t,t_0,y_0),B)| \leq M(\alpha)|d(x_0,A) - d(y_0,A)| \quad \text{for } x_0, y_0 \in \bar{S}(B,\alpha). \tag{5.5}$$

Then there exists a function $V \in \gamma$ such that
(i) for

$$(t,x),(t,y) \in IX\bar{S}(B,\alpha)$$
$$|V(t,x) - V(t,y)| \leq M(\alpha)|x-y|$$

(ii)

$$d(x,B) \leq V(t,x) \leq M(\alpha)d(x,A),$$

and
(iii)

$$D^+V(t,x) \leq \tfrac{V}{t} + \tfrac{V}{x}g(t,x) \leq 0, (t,x) \in IXS(B,\alpha).$$

Proof. For $(t,x) \in IX\bar{S}(B,\alpha)$, define

$$V(t,x) = \sup_{T\geq 0} d(x(t+T,t,x),B).$$

The continuity of V follows from standard arguments (31). If $y_0 \in A$ and B is C.I. set relative to A, we have, $d(y_0,A) = 0$ and $d(x(t,t_0,y_0),B) = 0$.
Thus from (5.5) we get

$$d(x(t,t_0,x_0),B) \leq M(\alpha)d(x_0,A) \tag{i}$$

whenever $d(x,B)$

Hence from the definition of V, for $x \in \bar{S}(B,\alpha)$ $d(x,B) \leq V(t,x) \leq M(\alpha)d(x,A)$, verifying (ii).
Let $x, y \in \bar{S}(B,\alpha)$.
Then

$$\begin{aligned}|V(t,x) - V(t,y)| &\leq \sup_{T\geq 0} |d(x(t+T,t,x),B) - d(x(t+T,t,y),B)| \\ &\leq \sup_{T\geq 0} M(\alpha)|d(x,A) - d(y,A)|, \text{ from (5.5)} \\ &\leq M(\alpha)|x-y| \text{ verifying (i)}\end{aligned}$$

The property follows since V is non-increasing along the trajectories of (5.1) and V satisfies Lipschitz's condition. □

Definition 5.1. *If the trajectories of system (5.1) satisfy (5.5), then the C.I. set B relative to A is said to be extremely uniformly stable for the system (5.1).*

Extreme uniform stability of the C.I. set B relative to A for the system (5.1) implies uniform stability of the C.I. set B relative to A for the system (5.1).

Definition 5.2. *The C.I. set B is said to be strongly uniformly asymptotically stable, for the system (5.1) if (5.5) is satisfied and for each $\alpha > 0$, there exists numbers $M(\alpha) > 0$ and a function $c \in L$ such that*

$$d(x(t,t_0,x_0),B) \leq M(\alpha)d(x_0,A)c(t-t_0) \tag{5.6}$$

whenever $d(x_0,B) \leq \alpha$ and $t_1 \geq 0$ independent of $(t_0,x_0) \in IX\bar{S}(B,\alpha)$ such that

$$d(x(t_0+t_1,t_0,x_0),B) \geq d(x_0,A) \tag{5.7}$$

It is easily seen that (5.6) implies uniform asymptotic stability of the C.I. set B relative to A for the system (5.1).

Theorem 5.2. *If the trajectories of (5.1) satisfy (5.5), (5.6) and (5.7), then for $(t,x) \in IXD$, there exists a function V such that for $(t,x) \in \bar{S}(B,\alpha)$*

(i) $|V(t,x) - V(t,y)| \leq M_1(\alpha)|x-y|$
(ii) $d(x,B) \leq V(t,x) \leq M_1(\alpha)d(x,A)$, and
(iii) $D^+V(t,x) \leq -\frac{1}{2}d(x,A)$.

Proof. Define

$$V_1 = \sup_{T\geq 0} d(x(t+T,t,x),B).$$

Let

$$V_2(t,x) = \int_{t+t_1}^{t+t_1+T} d(x(s,t,x),B)ds.$$

From definition of $V_2, V_2(t,x) \geq 0$.
Due to (5.6),

$$V_2(t,x) \leq \int_{t+t_1}^{t+t_1+T} M(\alpha)d(x,A)ds = M(\alpha)d(x,A)T.$$

Thus

$$0 \leq V_2(t,x) \leq M(\alpha)d(x,A)T \tag{i}$$

Now

$$|V_2(t,x)-V_2(t,y)| = \left| \int_{t+t_1}^{t+t_1+T} d(x(s,t,x),B)-d(x(s,t,y),B)\ ds \right|$$
$$\leq \int_{t+t_1}^{t+t_1+T} M(\alpha)|d(x,A)-d(y,A)|\ ds$$
$$\leq M(\alpha)|d(x,A)-d(y,A)|T$$
$$\leq M(\alpha)|x-y|T \quad \text{(ii)}$$

