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Stability of General Dynamical Systems

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ABSTRACT

In this paper comparison techniques are used to obtain sufficient condition for stability of an invariant set. Here sufficient conditions involving the stability of scalar differential equations and converse theorems for a reversible dynamical system proved and Two converse theorems for existence of a vector Lyapunov function in reversible dynamical system are proved.

Concept of conditional invariancy is introduced. Sufficient condition for stability of conditional invariant are proved. Here introduced notion of conditional stability of a compact set.

Keywords

Lyapunov function, Equistrict stability, Equistrict asymtotic stable, Reversible Dynamical System

1. INTRODUCTION

In this paper the author considers a general dynamical system (henceforth abbreviated as g.d.s.) in a locally compact seperable metric space E as a two-parameter family of transformations F $(t, t_0, .)$ of E into $A(E)$, the set of all subsets of E and obtains the existence of a V-function defined over $[0, \infty)XA(E)$, if reversibility is assumed. For this purpose the concepts of strict and asymptotic stability of an invariant set with respect to a g.d.s. are introduced and sufficient conditions in terms of V-function are obtained. It is to be noted that, eventhough such a definition of a V-function is most natural in view of the fact that a g.d.s. is defined in E into $A(B)$, in the literature (1,5,7,9,10) it is defined on $[0, \infty)XE$. Then a V-function in its natural setting is got.

In section (2), preliminaries are dealt with. Defining a g.d.s., concepts of reversibility of a g.d.s., stability of an invariant set A with respect to a g.d.s. and strict and asymptotic stability of sets with respect to a g.d.s. are introduced, in terms of Hausdorff metric on $A(E)$. Certain classes of monotonic functions introduced by W. Hahn (6) are recalled for future use.

In section (3), comparison techniques are used to obtain sufficient conditions for stability of an invariant set A with respect to a g.d.s. in terms of V-functions defined on $[0, \infty)XA(E)$.

Section (4), contains sufficient conditions, involving the stability of scalar differential equations and the existence of a V-function, for the stability of set A with respect to a g.d.s.

Section (5), present converse theorems (on the existence of V-functions) for a reversible dynamical system.

In section (6), concept of conditional invariancy (8) of a set B with respect to a set A in a g.d.s. and definitions of stability of a conditionally invariant set B with respect to a set A are introduced. Sufficient conditions for stability of conditional invariant set B with respect to a set A in a g.d.s. in terms of V-functions are proved and converse of some theorems in some form or the other for a reversible system are attempted and their relations to some theorems of section (5) are traced.

Section (7), introduces to the notion of conditional stability (7) of a compact set A with respect to a g.d.s. and a general comparison technique involving a vector Lyapunov function and the notion of quasi-monotonicity (which is developed in this section) is used to prove the sufficiency condition for conditional stability of set A for set M.

Section (8) contains two converse theorems for the existence of a vector Lyapunov function in a reversible dynamical system.

2. PRELIMINARIES

General Dynamical System (GDS)

Let I denote the half-line: $0 \leq t < \infty$ and $R_+ = [0, \infty)$. Let E be a locally compact seperable metric space. Consider a two-parameter family of transformations $F(t, t_0, p)$ of E into $A(E)$, the set of all subsets of E satisfying the following properties -

(i) For each $p_0 \in E$ and $t_0 \in I$, there is defined a set $F(t, t_0, p_0) \in A(E)$ for all $t \geq t_0$.

(ii) $F(t_0, t_0, p_0) = \{p_0\}$ and

(iii) For any $p_1 \in F(t_1, t_0, p_0)$, there is defined a set $F(t, t_1, p_1)$ such that

$\cup F(t, t_1, p_1) = F(t, t_0, p_0)$, for all $t \geq t_1 \geq t_0$, $p_1 \in F(t_1, t_0, p_0)$

For a fixed $p_0 \in E, F(t, t_0, p_0)$ is called a **motion**, while the set defined in (i) above is called the **trajectory** of the motion.

Definition 2.1. *The family of transformations $F(t, t_0, .)$ described thro' the properties (i), (ii) and (iii) above, is called a General Dynamical System (GDS) in E.*

The metric in $A(E)$: Let $d(p, q)$ denote the metric in $E, p, q \in E$. Let $d(A, B)$ where $A, B \in A(B)$ denote the Hausdorff distance between two sets A and B.

Then $d(A, B)$ is defined by

$$d(A, B) = \max\{d^*(A, B), d^*(B, A)\}$$

where $d^*(A, B) = \sup\{d(a, B), a \in A\}$

and $d(a, B) = \inf\{d(a, b), b \in B\}$

It is to be noted that in general $d^*(A, B) \neq d^*(B, A)$.

For any set $A \in A(E)$, the neighbourhoods $S(A, \in)$ and $\bar{S}(A, \in)$ are defined by

$$S(A, \in) = \{XA(E) : d(X, A) < \in\}$$
$$\bar{S}(A, \in) = \{XA(E) : d(X, A) \leq \in\},$$

respectively.

In what follows, we shall assume that the **flow** $F(t, t_0, X_0), X_0 \in A(E)$ is Hausdorff continuous in the triplet (t, t_0, X_0) where for any $X_0 \in A(E)$, we denote

$$F(t, t_0, X_0) = \bigcup_{p_0 \in X_0} F(t, t_0, X_0)$$

In these notations the properties (ii) and (iii) of a g.d.s. in E take the following form: (ii)* $F(t_0, t_0, X_0) = X_0, t_0 \in I$

(iii)* $F(t, t_1, F(t_1, t_0, X_0)) = F(t, t_0, X_0)$ for all $t \geq t_1 \geq t_0$.

Definition 2.2. *A G.D.S. in E in which $X = F(t, t_0, X_0)$ **iff** $X_0 = F(t_0, t, X)$, for $X_0, X \in A(E)$ and $t, t_0 \in I$, is called a **reversible dynamical system** (r.d.s.).*

Consequently,

$$F(t_0, t_1, F(t_1, t_0, X_0)) = F(t_0, t_0, X_0) = X_0.$$

Thus for a r.d.s.

$$F(t_0, t, F(t, t_0, X_0)) = X_0 \text{ for all } t \geq t_0.$$

In what follows X is compact in E.

Definition 2.3. *A set $X \in A(E)$ is called (Positively) invariant with respect to a g.d.s in E if*

$$F(t, t_0, X) \subset X \text{ for all } t \geq t_0.$$

Notation: Let $C(D, R)$ denote the class of all continuous functions $f : D \to R$.

Monotonic functions (due to W.Hahn (6))

Definition 2.4.

(i) *$a(r)$ is said to belong to the class K (whence we write $a \in K$) if $a \in C(I, R_+)$, $a(0) = 0$ and a is strictly monotonic increasing in r with $\lim_{r\to\infty} a(r) = \infty$.*

(ii) *$a(t, r)$ is said to belong to the class K^*(i.e.$a \in K^*$) if $a \in C(IXR_+, R_+)$ and $a \in K$ for each $t \in I$.*

(iii) *$b(s)$ is said to belong to the class L (i.e. $b \in L$) if $b \in C(I, R_+)$, b is strictly monotonic decreasing in s and $\lim_{s\to\infty} b(s) = 0$.*

(iv) *$b(t, s)$ is said to belong to the class L^* (i.e. $b \in L^*$) if $b \in C(I\ X\ R_+, R_+)$ and $b \in L$ for each $t \in I$.*

The following results on monotonic functions will be useful in the sequel:

(i) If $a = a(r) \in K$, then a^{-1} exists and $a^{-1} \in K$.

(ii) If $a_1 = a_1(t, r) \in K^*$, $a_2 = a_2(t, r) \in K^*$, then $a_3 = a_1(t, a_2(t, r)) \in K^*$.

(iii) If $a \in K^*$ and $b \in K$, the $b^{-1}a \in K^*$.

(iv) If $a \in K$ and $b \in L$, then $a^{-1}b \in L$.

(v) If $a \in K$ and $b \in L^*$, then $a^{-1}b \in L^*$.

(vi) If $a, b \in K$, then $ab \in K$.

In the following, $F(t, t_o, X_o)$ is assumed to be $H-$ continuous in the triplet (t, t_o, X_o) and the set A is compact in E.

Stability Definitions – 2.5: With respect to a g.d.s, the set A is said to be

$\mathbf{S_1}$: **Equi-stable**, if there exists $a \in K^*$ such that

$$d(F(t, T_o, X_o), A) \leq a(t_o, d(X_o, A)) \quad (2.1)$$

$\mathbf{S_2}$: **Equi-strict stable**, if there exists $a_1, a_2 \in K^*$ such that

$$a_1(t_o, d(X_o, A)) \leq d(F(t, t_o, X_o), A) \leq a_2(t_o d(X_o, A)) \quad (2.2)$$

$\mathbf{S_3}$: **Uniform stable**, if there exists $a \in K$ such that

$$d(F(t, t_o, X_o), A) \leq a(d(X_o, A)) \quad (2.3)$$

$\mathbf{S_4}$: **Uniform strict stable**, if there exists $a_1, a_2 \in K$ such that

$$a_1(d(X_o, A)) \leq d(F(t, t_o, X_o), A) \leq a_2(d(X_o, A)) \quad (2.4)$$

$\mathbf{S_5}$: **Equi-asymptotic stable**, if there exists $a \in K^*$ and $b \in L^*$ such that

$$d(F(t, t_o, X_o), A) \leq a(t_o, d(X_o, A))b(t_o, t - t_o) \quad (2.5)$$

$\mathbf{S_6}$: **Equi-strict asymptotic stable**, if there exists $a_1, a_2 \in K^*$ and $b_1, b_2 \in L^*$ such that

$$a_1(t_o, d(K_o, A))b_1(t_o, t - t_o) \leq d(F(t, t_o, X_o), A)$$
$$\leq a_2(t_o, d(X_o, A))b_2(t_o, t - t_o) \quad (2.6)$$

$\mathbf{S_7}$: **Uniform asymptotic stable**, if there exists $a \in K$ and $b \in L$ such that

$$d(F(t, t_o, X_o), A) \leq a(d(X_o, A))b(t - t_o) \quad (2.7)$$

$\mathbf{S_8}$: **Uniform strict asymptotic stable**, if there exists $a_1, a_2 \in K$ and $b_1, b_2 \in L$ such that

$$a_1(d(X_o, A))b_1(t - t_o) \leq d(F(t, t_o, X_o), A)$$
$$\leq a_2(d(X_o, A))b_2(t - t_o) \quad (2.8)$$

Note:

(1) It is assumed that the inequalities (2.1) to (2.8) hold for all $t \geq t_o, t_o \in I$ and $X_o \subset S(A, r)$ for some $r > 0$.

(2) (i) S_3 implies S_1 and S_4 implies S_2.

(ii) S_5 implies S_1 and S_7 implies S_3.

However, strict stability which corresponds to stability in some tube-like domain (9, 31) denies asymptotic stability.

3. COMPARISON THEOREMS

Theorem 3.1. *Let $V(t,X) \in C(I\ X\ A(E), R_+)$ be an auxiliary function (called a **LYAPUNOV FUNCTION**) and $X = X(t) = F(t,t_o,X_o) \in A(E)$ and $X_o \in A(E)$.*

Let

$$D^+V(t,X) = \lim_{h\to 0^+} \sup \left[\frac{1}{h}\left\{V(t+h, X(t+h)) - V(t,X)\right\}\right] \tag{3.1}$$

exist. Let $r(t,t_0,r_0)$ be the **maximal** solution of the scalar differential equation –

$$\left.\begin{aligned} r' &= g(t,r)('= d/dt) \\ r(t_0) &= r_0 \end{aligned}\right\} \tag{3.2}$$

and

$$D^+V(t,X) \leq g(t, V(t,X)) \tag{3.3}$$

for all $t \geq t_0$, $t_0 \in I$ and $g \in C(I\ X\ R_+, R_+)$.

Then

$$\left.\begin{aligned} V(t_0,X_0) &\leq r_0 \\ \text{implies} \qquad V(t,X) &\leq r(t,t_0,r_0) \end{aligned}\right\} \tag{3.4}$$

for all $t \geq t_0$.

The theorem asserts that the Lyapunov function V can be majorised by the maximal solution of the scalar differential equation (3.2).

Corollary 3.1. *If, in* (3.2)*, $g \equiv 0$, then we get*

$$V(t,X) \leq V(t_0,X_0) \tag{3.5}$$

Theorem 3.2. *With the notation as in theorem (3.1), let V exist and let*

$$D^-V(t,X) = \lim_{h\to 0^+} \inf \left[\frac{1}{h}\left\{V(t+h, X(t+h)) - V(t,X)\right\}\right] \tag{3.6}$$

Let $u(t,t_0,u_0)$ be the **minimal** solution of the scalar differential equation–

$$\left.\begin{aligned} u' &= h(t,u)('= d/dt) \\ u(t_0) &= u_0 \end{aligned}\right\} \tag{3.7}$$

and

$$D^-V(t,X) \geq h(t, V(t,X)) \tag{3.8}$$

for all $t \geq t_0$, $t_0 \in I$ and $h \in C(I\ X\ R_+, R_+)$.

Then

$$\left.\begin{aligned} V(t_0,X_0) &\geq u_0 \\ \text{implies} \qquad V(t,X) &\geq u(t,t_0,u_0) \end{aligned}\right\} \tag{3.9}$$

for all $t \geq t_0$.

Corollary 3.2. *If, in* (3.7)*, $h \equiv 0$, then we get*

$$V(t,X) \geq V(t_0,X_0). \tag{3.10}$$

Theorems on the stability of a set A with respect to a g.d.s.

Theorem 3.3. *Let $V(t,X) \in C(I\ X\ A(E), R_+)$ exist such that*

1. *for all $(t,X) \in IX\ A(E)$,*

$$b(d(X,A)) \leq V(t,X) \leq a(t, d(X,A)) \tag{3.11}$$

where $a \in K^*$ and $b \in K$, and

2. *the inequality* (3.3) *hold with $g \equiv 0$.*

Then the set A is equistable with respect to the g.d.s.

Proof. By (3.5) and due to (2)

$$V(t_0,X_0) \geq V(t,X) = V(t, F(t,t_0,X_0)) \geq b(F(t,t_0,X_0), A)$$

Hence by (3.11),

$$\begin{aligned} b(d(F(t,t_0,X_0),A)) &\leq V(t_0,X_0) \leq a(t_0, d(X_0,A)) \\ \text{or} \qquad d(F(t,t_0,X_0),A) &\leq b^{-1}a(t_0, d(X_0,A)) \\ &= c(t_0, d(X_0,A)) \\ \text{where} \qquad c = b^{-1}a \in K^*, &\quad \text{for all } t \geq t_0. \end{aligned}$$

Hence A is equistable with respect to the g.d.s. □

Theorem 3.4. *Let there exist functions*

$$V_1, V_2 \in C(I\ X\ A(E), R_+) \text{ for } (t,X) \in I\ X\ A(E).$$

Further let V_1 satisfy the hypotheses of theorem (3.3) while V_2 satisfies–

(1)

$$b_1(t, d(X,A)) \leq V_2(t,X) \leq a_1(d(S,A)) \tag{3.12}$$

where $a_1 \in K$ and $b_1 \in K^*$ and

(2) (3.8) hold with $h \equiv 0$ and V_2 replacing V.

Then the set A is equi-strict stable with respect to a g.d.s.

Proof. As the conditions of theorems (3.1) and (3.2) hold with both g and h identically vanishing, (3.5) and (3.10) with V replaced by V_1 and V_2 respectively hold.

By (3.11),

$$\begin{aligned} V_1(t_0,X_0) &\leq a(t_0, d(X_0,A)) \\ \text{and} \qquad V_1(t, F(t,t_0,X_0)) &\geq b(d(F(t,t_0,X_0),A)) \end{aligned}$$

so that

$$\begin{aligned} b(d(F(t,t_0,X_0),A)) &\leq V_1(t, F(t,t_0,X_0)) \\ &\leq V_1(t_0,X_0) \\ &\leq a(t_0, d(X_0,A)) \end{aligned}$$

Thus

$$d(F(t,t_0,X_0),A) \leq b^{-1}a(t_0, d(X_0,A)) = c(t_0, d(X_0,A)) \quad \text{(i)}$$

where $c = b^{-1}a \in K^*$.

Again, by (3.12)

$$\begin{aligned} V_2(t_0,X_0) &\geq b_1(t_0, d(X_0,A)) \\ \text{and} \quad V_2(t, F(t,t_0,X_0)) &\leq a_1(d(F(t,t_0,X_0),A)). \end{aligned}$$

But $V_2(t, F(t,t_0,X_0)) \geq V_2(t_0,X_0)$, because of hypothesis (2).

Hence

$$\begin{aligned} a_1(d(F(t,t_0,X_0),A)) &\geq V_2(t,F(t,t_0,X_0)) \\ &\geq V_2(t_0,X_0) \\ &\geq b_1(t_0,d(X_0,A)) \\ \text{or} \quad d(F(t,t_0,X_0),A) &\geq a_1^{-1}b_1(t_0,d(X_0,A)) \\ &= c_1(t_0,d(X_0,A)) \qquad \text{(ii)} \end{aligned}$$

where $c_1 \in K^*$. □

Steps (i) and (ii) together imply equi–strict stability of A with respect to g.d.s.

Theorem 3.5. *Let the hypotheses of theorem (3.3) hold with*

$$b(d(X,A)) \leq V(t,X) \leq a(d(X,A)) \tag{3.13}$$

in place of (3.11)*, where $a, b \in K$. Then the set A is uniformly stable with respect to the g.d.s.*

Proof. The proof follows on the same lines as the proof of theorem (3.3) except that $a \in K$. □

Theorem 3.6. *Let there exist functions $V_1, V_2 \in C(IXA(E), R_+)$ for all $(t,X) \in IXA(E)$, and*

$$b_i(d(X,A)) \leq V_i(t,X) \leq a_i(d(X,A)) \tag{3.14}$$

$a_i, b_i \in X$, $(i = 1, 2)$, V_i satisfying the conditions (3.3) *with $g \equiv 0$ and* (3.8) *with $\equiv 0$ respectively.*

Then A is uniformly strictly stable with respect to the g.d.s.

Proof. Similar to that for theorem (3.4). □

4. SUFFICIENCY CONDITIONS FOR STABILITY

Results of asymptotic stability of sets can be obtained by using comparison techniques. For this purpose, we state the following definitions for the stability of solution and the properities of the solution of the comparison equations (3.2) and (3.7) respectively.

Definition 4.1. *The trivial solution of* (3.2) *is said to be*

S_1^*: **Equistable**, if exists a function $a \in K^*$ such that

$$r(t,t_0,r_0) \leq a(t_0,r_0) \tag{4.1}$$

for all $t \geq t_0, r(t,t_0,r_0)$ being any solution of (3.2) and the inequality (4.1) holding for $r_0 \leq p, p > 0$.

S_3^*: **Uniformly stable**, if there exists a function $a \in K$ such that

$$r(t,t_0,r_0) \leq a(r_0) \tag{4.2}$$

for all $t \geq t_0$, $r(t,t_0,r_0)$ being any solution of (3.2), the inequality (4.2) holding for $r_0 \leq p, p > 0$.

S_5^*: **Equi-asymptotic stable**, if there exist functions $a \in K^*$ and $b \in L^*$ and a number $p > 0$ such that

$$r(t,t_0,r_0) \leq a(t_0,r_0)b(t_0,t-t_0) \tag{4.3}$$

for all $t \geq t_0$, $r(t,t_0,r_0)$ being any solution of (3.2), the inequality (4.3) holding for $r_0 \leq p$.

S_7^*: **Uniform asymptotic stable**, if there exist functions $a \in K$ and $b \in L$ and a number $p > 0$ such that

$$r(t,t_0,r_0) \leq a(r_0)b(t-t_0) \tag{4.4}$$

for all $t \geq t_0$, $r(t,t_0,r_0)$ being any solution of (3.2), the inequality (4.4) holding for $r_0 \leq p$.