$$V_2(t,x(t,t_0,x_0)) = \int_{t+t_1}^{t+t_1+T} d(x(s,t,x(t,t_0,x(t,t_0,x_0))),B)ds$$
$$= \int_{t+t_1}^{t+t_1+T} d(x(s,t_0,x_0),B)ds$$

$$V_2(t+h,x(t+h,t_0,x_0)) = \int_{t+h+t_1}^{t+h+t_1+T} d(x(s,t_0,x_0),B)ds$$

due to the uniqueness of the trajectories of (5.1).
Hence

$$\lim_{h\to 0^+} \sup \frac{1}{h}[V_2(t+h,x(t+h,t_0,x_0))-V_2(t,x(t,t_0,x_0))]$$
$$= \lim_{h\to 0^+} \sup \frac{1}{h}\left[\int_{t+h+t_1}^{t+h+t_1+T} d(x(s,t_0,x_0),B)ds - \int_{t+t_1}^{t+t_1+T} d(x(s,t_0,x_0),B)ds\right]$$
$$= \lim_{h\to 0^+} \sup \frac{1}{h}\left[\int_{t+t_1+T}^{t+h+t_1+T} d(x(s,t_0,x_0),B)ds - \int_{t+t_1}^{t+t_1+h} d(x(s,t_0,x_0),B)ds\right]$$

$$\leq d(x(t+t_1+T,t_0,x_0),B)-d(x(t+t_1,t_0,x_0),B)$$
$$\leq d(x(t+t_1+T,t,x),B)-d(x(t+t_1,t,x),B)$$
$$\leq d(x,A)M(\alpha)c(t_1+T)-d(x,A),$$

due to (5.6) and (5.7).
Using Lipschitz condition for V_2

$$D^+V_2(t,x) \leq M(\alpha)d(x,A)c(t_1+T)-d(x,A)$$

Now choose T such that $c(t_1+T) \leq \frac{1}{2M(\alpha)}$; T is independent of $t_0, x_0 \in S(B,\alpha)$ as t_1 is also independent of t_0, $x_0 \in S(B,\alpha)$.
Then

$$D^+V_2(t,x) \leq -\frac{1}{2}d(x,A) \quad \text{(iii)}$$

Define $V(t,x) = V_1(t,x)+V_2(t,x)$.
From the properties of V_1 in theorem (5.1), steps (i), (ii) and (iii) with $M_1(d) = (1+T)M(\alpha)$, it is seen that the properties (i), (ii) and (iii) are satisfied by V. ☐

Theorem 5.3. *Let the C.I. set B relative to A for the system (5.1) be extremely uniformly stable and*

$$R(t,x,u) = \alpha(t)a(d(x,B)), a \in K \quad (5.8)$$

whenever $x \in \bar{S}(B,\alpha)$, $u \in E$ and $\alpha \in C(I,I)$ is integrable. Then the set B is also conditionally invariant relative to A for the system (2.1) and it is uniformly stable, provided that

$$\int_{t_0}^{\infty} \frac{dr}{a(r)} \neq \infty, t_0 > 0.$$

Proof. From theorem (5.1) there exists a function V satisfying the hypotheses of theorem (5.1) due to the extreme uniform stability of the C.I. set B relative to A for the system (5.1).
Then for sufficiently small h,

$$\lim_{h\to 0^+} \sup \frac{1}{h}[V(t+h,x+hg(t,x)+hR(t,x,u))-V(t,x)]$$
$$= \lim_{h\to 0^+} \sup \frac{1}{h}[\{V(t+h,x+hg(t,x))-V(t,x)\}+$$
$$\{V(t+h,x+hg(t,x)+hR(t,x,u))-V(t+h,x+hg(t,x))\}]$$
$$\leq \lim_{h\to 0^+} \sup \frac{1}{h}[V(t+h,x+hg(t,x))-V(t,x)]+\frac{LhR(t,x,u)}{h}$$
$$\leq 0+LR(t,x,u) \leq L\alpha(t)a(d(x,B)) \leq L\alpha(t)a(V(t,x)),$$

from the property of V, $a \in K$ and $d(x,B) \leq V(t,x)$.
The trivial solution of the scalar equation $r' = L\alpha(t)a(r)$, is uniformly stable due to the assumption $\int_{t_0}^{\infty} \frac{dr}{a(r)} \neq \infty$. Then by comparison theorem (3.2), it follows that set B is uniformly stable relative to A for the system (2.1). The fact that B is also C.I. relative to A, for the system (2.1) now follows from the uniform stability of the set B relative to A. ☐