Note:

These definitions correspond to S_1, S_3, S_5 and S_7 of definitions (2.5). To prove strict results we require the following properties of the solution of the equation (3.7).

Properties 4.1.

S_2^* : *There exists a function $a \in K^*$ such that for any solution $u(t,t_0,u_0)$ of* (3.7) *with*

$$u_0 \leq q, q > 0, u(t,t_0,u_0) \geq a(t_0,u_0), \textit{ for all } t \geq t_0 \tag{4.5}$$

S_4^* : *There exists a function $a \in K$ such that for any solution $u(t,t_0,u_0)$ of* (3.7) *with*

$$u_0 \leq q, q > 0, u(t,t_0,u_0) \geq a(u_0) \textit{ for all } t \geq t_0 \tag{4.6}$$

S_6^* : *There exists a functions $a \in K^*$ and $b \in L^*$ such that for any solution $u(t,t_0,u_0)$ of* (3.7) *with*

$$u_0 \leq q, q > 0, u(t,t_0,u_0) \geq a(t_0,u_0), b(t_0,t-t_0) \textit{ for all } t \geq t_0 \tag{4.7}$$

S_8^* : *There exists a functions $a \in K$ and $b \in L$ such that for any solution $u(t,t_0,u_0)$ of* (3.7) *with*

$$u_0 \leq q, q > 0, u(t,t_0,u_0) \geq a(u_0)b(t-t_0) \textit{ for all } t \geq t_0 \tag{4.8}$$

Note: *S_2^*, S_4^*, S_6^* and S_8^** ***do not*** *reflect the properties corresponding to S_2, S_4, S_6 and S_8 of definitions* (2.5).

Theorem 4.1. *Assume the existence of a function $V \in C(IXA(E), R_+)$ satisfying–*

(1) *the hypothesis (1) of theorem (3.3) and*

(2) *the inequality* (3.3).

Then the equistability of the trivial solution of (3.2) implies the equistability of the set A with respect to the g.d.s.

Proof. Given $t_0 \in I$, since the trivial solution of (3.2) is equistable, there exists $a_1 \in K^*$ and a positive number p such that

$$r_0 \leq p \tag{i}$$

implies $\quad r(t,t_0,r_0) \leq a_1(t_0,r_0)$ for all $t \geq t_0 \qquad$ (ii)

$a_1 \in K^*$, where $r(t,t_0,r_0)$ is any solution of (3.2) through (t_0,r_0).

Due to the properties of the function a in the inequality (3.11)

(viz., $b(d(X,A)) \leq V(t,X) \leq a(t,d(X,A)), a \in K^*$ and $b \in K$)

there exists a number $p_1 \equiv p_1(t_0,p) > 0$ such that $d(X_0,A) \leq p_1$ and $a(t_0,d(X_0,A)) \leq p$ (iii), hold simultaneously. Choosing $V(t_0,X_0) \leq a(t_0,d(X_0,A)) = r_0$ and letting $d(X_0,A) \leq p_1$, step (iii) above implies the verification of step (i) so that step (ii) holds.

The choice of r_0 and the theorem (3.1) show that

$$\begin{aligned} V(t,F(t,t_0,X_0)) &\leq r_{\max}(t,t_0,r_0) \\ &\leq r_{\max}(t,t_0,a(t_0,d(X_0,A))) \end{aligned}$$

so that

$$\begin{aligned} b(d(F(t,t_0,X_0),A)) &\leq V(t,F(t,t_0,X_0)) \\ &\leq r_{\max}(t,t_0,a(t_0,d(X_0,A))) \\ &\leq a_1(t_0,a(t_0,d(X_0,A))) \end{aligned}$$

implying $d(F(t,t_0,X_0),A) \leq b^{-1}a_1(t_0,a(t_0,d(X_0,A)))$ putting $a_2 = b^{-1}a_1 \in K^*$, $d(F(t,t_0,X_0),A) \leq a_2(t_0,a(t_0,d(X_0,A))) = a_3(t_0,d(X_0,a))$ where $a_3 \in K^*$.

Hence A is equistable with respect to the g.d.s. □

Theorem 4.2. *Assuming the hypotheses of theorem (4.1) with* (3.13) *replacing* (3.11)*, the uniform stability of the trivial solution of* (3.2) *implies the uniform stability of the set A with respect to the g.d.s.*

Proof. The trivial solution of (3.2) being uniformly stable, there exists $a_1 \in K$ and a positive number $p > 0$ such that $r_0 \leq p$

(i) implies that $r(t,t_0,r_0) \leq a_1(r_0)$

(ii) for all $t \geq t_0$, $r(t,t_0,r_0)$ being any solution of (3.2) through (t_0,r_0).

Due to the properties of function a in (3.13)

$$(\text{viz., } b(d(X,A)) \leq V(t,X) \leq a(d(X,A)), a,b \in K)$$

there exists a number $p_1 = p_1(p) > 0$ such that $d(X_0,A) \leq p_1$ and $a(d(X,A)) \leq p$ (iii) hold simultaneously. Choosing $V(t_0,X_0) \leq a(d(X_0,A)) = r_0$ and letting $d(X_0.A) \leq p_1$. (iii) implies the verification of (i) so that (ii) holds.

The choice of r_0 and the theorem (3.1) show that

$$\begin{aligned} V(t,F(t,t_0,X_0)) &\leq r_{\max}(t,t_0,r_0) \\ &\leq a_1(a(d(X_0,A))), \text{ so that} \\ b(d(F(b,t_0,X_0),A)) &\leq V(t,F(t,t_0,X_0)) \\ &\leq a_1(a(d(X_0,A))) \\ &= a_2(d(X_0,A)) \text{ where } a_2 = a_1 a \in K \end{aligned}$$

implying $d(F(t,t_0,X_0),A)b^{-1}a_2(d(X_0,A)) = a_3(d(X_0,A))$
where $a_3 = b^{-1}a_2K$.

Hence A is uniformly stable with respect to the g.d.s. □

Theorem 4.3. *Assume that the conditions of theorem (4.1) are satisfied. Also let there exist a function $V_2 \in C(IXA(E),R_+)$ satisfying the theorem (3.2) and the condition* (3.12) *of theorem (3.4). Then S_1^* (Equistability of the trivial solution of the equation* (3.2)*) together with S_2^* imply the equistrict stability of the set A with respect to the g.d.s.*

Proof. As the conditions of theorem (4.1) hold with S_1^*, for $d(X_0,A) \leq p_1$, $p_1 > 0$ implies the conclusion of the theorem (4.1) that $d(F(t,t_0,X_0),A) \leq a_3(t_0,d(X_0,A))$ (i) for all $t \geq t_0$, where $a_3 \in K^*$. □

As S_2^* holds, there exists a number $q > 0$ such that for $u_0 \leq q$, $u(t,t_0,u_0) \geq a_4(t_0,u_0)$ (ii) where $a_4 \in K^*$ and $t \geq t_0$, $u(t,t_0,u_0)$ being any solution of the equation (3.7). From (3.12) and the properties of the function $b_1 \in K^*$, there exists a number $q_1 = q_1(t_0,q) > 0$ such that $d(X_0,A) \leq q_1$ and $b_1(t_0,d(X_0,A)) \leq q$ hold simultaneously. Define $q_2 = \min(p_1,q_1)$.

Then (i) above holds for all X_0 such that $d(X_0,A) \leq q_2$. Choose u_0 so that $V_2(t_0,X_0) \geq b_1(t_0,d(X_0,A)) = u_0$. Then from the theorem (3.2)

$$\begin{aligned} V_2(t,F(t,t_0,X_0)) &\geq u_{\min}(t,t_0,u_0) \\ &= u_{\min}(t,t_o,b_1(t_0,d(X_0,A))) \end{aligned} \quad \text{(iii)}$$

It follows from (3.12), (iii) and (ii) above, that

$$\begin{aligned} a_1(d(F(t,t_0,X_0),A)) &\geq V_2(t,F(t,t_0,X_0)) \\ &\geq a_4(t_0,b_1(t_0,d(X_0,A))) \\ &= a_5(t_0,d(X_0,A)) \end{aligned} \quad \text{(iv)}$$

Therefore

$$d(F(t,t_0,X_0),A) \geq a^{-1}, a_5(t_0,d(X_0,A)) = a_6(t_0,d(X_0,A)) \quad \text{(v)}$$

for all $t \geq t_0$, where $a_6 \in K^*$.

(i) and (iv) together imply the equistrict stability of A with respect to the g.d.s.

Theorem 4.4. *Let the assumptions of theorems (4.2) and (4.3) hold with the condition* (3.12) *replaced by*

$$b_3(d(X,A)) \leq V_2(t,X) \leq a_3(d(X,A)) \quad (4.9)$$

where a_3 and $b_3 \in K$. Then S_3^* (uniform stability of the trivial solution of the equation (3.2)*) together with S_4^* imply the uniform strict stability of the set A with respect to the g.d.s.*

Proof. As the conditions of theorem (4.2) are valied with S_3^*, for $d(X_0,A) \leq p_1, p_1 > 0$, the conclusion of theorem (4.2) is immediate:

$$\text{viz., } d(F(t,t_0,X_0),A) \leq a_3(d(X_0,A)) \quad \text{(i)}$$

for all $t \leq t_0$, where $a_3 \in K$.

As S_4^* holds, there exists a number $q > 0$ such that for

$$u_0 \leq q, u(t,t_0.u_0) \geq a_4(t_0,u_0) \quad \text{(ii)}$$

where $a_4 \in K$ and $t \geq t_0$, $u(t,t_0,u_0)$ being any solution of the equation (3.7). From (4.9) and the properties of b_3K, there exists a number $q_1 = q_1(q) > 0$ such that $d(X_0,A) \leq q_1$ and $b_3(d(X_0,A)) \leq q$ hold simultaneously.

Define $q_2 = \min(p_1,q_1)$. Then (i) above holds for all X_0 such that $d(X_0,A) \leq q_2$.

Then from theorem (3.2)

$$V_2(t,F(t,t_0,X_0)) \geq u_{\min}(t,t_0,u_0) = u_{\min}(t,t_0,b_3(d(X_0,A))) \quad \text{(iii)}$$

It follows from (4.9), (iii) and (ii) above that

$$\begin{aligned} a_3(d(F(t,t_0,X_0),A)) &\geq V_2(t,F(t,t_0,X_0)) \\ &= a_4(u_0) = a_4(b_3(d(X_0,A))) \\ &= a_5(d(X_0,A)) \end{aligned} \quad \text{(iv)}$$

Therefore

$$d(F(t,t_0,X_0),A) \geq a_3^{-1}a_5(d(X_0,A)) = a_6(d(X_0,A)) \quad \text{(v)}$$

for all $t \geq t_0$, where $a_6 \in K$.

(i) and (v) together imply the uniform strict stability of the set A with respect to the g.d.s. □

Theorem 4.5. *Let the assumptions of theorem (4.1) hold. Then the equiasymptotic stability of the trivial solution of the equation* (3.2) *implies the equiasymptotic stability of the set A with respect to the g.d.s.*

Proof. Let $t_o \in I$ be given. As the trivial solution of the equation (3.2) is equiasymptotically stable, there exist functions $a_1 \in K^*$ and $b_1 \in L^*$ and a number $p > 0$ such that

$$r_0 \leq p \qquad \text{(i)}$$

implies that
$$r(t, t_0, r_0) \leq a_1(t_0, r_0) b_1(t_0, t - t_0) \qquad \text{(ii)}$$

for all $t \leq t_0$, where $r(t, t_0, r_0)$ is a solution of the equation (3.2). As in the proofs of earlier theorems we can determine a number $p_1 = p_1(t_0, p) > 0$ such that $d(X_0, A) \leq p_1$ and $a(t_0, d(X_0, A)) \leq p$ hold simultaneously.

Let X_0 be such that $d(X_0, A) \leq p_1$ and choose

$$V(t_0, X_0) \leq a(t_0, d(X_0, A)) = r_0.$$

The choice of r_0 verifies (i); thus (ii) holds. From theorem (3.1). (ii) and the inequality (3.11) it follows

$$\begin{aligned} b(d(F(t, t_0, X_0), A)) &\leq V(t, F(t, t_0, X_0)) \\ &\leq r(t, t_0, a(t_0, d(X_0, A))) \\ &\leq a_1(t_0, a(t_0, d(X_0, A))) b_1(t_0, t - t_0) \\ &\leq a_2(t_0, d(X_0, A)) b_1(t_0, t - t_0) \end{aligned}$$

Hence

$$d(F(t, t_0, X_0), A) \leq b^{-1}(a_2(t_0, d(X_0, A)) b_1(t_0, t - t_0)) \qquad \text{(iii)}$$

Now $b_1 \in L^*$; hence $b_1(t_0, t - t_0) \leq b_1(t_0, 0)$.

Then from (iii) above,

$$\begin{aligned} d(F(t, t_0, X_0), A) &\leq b^{-1}(a_2(t_0, d(X_0, A)) b_1(t_0, 0)) \\ &\leq b^{-1} a_3(t_0, d(X_0, A)) \\ &\leq a_4(t_0, d(X_0, A)) \text{ where } a_4\ k^* \qquad \text{(iv)} \end{aligned}$$

Also from (iii) above and the fact $d(X_0, A) \leq p_1$

$$\begin{aligned} d(F(t, t_0, X_0), A) &\leq b^{-1}(a_2(t_0, p_1) b_1(t_0, t - t_0)) \\ &\leq b^{-1}(b_2(t_0, t - t_0)) \\ &\leq b_3(t_0, t - t_0) \text{ where } b_3 \in L^*. \qquad \text{(v)} \end{aligned}$$

Combining (iv) and (v)

$$d(F(t, t_0, X_0), A) \leq [a_4(t_0, d(X_0, A) b_3(t_0, t - t_0))]^{1/2}$$

or
$$d(F(t, t_0, X_0), A) \leq a_5(t_0, d(X_0, A) b_4(t_0, t - t_0)) \qquad \text{(vi)}$$

where $a_5 \in K^*$ and $b_4 \in L^*$. This means that A is equiasymptotically stable with respect to the g.d.s. □

Theorem 4.6. *Let the assumptions of theorem (4.3) hold. Them S_5^* (i.e., equi-asymptotic stability of the trivial solution of the equation* (3.2)*) together with S_6^* imply equistrict asymptotic stability of the set A with respect to the g.d.s.*

Proof. The assumptions of theorem (4.3) include those of theorem (4.1). Thus the theorem (4.5) holds. Hence for $d(X_0, A) \leq p_1, p_1 > 0$, the conclusion (vi) of the previous theorem holds.

Since S_6^* is given, there exists a number $p_2 > 0$ such that

$$u_0 \leq p_2 \text{ implies } u(t, t_0, u_0) \geq c(t_0, u_0) d(t_0, t - t_0) \qquad \text{(i)}$$

for all $t \geq t_0$, where $c \in K^*$ and $d \in L^*, u(t, t_0, u_0)$ being a solution of the equation (3.7).

As before from the inequality (3.12) and the properties of b_1, we determine a number $p_3 = p_3(p_2, t_0) > 0$ such that $d(X_0, A) \leq p_3$ and $b_1(t_0, d(X_0, A)) \leq p_2$ hold simultaneously.

Let $p_4 = \min(p_1, p_3)$.

Thus for $d(X_0, A) \leq p_4$, the conclusion (vi) of the previous theorem (4.5) and the step (i) (in this theorem) are both satisfied.

Now choose u_0 such that

$$V_2(t_0, X_0) \geq b_1(t_0, d(X_0, A)) = u_0.$$

Them from the inequality (3.12), theorem (3.2), (i) above and (vi) of the previous theorem (4.5), it follows that

$$\begin{aligned} a_1(d(F(t, t_0, X_0), A)) &\geq V_2(t, F(t, t_0, X_0)) \\ &\geq u(t, t_0, b_1(t_0, d(X_0, A))) \\ &\geq c(t_0, b_1(t_0, d(X_0, A))) d(t_0, t - t_0) \end{aligned}$$

implying

$$\begin{aligned} d(F(t, t_0, X_0), A) &\geq a_1^{-1} \left[c(t_0, b_1(t_0, d(X_0, A) d(t_0, t - t_0))) \right] \\ &\geq c_1(t_0, d(X_0, A) d_1(t_0, t - t_1)) \qquad \text{(ii)} \end{aligned}$$

where $c_1 \in K^*$ and $d_1 \in L^*$.

The conclusion (vi) of the previous theorem (which has already been seen to hold) and (ii) above, imply that the set A is equistrict asymptotic stable with respect to the g.d.s. □

Theorem 4.7. *Let the assumptions of theorem (4.2) hold. Then the uniform asymptotic stability of the trivial solution of the equation* (3.2) *implies the uniform asymptotic stability of the set A with respect to the g.d.s.*

Proof. The trivial solution of the equation (3.2) is uniform asymptotic stable. Therefore, there exists a number $p > 0$ such that

$$r_0 \leq p \qquad \text{(i)}$$

implies
$$r(t, t_0, r_0) \leq a_1(r_0) b_1(t - t_0) \qquad \text{(ii)}$$

where $a_1 \in K$ and $b_1 \in L$, for all $t \geq t_0$, $r(t, t_0, r_0)$ being a solution of the equation (3.2).

As in the proofs of earlier theorems (See theorem (4.5)) we can determine $p_1 = p_1(p) > 0$ such that $d(X_0, A) \leq p_1$ and $a(d(X_0, A)) \leq p$, $a \in K$ hold simultaneously.

Let X_0 be such that $d(X_0, A) p_1$ and choose r_0 such that $V(t_0, X_0) \leq a(d(X_0, A)) = r_0$. The choice of r_0 verifies (i) from which (ii) is implied. From theorem (3.1), (ii) above and the inequality (3.15) it follows that

$$\begin{aligned} b(d(F(t,t_0,X_0),A)) &\leq V(t,F(t,t_0,X_0)) \\ &\leq r(t,t_0,r_0) \\ &\leq r(t,t_0,a(d(X_0,A))) \\ &\leq a_1(ad(X_0,A))b_1(t-t_0) \\ &\leq a_2(d(X_0,A)b_1(t-t_0)), \\ &\quad \text{where } a_2 = a_1 a \in K. \end{aligned}$$

Therefore,

$$\begin{aligned} d(F(t,t_0,X_0),A) &\leq b^{-1}(a_2(d(X_0,A))b_1(t-t_0)) \qquad \text{(iii)} \\ &\leq b^{-1}(a_2(d(X_0,A)b_1(0))) \end{aligned}$$

since $b_1(t-t_0) \leq b_1(0)$, so that

$$d(F(t,t_0,X_0),A) \leq a_3(d(X_0,A)), a_3 \in K \qquad \text{(iv)}$$

Also from (iii) above, and $d(X_0,A) \leq p_1$, $a_2 \in K$,

$$\begin{aligned} d(F(t,t_0,X_0),A) &\leq b^{-1}(a_2(p_1)b_1(t-t_0)) = b^{-1}(b_2(t-t_0)) \\ &\leq b_3(t-t_0) \text{ where } b_3 = b^{-1}b_2 \in L. \qquad \text{(v)} \end{aligned}$$

(iv) and (v) together imply

$$\begin{aligned} d(F(t,t_0,X_0),A) &\leq [a_3(d(X_0,A))b_3(t-t_0)]^{1/2} \\ &\leq a_4(d(X_0,A))b_4(t-t_0) \qquad \text{(vi)} \end{aligned}$$

where $a_4 \in K$ and $b_4 \in L$. □

Thus the set A is uniform asymptotic stable with respect to the g.d.s.