Theorem 5.4. *Let the C.I. set B relative to A for the system (5.1) be strongly uniformly stable. Then $\lim_{t_0 \geq 0} -t+L\int_{t_0}^{t_0+t} \eta(s)ds \to -\infty$ as $t\to\infty$, $L>0$, implies that C.I. set B is uniformly asymptotically stable relative to the system (2.1).*

Proof. From (5.4), under the hypothesis of strong uniform asymptotic stability of the C.I. set B relative to A for the system (5.1), we have the existence of a function V satisfying conditions (i), (ii) and (iii) of theorem (5.2). Then proceeding as in theorem (5.3), we show that

$$\lim_{h\to 0^+} \frac{1}{h}[V(t+h,x+hf(t,x,u))-V(t,x)]$$
$$= \frac{1}{2M_1(\alpha)}[L_1\eta(t)-1]V(t,x), \text{ where } L_1 = 2LM_1().$$

The trivial solution of the corresponding scalar equation:

$$r' = \frac{1}{2M_1(\alpha)}[L_1\eta(t)-1]r,$$

is uniformly asymptotically stable, due to the hypothesis on $\eta(t)$. Thus from the theorem (4.12), the uniform asymptotic stability of C.I. set B relative to A for the system (2.1) follows. ☐

6. OPTIMISATION PROBLEM

In the system (2.1) let us assume that the controls $u = u(t,x) \in C(IXD, E)$. We study an optimisation problem where

$$J = \int_{t_0}^{\infty} W(t, x(t)), u(t, x(t))dt,$$

measures the performance of u along the trajectories $x(t)$ of (2.1), for a fixed $u \in C(IXD, E)$. $W(t,x,u)$ is an integrable function from $IXDXU$ into E where U is the space of continuous functions from IXD into E.
We consider the optimal control problem that generalises the corresponding problem discussed in (56).
To determine a function $u = u(t,x) \in U$ such that the C.I. set B is asymptotically stable for (2.1) and for any other $u \in u$ and $t_0 \geq 0$,

$$\int_{t_0}^{\infty} W(t, x^0(t), u^0(t, x^0(t)))dt \geq \int_{t_0}^{\infty} W(t, x(t), u(t, x(t)))dt \tag{6.1}$$

We have

Theorem 6.1. *Suppose that*

(1) There exists a function $V = V(t,x) \in \gamma'$ such that (2.8) and (2.9) are satisfied,
(2) $V'(t,x) + W(t,x,u) = 0, W(t,x,u) \geq c(d(x,A))$ for $(t,x,u) \in IXDXU$, where $c \in K$.
(3) there exists a control $u^0 \in U$ such that

$$\frac{V}{t} + \frac{V}{|x|} f(t, x, u^0(t,x)) + W(t, x, u^0(t,x)) = 0$$

Then the function u^0 is such that for the system (2.1), with this u^0, C.I. set B is asymptotically stable relative to A and (6.1) is satisfied with

$$\int_{t_0}^{\infty} W(t, x^0(t), u^0(t, x^0(t)))dt = \max \int_{t_0}^{\infty} W(t, x(t), u(t, x(t)))dt = V(t_0, x^0(t)) \tag{6.2}$$

Proof. With $u^0(t,x)$ in (2.1), hypothesis (2) shows that

$$V'(t,x) = -W(t,x,u(t,x)) \leq -c(d(x,A)) \tag{i}$$

With hypothesis (1) and step (i) above, we have the uniform asymptotic stability of C.I. set B relative to A by theorem (4.11). Integrating (i) along the trajectories of (2.1) with $u = u^0(t,x)$,

$$V(t, x^0(t)) - V(t_0, x_0) = -\int_{t_0}^{t} W(s, x(s), u^0(s, x(s)))ds$$

As $t \to \infty$, we have

$$V(t, x^0(t, t_0, x_0)) = 0.$$

Hence

$$V(t_0, x_0) = \int_{t_0}^{\infty} W(s, x(s), u^0(s, x(s)))ds \tag{ii}$$

Now for any other control $u \in U$, we have from (i), with trajectory x_u through the same point (t_0, x_0),

$$V(t, x_u(t, t_0, x_0)) - V(t_0, x_0) = -\int_{t}^{\infty} W(s, x_u(s), u(s, x_u(s)))ds$$

So that

$$V(t_0, x_0) \geq \int_{t_0}^{\infty} W(s, x_u(s), u(s, x_u(s)))ds \tag{iii}$$

Hence (ii) and (iii) verify - (6.2). □

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