Theorem 4.8. *Let the assumptions of theorem (4.4) hold. Then S_7^* (the uniform asymptotic stability of the trivial solution of the equation (3.2)) together with S_8^* implies the uniform strict asymptotic stability of the set A with respect to the g.d.s.*

Proof. The assumptions of the theorem (4.4) include those of theorems (4.2) and (4.3) with the inequality (4.9) (viz., $b_3(d(X,A)) \leq V_2(t,X) \leq a_3(d(X,A))$) replacing (3.12). Theorem (4.7) holds.

Hence, for $d(X_0,A) \leq p_1$, the conclusion (vi) of the previous theorem (4.7) holds.

Because of S_8^*, for any solution $u(t,t_0,u_0)$ of the equation (3.7), there exists $p_2 > 0$ such that $u_0 \leq p_2$ implies

$$u(t,t_0,u_0) \geq a(u_0)b(t-t_0), \text{ for all } t \geq t_0, \qquad \text{(i)}$$

where $a \in K$ and $b \in L$.

From (4.9) and the properties of b_3, we define $p_3 = p_3(p_2) > 0$ such that $d(X_0,A) \leq p_3$ and $b_3(d(X_0,A)) \leq p_2$ hold simultaneously.

Let $p_4 = \min(p_1,p_3)$. Thus $d(X_0,A) \leq p_4$, (4.9) and (i) above are satisfied.

Now choose u_0 such that $V_2(t_0,X_0) \geq b_3(d(X_0,A)) = u_0$. Then from theorem (3.2), (i) above and the inequality (4.9).

$$\begin{aligned} a_3(d(F(t,t_0,X_0),A)) &\geq V_2(t,F(t,t_0,X_0)) \\ &\geq u(t,t_0,b_3(d(X_0,A))) \\ &\geq a(b_3(d(X_0,A))b(t-t_0)) \\ &\geq a_1(d(X_0,A)b(t-t_0)), a_1 \in K. \end{aligned}$$

Therefore

$$d(F(t,t_0,X_0),A) \geq a_3^{-1}(a_1(d(X_0,A)b(t-t_0))) \qquad \text{(ii)}$$

The conclusion (vi) of the previous theorem (4.7) together with (ii) above implies the uniform strict asymptotic stability of the set A with respect to the g.d.s. □

5. CONVERSE THEOREMS(ON THE EXISTENCE OF LYAPUNOV FUNCTIONS IN A REVERSIBLE DYNAMICAL SYSTEM)

From the definition (2.2) it follows that a g.d.s. in which $X = F(t,t_0,X_0)$ **iff** $X_0 = F(t_0,t,X)$ is called a Reversible Dynamical System in E, which we henceforth denote by r.d.s.

In theorems that follow, the converse theorems in which the existence of $V-$function is sought are proved in r.d.s.

Theorem 5.1. *If a set A is equistrict stable with respect to a r.d.s. in E, then there exists a $V-$function satisfying all the assumptions of theorem (3.4) (i.e., the $V-$function satisfies (1) and (2) of theorem (3.3) as well as the inequality (3.12) in which V_2 is replaced by V).*

Proof. Let us define $V(t,X) = d(F(t_0,t,X),A)$, $t_0 \in I$. $V \in C(IXA(E),R_+)$ follows from the continuity of the flow F. By the reversibility condition we have $X = F(t,t_0,F(t_0,t,X))$.

Equivalently $X = F(t,t_0,X_0)$ **iff** $X_0 = F(t_0,t,X)$.

By the equistrict stability of the set A with respect to the r.d.s., there exist a_1, $a_2 \in K^*$, satisfying

$$\begin{aligned} & a_1(t_0,d(X_0,A)) \leq d(F(t,t_0,X_0),A) \leq a_2(t_0,d(X_0,A)) \\ \text{i.e.,}\quad & a_1(t_0,d(F(t_0,t,X),A)) \leq d(F(t,t_0,F(t_0,t,X)),A) \\ & \qquad \leq a_2(t_0,d(F(t_0,t,X),A)) \qquad \text{(i)} \end{aligned}$$

With what we have defined as V,

$$a_1(t_0,V(t,X)) \leq d(X,A) \leq a_2(t_0,V(t,X))$$

Equivalently,

$$a_1(t_0,d(X_0,A)) \leq d(X,A) \leq a_2(t_0,d(X_0,A))$$

implying $\quad d(X_0,A) \leq a_1^{-1}(t_0,d(X,A))$

and $\quad d(X_0,A) \leq a_2^{-1}(t_0,d(X,A))$

or $\quad a_2^{-1}(t_0,d(X,A)) \leq d(X_0,A) \leq a_1^{-1}(t_0,d(X,A))$.

The inequality (3.12) is thus verified, since $a_1^{-1}, a_2^{-1} \in K^*$.

Now for $h > 0$,

$$\begin{aligned} V(t+h,F(t+h,t_0,X_0)) &= d(F(t_0,t+h,F(t+h,t_o,X_0)),A) \\ &= d(X_0,A) \end{aligned}$$

so that

$$V(t+h,F(t+h,t_0,X_0)) - V(t,F(t,t_0,X_0)) = 0$$

Thus

$$\left.\begin{aligned} D^+V(t,X) &= 0 \\ \text{and} \qquad D^-V(T,X) &= 0 \end{aligned}\right\}$$

which verifies the inequalities (3.3) and (3.8) with the functions of g and h identically vanishing. □

Theorem 5.2. *If the set A is uniform strict stable for the r.d.s. in E, then there exists a $V-$function satisfying the assumptions of theorem (3.6).*

Proof. Define $V(t,X) = d(F(O,T,X),A)$. Equivalently, $V(t,X) = d(X_0,A)$. Clearly $V \in d(IXA(E),R_+)$.

$$a_1(d(X_0,A)) \leq d(F(t,O,X_0),A) \leq a_2(d(X_0,A))$$

or $$a_1(V(t,X)) \leq d(X,A) \leq a_2(V(t,X)),$$
since $X = F(t,O,X_0)$.

This implies

$$a_2^{-1}(d(X,A)) \leq V(t,X) \leq a_1^{-1}(d(X,A)),$$

where $a_1^{-1}, a_2^{-1} \in K$.

Thus (3.14) of theorem (3.6) is verified.

Now for $h > 0$,

$$\begin{aligned} V(t+h,F(t+h,O,X_0)) &= d(F(O,t+h,X),A) \\ &= d(F(O,t+h,F(t+h,O,X_0)),A) \\ &= d(X_0,A). \end{aligned}$$

Also $V(t,F(t,O,X_0)) = d(X_0,A)$.

Hence $D^+V(t,X) = 0 = D^-V(t,X)$, which satisfy (3.3) and (3.8), with the functions g and h identically vanishing. i.e., V satisfies (3.5) and (3.10) simultaneously. □

Theorem 5.3. *Assume that –*

(1) *the set A is uniformly strict stable, so that for some $p > 0$, $d(X_0,A) \leq p$ implies*

$$a_1(d(X_0,A)) \leq d(F(t,O,X_0),A) \leq a_2(d(X_0,A)) \quad (5.1)$$

where a_1, $a_2 \in K$.

(2) *Let $g \in C(IXR_+,R_+)$, $g(t,O) = 0$ and that the trivial solution of $r' = g(t,r)$ is uniformly strictly stable, so that for $u_0 \leq p$, $p > 0$*

$$b_1(u_0) \leq u(t,O,u_0) \leq b_2(u_0) \quad (5.2)$$

where $b_1, b_2 \in K$ and $u(t,O,u_0)$ is any solution of

$$r' = g(t,\gamma) \text{ with } u(O) = u_0 \quad (5.3)$$

Then there exists a function $V = V(t,X)$ such that

(i) $V = V(t,X) \in C(I\ X\ S(A,\delta),R_+)$

(ii) $b(d(X,A)) \leq V(t,X) \leq a(d(X,A))$ *for* $(t,X) \in I\ X\ S(A,\delta)$ *and* $a,b \in K$.

(iii) $D^+V(t,X) = D^-V(t,X) = g(t,V(t,X))$, *for all* $t \geq t_0$, *for which* $X\ S(A,\delta)$.

Proof. Due to the reversibility of the system

$$X_0 = F(O,t,X) \text{ iff } X = F(t,O,X_0).$$

Choose any function $\mu \in C(S(A,\delta),R_+)$ such that

$$c_1(d(X,A)) \leq \mu(X) \leq c_2(d(X,A)) \quad (5.4)$$

where $c_1, c_2 \in K$.

Define

$$V(t,X) = u(t,O,\mu(F(O,t,X))) \quad (5.5)$$

where $u(t,0,u_0)$ is a solution of the eq.(5.4)

Due to the continuity of μ_1, (i) follows.

Also for $(t,X) \in I\ X\ S(A,\delta)$, we have

$$\begin{aligned} V(t,X) &= u(t,O,\mu(F(O,t,X))) \\ &\leq b_2(\mu(F(O,t,X))) \\ &\leq b_2c_2(d(F(O,t,X),A)) = b_2c_2d(X_0,A) \end{aligned}$$

and $$d(X_0,A) \leq a_1^{-1}d(F(t,O,X_0),A) = a_1^{-1}(d(X,A))$$

Hence

$$V(t,X) \leq b_2c_2a_1^{-1}(d(X,A)) = a(d(X,A)),$$

where $a = b_2c_2a_1^{-1} \in K$.

Again,

$$\begin{aligned} V(t,X) &= u(t,O,\mu(F(O,t,X))) \\ &\geq b_1\mu(F(O,t,X)) \\ &\geq b_1c_1d(F(O,t,X),A) \\ &\geq b_1c_1a_2^{-1}d(X,A) = b(d(X,A)) \text{ where } b = b_1c_1a_2^{-1} \in K. \end{aligned}$$

Thus, $b(d(X,A)) \leq V(t,X) \leq a(d(X,A))$, which proves (ii).

Finally, so long as $F(t,t_0,X_0) \in S(A,\delta)$, we have

$$V(t,F(t,t_0,X_0)) = u(t,O,\mu(F(O,t,F(t,t_0,X_0))))$$

Hence

$$\begin{aligned} V(t+h,F(t+h,t_0,X_0)) &= u(t+h,O,\mu(F(O,t+h,F(t+h,t_0,X_0)))) \\ &= u(t+h,O,\mu(F(O,t,F(t,t_0,X_0)))) \end{aligned}$$

Consequently,

$$\begin{aligned} D^+V(t,F(t,t_0,X_0)) &= D^-V(t,F(t,t_0,X_0)) \\ &= \lim_{h\to 0}\Big[\frac{1}{h}\Big\{u(t+h,O,\mu(F(O,t,F(t,t_0,X_0))) \\ &\qquad - u(t,O,\mu(F(O,t,F(t,t_0,X_0))))\Big\}\Big] \\ &= u'(t,O,\mu(O,t,F(t,t_0,X_0))) \\ &= g(t,V(t,F(t,t_0,X_0))) \end{aligned}$$

due to the differentiability of u. Thus (iii) is proved. This establishes the throrem showing the existence of a $V-$function in place of V_1, V_2 of theorem (4.4). □

Theorem 5.4. *Suppose that*

(1) *The set A is strict uniform asymptotic stable with respect to the r.d.s. on E. i.e., for all $t \in I$ and $X_0 \in S(A,\delta)$,*

$$a_1(d(X_0,A))b_1(t) \leq d(F(t,O,X_0),A) \leq a_2(d(X_0,A))b_2(t) \quad (5.6)$$

where $a_i \in K$ and $b_i \in L$, $i = (1,2)$.

(2) $g \in C(I\ X\ R_+,R_+)$, $g(t,O) = 0$ *ensures the existence, uniqueness and continuous dependance of solutions of* $r' = g(t,r)$ *on initial conditions and the trivial solution of the equation is strict uniform asymptotic stable. i.e., there exists functions* a_3, $a_4 \in K$ *and* b_3, $b_4 \in L$ *such that*

$$a_3(u_0)b_3(t) \leq u(t,O,u_0) \leq a_4(u_0)b_4(t) \quad (5.7)$$

for all $t \in I$, $u(t, O, u_0)$ being a solution of $r' = g(t, r)$ through (O, u_0).

(3) a_3 *is differentiable and* $a_3'(r) \leq \lambda > 0$ *for all* $r \in R_+$.

(4) $b_3(t) = \lambda_1 b_2(t)$, $\lambda_1 > 0$ *for all* $t \in I$.

Then there exists a $V-$function satisfying–

(i) $V = V(t, X) \in c(I\ X\ S(A, \delta_1), R_+)$, $\delta_1 = a_1(\ )b_1(O)$

(ii) for $(t, X) \in I\ X\ S(A, \delta_1)$

$$b(d(X, A)) \leq V(t, X) \leq a(t, d(X, A)),\ \text{for all } t \in I \tag{5.8}$$

where $a \in K^*$ and $b \in L$ are defined on $[O, \delta_1]$

(iii) $D^+V(t, X) = D^-V(t, X) = g(t, V(t, X))$ for all $t \geq t_0$ for which $X = F(t, t_0, X_0) \in S(A, \delta_1)$.

Proof. Let $X \in S(A, \delta_1)$ and $X_0 = F(O, t, X)$ so that $X = F(t, O, X_0)$, by the reversibility condition.

By (5.6), $X_0 \in S(A, \delta)$ implies $X \in S(A, \delta_1)$.

Choose any function $\mu \in c(S(A, \delta_1), R_+)$ such that

$$a_2(d(X, A)) \leq \mu(X) \leq a_5(d(X, A)) \tag{i}$$

where a_2 is defined in (5.6) and $a_5 \in k$

Define the Lyapunov function by

$$V(t, X) = u(t, O, \mu(F(O, t, X))) \tag{ii}$$

As in theorem (5.3) V satisfies (i) and (iii) of the conclusions of the theorem and to complete the proof of the theorem the verification of (ii) alone is required.

$F(O, t, X) = X_0$. Hence from (5.6)

$$a_1(d(X_0, A))b_1(t) \leq d(X, A) \leq a_2(d(X_0, A))b_2(t),$$

so that

$$a_2^{-1}\left(\frac{d(X, A)}{b_2(t)}\right) \leq d(X_0, A) \leq a_1^{-1}\left(\frac{d(X, A)}{b_1(t)}\right) \tag{iii}$$

Using (5.7), step (i) and step (iii) together with the assumptions (3) and (4) we have, since $V(t, X) = u(t, O(F(O, t, X)))$, by (ii)

$$\begin{aligned} V(t, X) &\geq a_3(\mu(F(O, t, X)))b_3(t) \\ &\geq a_3(a_2(d(F(O, t, X), A)))b_3(t) \\ &\geq a_3(a_2(d(X_0, A)))b_3(t) \\ &\geq a_3\left(a_2 a_2^{-1}\left(\frac{d(X, A)}{b_2(t)}\right)\right)b_3(t) \\ &= a_3\left(\frac{d(X, A)}{b_2(t)}\right)b_3(t) = \lambda_1 a_3(d(X, A)) = b(d(X, A)) \end{aligned}$$

Thus $V(t, X) \geq b(d(X, A))$ when $b \in K$.

Similarly

$$\begin{aligned} V(t, X) &= u(t, O, \mu(F(O, t, X))) \\ &\leq a_4(\mu(F(O, T, X)))b_4(t) \\ &\leq a_4\left(a_5 a_1^{-1}\left(\frac{d(X, A)}{b_1(t)}\right)\right)b_4(t) \\ &\leq a(t, d(X, A)) \text{ where } a \in K^*. \end{aligned}$$

Thus the conclusion (ii) is verified. □

Remarks:

(1) While proving the sufficiency criteria in theorems (3.4), (3.6) and (4.4), two $V-$functions were used, because of the nature of inequalities (3.4) and (3.9). However, in theorems of this section (i.e., theorems (5.1) through (5.4)), a stronger result, in the form of a single Lyapunov function satisfying both conditions satisfied by individual functions of theorems (3.4), (3.6) and (4.4) is proved.

(2) Theorems (5.1), (5.2) and (5.3) are converse theorems for a r.d.s. on E wherein, the existance of a single Lyapunov function is established. But, theorem (5.4) is not a converse of theorem (4.6) or theorem (4.8), for, assuming uniform strict asymptotic stability for the set A, we have got a Lyapunov function that yields only equi-strict-asymptotic stability. Thus a weaker result is obtained.

6. CONDITIONAL INVARIANCY OF SET B WITH RESPECT TO SET A FOR A G.D.S. IN E

Definition 6.1. *A set B is said to be conditionally invariant with respect to the set A for a g.d.s. in E, if $F(t, t_0, A) \subset B$ for all $t \geq t_0$.*

Note: 1. If B is conditionally invariant with respect to A for a g.d.s. in E and $B \subset C$, then C is also conditionally invariant with respect to A. i.e., any super set to B is also conditionally invariant with respect to A. This is evident since $F(t, t_0, A) \subset B$ and $B \subset C$ implies $F(t, t_0, A) \subset C$.

Note: 2. An invariant set A for a g.d.s. in E is self invariant. (i.e., $F(t, t_0, A) \subset A$ for all $t \geq t_0$). If A is self invariant, then A is conditionally invariant with respect to any subset of A.

In the following, B is conditionally invariant with respect to A for the g.d.s. in E.

Definition 6.2. *With respect to A, for the g.d.s. in E, B is said to be*

(1) ***Equistable,*** *if*

$$d(F(t, t_0, X_0), B) \leq a(t_0, d(X_0, A)) \tag{6.1}$$

for all $t \geq t_0$, where $a \in K^*$.

(2) ***Equistrict stable,*** *if*

$$a_1(t_0, d(X_0, B)) \leq d(F(t, t_0, X_0), B) \leq a_2(t_0, d(X_0, A)) \tag{6.2}$$

where $a_1, a_2 \in K^*$, for all $t \geq t_0$.

(3) ***Uniform stable,*** *if*

$$d(F(t, t_0, X_0), B) \leq a(d(X_0, A)) \tag{6.3}$$

where $a \in K$, for all $t \geq t_0$.

(4) ***Uniform strict stable,*** *if*

$$a_1(d(X_0, B)) \leq d(F(t, t_0, X_0), B) \leq a_2(d(X_0, A)), \tag{6.4}$$

where $a_1, a_2 \in K$, for all $t \geq t_0$.

(5) ***Equiasymptotic stable,*** *if*

$$d(F(t, t_0, X_0), B) \leq a(t_0, d(X_0, A))b(t_0, t - t_0), \tag{6.5}$$

where $a \in K^*$ and $b \in L^*$, for all $t \geq t_0$.

(6) ***Uniform asymptotic stable,*** *if*

$$d(F(t,t_0,X_0),B) \leq a(d(X_0,B))b(t-t_0), \tag{6.6}$$

where $a \in K$ *and* $b \in L$, *for all* $t \geq t_0$.

(7) ***Equistrict asymptotic stable,*** *if*

$$a_1(t_0,d(X_0,B))b_1(t_0,t-t_0) \leq d(F(t,t_0,x_0),B) \leq a_2(t_0,d(X_0,A))b_2(t_0,t-t_0) \tag{6.7}$$

where $a_1,a_2 \in K^*$ *and* $b_1,b_2 \in L^*$, *for all* $t \geq t_0$.

(8) ***Uniform strict asymptotic stable,*** *if*

$$a_1(d(X_0,B))b_1(t-t_0) \leq d(F(t,t_0,X_0),B) \leq a_2(d(X_0,A)b_2(t-t_0)), \tag{6.8}$$

where $a_1,a_2 \in K$ *and* $b_1,b_2 \in L$, *for all* $t \geq t_0$.

Remark:

In the above definitions, we use d^*, where $d^*(A,B) = \sup\{d(a,B), a \in A\}$, $d(a,B) = \inf\{d(a,b), b \in B\}$ instead of the Hausdorff distance d defined as $d(A,B) = \max\{d^*(A,B), d^*(B,A)\}$. In order to see the reason for this, let us suppose, that the Hausdorff distance d is used in the definition of, say, equistability of B with respect to A.

Then $X_0 = A$ implies $d(F(t,t_0,A),B) = 0$, since $a \in K^*$ and $d(X_0,A) = 0$. In particular at $t = t_0$, this means $d(A,B) = 0$ implying $A = B$. Thus the equistability condition (6.1) with Hausdorff distance d implies equality of sets A and B.

Moreover the definition for conditional invariancy is in terms of 'subset of' relation. However, in the Hausdorff distance, there is no way of inferring subset relation between the two sets. On the other hand $d^*(A,B) = 0$ implies $A \subset B$.

We shall henceforth abbreviate **Conditionally Invariant Set** B**'** by **'C.I. set** B**'**

Theorem 6.1. *Let the assumptions of theorem (3.3) be satisfied except that* (3.11) *is replaced by*

$$b(d^*(X,B)) \leq V(t,X) \leq a(t,d^*(X,A)), \text{ for all } t \geq t_0 \tag{6.9}$$

where $a \in K^*$, $b \in K$ *and* d^* *is the distance as explained in the remark above. Then the C.I. set* B *is equistable with respect to* A *for the g.d.s. in* E.

Proof. Due to the condition (2) of theorem (3.3) $V(t,F(t,t_0,X_0)) \leq V(t_0,X_0)$. From (6.9)

$$b(d^*(F(t,t_0,X_0),B)) \leq V(t,F(t,t_0,X_0)) \leq V(t_0,X_0) \leq a(t_0,d^*(X_0,A))$$

implying

$$d^*(F(t,t_0,X_0),B) \leq b^{-1}a(t_0,d^*(X_0,B)) = c(t_0,d^*(X_0,B)) \text{ where } c = b^{-1}a \in K^*.$$

Hence the equistability of the C.I. set B with respect to A for the g.d.s. in E. □

Theorem 6.2. *Let the assumptions of theorem (3.4) hold except that the conditions* (3.11) *and* (3.12) *are replaced by*

$$b_1(d^*(X,B)) \leq V_1(t,X \leq a_1(t,d^*(X,A))) \tag{6.10}$$

and $$b_2(t,d^*(X,B)) \leq V_2(t,X) \leq a_2(d^*(X,B)) \tag{6.11}$$

for all $t \geq t_0$, *where* $b_1,a_2 \in K$ *and* $a_1,b_2 \in K^*$, d^* *being as explained under the remark. Then the C.I.set* B *is equistrict stable with respect to* A *for the g.d.s. in* E.

Proof. As in theorems (3.3) and (3.4)

$$b_1(d^*(F(t,t_0,X_0),B)) \leq V_1(t,F(t,t_0,X_0)) \leq V_1(t_0,X_0) \leq a_1(t_0,d^*(X_0,A))$$

so that

$$d^*(F(t,t_0,X_0),B) \leq b_1^{-1}a_1(t_0,d^*(X_0,A)) \tag{i}$$

Again

$$b_2(t_0,d^*(X_0,B)) \leq V_2(t_0,X_0) \leq V_2(t,F(t,t_0,X_0)) \leq a_2(d^*(F(t,t_0,X_0),B))$$

so that

$$a_2^{-1}b_2(t_0,d^*(X_0,B)) \leq d^*(F(t,t_0,x_0),B) \tag{ii}$$

putting $a_2^{-1}b_2 = c_2$ and $b_1^{-1}a_1 = c_1$ where $c_1,c_2 \in K^*$, we get from (i) and (ii) above

$$c_2(t_0,d^*(X_0,B)) \leq d^*(F(t,t_0,X_0),B) \leq c_1(t_0,d^*(X_0,A))$$

which means the equistrict stability of the C.I.set B with respect to A for the g.d.s. in E. □

Theorem 6.3. *Let the assumptions of theorem (6.1) be satisfied except that (6.9) is replaced by*

$$b(d^*(X,B)) \leq V(t,X) \leq a(d^*(X,A)) \tag{6.12}$$

for all $t \geq t_0$, *where* $a,b \in K$. *Then the C.I.set* B *is uniform stable with respect to* A, *for the g.d.s. in* E.

Proof. By the assumptions, as in theorem (6.1)

$$b(d^*(F(t,t_0,X_0),B)) \leq V(t,F(t,t_0,X_0)) \leq V(t_0,X_0) \leq a(d^*(X_0,A))$$

implying

$$d^*(F(t,t_0,X_0),B) \leq b^{-1}a(d^*(X_0,A)) = c(d^*(X_0,A))$$

where $c \in K$. Hence the uniform stability of the C.I.set B with respect to A for the g.d.s. in E. □

Theorem 6.4. *Let the assumptions of theorem (6.2) be satisfied with the conditions* (6.10) *and* (6.11) *replaced by*

$$b_1(d^*(X,B)) \leq V_1(t,X) \leq a_1(d^*(X,A)) \tag{6.13}$$

and $$b_2(d^*(X,B)) \leq V_2(t,X) \leq a_2(d^*(X,B)) \tag{6.14}$$

for all $t \geq t_0$, *where* $a_1,b_1 \in K$, $(i = 1,2)$. *Then the C.I.set* B *is uniform strict stable with respect to* A *for the g.d.s. in* E.

Proof. As in the proof of the theorem (6.2),

$$V_1(t, F(t, t_0, X_0)) \leq V_1(t_0, X_0) \quad \text{(i)}$$

and $$V_2(t, F(t, t_0, X_0)) \leq V_2(t_0, X_0) \quad \text{(ii)}$$

for all $t \geq t_0$. Consequently, by (6.15) and (i) above

$$b_1(d^*(F(t, t_0, X_0), B)) \leq V_1(t, F(t, t_0, X_0)) \leq V_1(t_0, X_0) \leq a_1(d^*(X_0, A))$$

implying $$d^*(F(t, t_0, X_0), B) \leq b_1^{-1} a_1(d^*(X_0, A)) \quad \text{(iii)}$$

Likewise, by (6.16) and (ii) above

$$a_2(d^*(F(t, t_0, X_0), B)) \geq V_2(t, F(t, t_0, X_0)) \geq V_2(t_0, X_0) \geq b_2(d^*(X_0, B))$$

implying $$d^*(F(t, t_0, X_0), B) \geq a_2^{-1} b_2(d^*(x_0, B)) \quad \text{(iv)}$$

Putting $b_1^{-1} a_1 = c_1 \in K$ and $a_2^{-1} b_2 = c_2 \in K$ in (iii) and (iv) above respectively

$$c_2(d^*(X_0, B)) \leq d^*(F(t, t_0, X_0), B) \leq c_1(d^*(X_0, A)).$$

Thus, the C.I. set B is uniform strict stable with respect to A for the g.d.s. in E. □

Theorem 6.5. *Let the assumptions (1) and (2) of theorem (4.1) hold except that the inequalilty of (1) is replaced by (6.9) of theorem (6.1).*

Then (i) equistability of the trivial solution of the equation (3.2) implies the equistability of the C.I. set B with respect to A and (ii) equi-asymptotic stability of the trivial solution of the equation (3.2) implies the equi-asymptotic stability of the C.I. set B with respect to A, for the g.d.s. in E.

Proof.

(i) The proof is the same as in theorem (4.1) except that d^* replaces d. Moreover the conditional invariancy of the set B with respect to A implies $d^*(F(t, t_0, X_0), B) \leq d^*(F(t, t_0, X_0), A)$. This consideration leads to the required conclusion.

(ii) The proof of this part is the same as in theorem (4.5) where d^* replaces d.

<u>**Note:**</u> $d^* \leq d$. □

Theorem 6.6. *Assume the conditions (1) and (2) of theorem (4.1) with the inequality of (1) replaced by (6.12). Then (i) uniform stability of the trivial solution of the equation* (3.2) *implies the uniform stability of the C.I. set B with respect to A and (ii) uniform asymptotic stability of the trivial solution of the equation* (6.2) *implies the uniform asymptotic stability of the C.I. set B with respect to A for the g.d.s. in E.*

Proof.

(i) The proof of uniform stability of the C.I. set B with respect to A is parallel to that given in theorem (4.2) with d^* in place of d. Further the conditional invariance of the set B with respect to A implies $d^*(F(t, T_0, X_0), B) \leq d^*(F(t, t_0, X_0), A)$. These considerations lead to the required conclusion.

(ii) The proof of uniform asymptotic stability of the C.I. set B with respect to A runs parallel to that of theorem (4.7) with d^* in place of d.

□

Theorem 6.7. *Let the assumptions of theorem (4.3) hold except that (3.11) and (3.12) are replaced by (6.10) and (6.11) of theorem (6.2) respectively. Then (i) S_1^* and S_2^* imply equistrict stability and (ii) S_5^* and S_6^* imply equistrict asymptotic stability of the C.I. set B with respect to A for the g.d.s. in E.*

Proof. In view of the assumptions of theorem (4.3), the condition (6.10) and S_1^*, which means the equistability of the C.I. set B with respect to A is implied by the theorem (6.5) – (i).

Again, since S_5^* means equiasymptotic stability of the trivial solution of the equation (3.2), equiasymptotic stability of the C.I. set B with respect to A is implied by the theorem (6.5) – (ii)

We now prove the 'STRICT' results –

(1) By the equistability of the C.I. set B with respect to A we have

$$d^*(F(t, t_0, X_0), B) \leq c_1(t_0, d^*(X_0, A)) \quad \text{(i)}$$

for all $t \geq t_0$, where $c \in K^*$.

By S_2^* there exists $p > 0$, such that $u_0 \leq p$, $u(t, t_0, u_0) \geq c_3(t_0, u_0)$, $c_3 \in K^*$ and $t \geq t_0$ for any solution $u(t, t_0, u_0)$ of the equation (3.7).

By the property of b_2 in (6.11)

viz: $b_2(t, d^*(X, B)) \leq V_2(t, X) \leq a_2(d^*(x, B))$

there exists $p_1 = p_1(t_0, p) > 0$ such that $d^*(X_0, B) \leq p_1$ and $b_2(t, d^*(X_0, B)) \leq p$ hold simultaneously.

Let $q = \min(p_1, p)$. Then (6.11) holds for all X_0 such that $d(X_0, B) \leq q$. Choose u_0 so that $V_2(t_0, X_0)$ $b_2(t_0, d^*(X_0, B)) = u_0$. As all the conditions of theorem (3.2) are satisfied

$$V_2(t, F(t, t_0, X_0)) \geq u(t, t_0, X_0) = u(t, t_0, b_2(t_0, d^*(X_0, B))) \geq c_3(t_0, b_2(t_0, d^*(X_0, B))) \geq c_4(t_0, d^*(X_0, B))$$

But

$$a_2(d^*(F(t, t_0, X_0), B)) \geq V_2(t, F(t, t_0, X_0)),$$

so that $$a_2(d^*(F(t, t_0, X_0), B)) \geq c_4(t_0, d^*(X_0, B))$$

or $$d^*(F(t, t_0, X_0), B) \geq a_2^{-1} c_4(t_0, d^*(X_0, B))$$

i.e., $$d^*(F(t, t_0, X_0), B) \geq c_2(t_0, d^*(X_0, B)) \quad \text{(ii)}$$

where $c_2 = a_2^{-1} c_4 \in K^*$, for all $t \geq t_0$.

The steps (i) and (ii) above imply equistrict stability of the C.I. set B, with respect to A for the g.d.s. in E.

(2) Because of equiasymptotic stability of the C.I. set B with respect to A we have

$$d^*(F(t, t_0, X_0), B) \leq c_1(t_0, d^*(X_0, A)) d_1(t_0, t - t_0) \quad \text{(iii)}$$

for all $t \geq t_0$, $c_1 \in K^*$, $d_1 \in L^*$.

By S_6^* there exists a number $p > 0$ with $u_0 \leq p$ such that, for any solution $u(t, t_0, u_0)$ of the equation (3.7) $u(t, t_0, u_0) \geq c_3((t_0, u_0))d_3(t_0, t - t_0)$, for all $t \geq t_0$, where $c_3 \in K^*$ and $d_3 \in L^*$.

By the property of b_2 in (6.11), there exists $p_1 = p_1(t_0, p) > 0$, such that $d^*(X_0, B) \leq p_1$ and $b_2(t, d^*(X_0, B)) \leq p$ hold simultaneously. Let $q = \min(p, p_1)$. Then (6.11) holds for all X_0 such that $d^*(X_0, B) \leq q$. Choose u_0 such that $V_2(t_0, X_0) \geq b_2(t_0, d^*(X_0, B)) = u_0$. As all the conditions of theorem (3.2) are satisfied,

$$\begin{aligned} V_2(t, F(t, t_0, u_0)) &\geq u(t, t_0, u_0) = u(t, t_0, b_2(t_0, d^*(X_0, B))) \\ &\geq c_3(t_0, b_2(t_0, d^*(X_0, B)))d_3(t_0, t - t_0) \end{aligned}$$

But

$$\begin{aligned} a_2(d^*(F(t, t_0, X_0), B)) \geq V_2(t, F(t, t_0, X_0)) \\ &\geq c_3(t_0, b_2(t_0, d^*(X_0, B)))d_3(t_0, t - t_0) \\ &\geq c_4(t_0, d^*(X_0, B))d_3(t_0, t - t_0) \end{aligned}$$

implying

$$d^*(F(t, t_0, X_0), B) \geq a_2^{-1}\left[c_4(t_0, d^*(X_0, B))d_3(t_0, t - t_0)\right]$$

$$\text{i.e.,} \quad d^*(F(t, t_0, X_0), B) \geq c_2(t_0, d^*(X_0, B))d_2(t_0, t - t_0) \qquad \text{(iv)}$$

for all $t \geq t_0$, where $c_2 \in K^*$ and $d_2 \in L^*$.

The steps (iii) and (iv) above imply equistrict asymptotic stability of the C.I. set B with respect to A for the g.d.s. in E. □

Theorem 6.8. *Assume that the conditions of theorem (4.3) hold, with conditions (3.11) and (3.12) replaced by (6.13) and (6.14) respectively.*

Then (i) s_3^* and S_4^* imply uniform strict stability of the C.I. set B with respect to A and (ii) S_7^* and S_8^* imply uniform strict asymptotic stability of the C.I. set B with respect to A, for the g.d.s. in E.

Proof. All the conditions of theorem (6.6) – (1) and (2) are satisfied, since S_3^* and S_7^* mean the uniform and uniform asymptotic stability of the trivial solution of the equation (3.2).

Accordingly, the C.I. set B is uniform stable/uniform asymptotic stable with respect to A.

Considering $S_4^* : u(t, t_0, u_0) \geq c_3(u_0)$, for all $t \geq t_0$ with $u_0 \leq p$, $p > 0$ and $S_8^* : u(t, t_0, u_0)\ c_3(u_0)d_3(t_0, t - t_0)$ with $u_0 \leq p$, $p > 0$, the 'strict' results of the stability of the C.I. set B with respect to A for the g.d.s. in E can be proved on the same lines as in theorem (6.7). □

Note:

Theorems (6.5) – (1) and (2)
(6.6) – (1) and (2)
(6.7) – (1) and (2)
and (6.8) – (1) and (2)

correspond, in order, to theorems (4.1) – (4.5); (4.2) – (4.7); (4.3) – (4.6) and (4.4) – (4.8).

Converse theorems on the existence of Lyapunov functions for the stability properties of the C.I. set B with respect to A for a r.d.s. can be proved on similar lines of theorems in section 5.

We state and prove a theorem corresponding to theorem (5.2).

Theorem 6.9. *If the set B is uniform strict stable with respect to A for a r.d.s. in E, there exist a pair of Lyapunov functions $V_i(i = 1, 2)$ satisfying the hypotheses of theorem (6.4).*

Proof. Define the functions V_1 and V_2 as follows –

$$V_1(t, X) = \inf_{O \leq T \leq t} d^*(F(T, t, X), A)$$

and

$$V_2(t, X) = \sup_{O \leq T \leq t} d^*(F(T, t, X), B).$$

Since the C.I. set B is uniform strict stable with respect to A,

$$a_2(d^*(X_0, B)) \leq d^*(F(t, t_0, X_0), B) \leq a_1(d^*(X_0, A)), t \geq t_0, a_i(i = 1, 2) \in K.$$

Then for $T \leq t$, by the reversibility condition, $X = F(t, T, X(T))$, we have

$$a_2(d^*(X(T), B)) \leq d^*(X, B) \leq a_1(d^*(X(T), A)) \qquad \text{(i)}$$

where $X(T) = F(T, t, X)$. Hence for each T such that

$$O \leq T \leq t, d^*(X(T), A) \geq a_1^{-1}(d^*(X, B)) \qquad \text{(ii)}$$

so that

$$V_1(t, X) = \inf_{O \leq T \leq t} d^*(X(T), A) \geq a_1^{-1}(d^*(X, B)) \qquad \text{(iii)}$$

Also trivially

$$V_1(t, X) \leq d^*(X, A) \qquad \text{(iv)}$$

(iii) and (iv) together give

$$a_1^{-1}(d^*(X, B)) \leq V_1(t, X) \leq d^*(X, A), O \leq T \leq t.$$

This verifies (6.13) of theorem (6.4).

$$V_2(t, X) \geq d^*(X, B) \qquad \text{(v)}$$

Also from (i)

$$d^*(X(T), B) \leq a_2^{-1}(d^*(X, B))$$

so that

$$V_2(t, X) = \sup_{O \leq T \leq t} d^*(X(T), B) \leq a_2^{-1}(d^*(X, B)) \qquad \text{(vi)}$$

(v) and (vi) together give

$$d^*(X, B) \leq V_2(t, X) \leq a_2^{-1}(d^*(X, B))$$

which verifies (6.14) of theorem (6.4).

$V(t, X)$ and $V_2(t, X)$ satisfy the inequality (3.3) with $g \equiv 0$ and the inequality (3.8) with $h \equiv 0$.

To see this,

$$\begin{aligned} V_1(t, F(t, t_0, X_0)) &= \inf_{O \leq T \leq t} d^*(F(T, t, X), A) \\ &= \inf_{O \leq T \leq t} d^*(X(T), A) \end{aligned}$$

Also for $h > 0$,

$$V_1(t + h, F(t + h, t_0, X_0)) = \inf_{O \leq T \leq t+h} (d^*(X(T), A))$$

clearly, $V_1(t + h, F(t + h, t_0, X_0)) \leq V_1(t, F(t, t_0, X_0))$

so that $D^+V_1(t, X) \leq 0$

which is (3.3) with $g \equiv 0$.

$$V_2(t+h, F(t+h, t_0, X_0)) = \sup_{Om \le T \le t+h} d^*(X(T), B)$$

$$\sup_{O \le T \le t} d^*(X(T), B) = V_2(t, F(t, t_0, X_0))$$

Thus $D^- V_2(t, X) \ge 0$ which is (3.8) with $h \equiv 0$.

Thus V_1 and V_2 satisfy all the conditions of the theorem (6.4). □

Remarks:

(1) The function V_1 shows that this is a Lyapunov function of theorem (6.3).

(2) One can easily see that theorem (6.9) with $B = A$ and d^* replaced by d, is a converse for theorem (4.4) giving two different functions V_1 and V_2 unlike theorem (5.2).

7. CONDITIONAL (OR RELATIVE) STABILITY OF A COMPACT SET A WITH RESPECT TO A G.D.S. IN E

Let the set $A \in A(E)$ be compact in E and M be a subset of E such that $A \subset M \subset E$. We state the definitions of conditional stability of the set A with respect to a g.d.s. in E. Lyapunov (vector) function defined on $I\ X\ A(E)$ is used to determine the sufficient conditions for conditional stability of A with respect to a g.d.s. in E. This concept (i.e., conditional stability of ...) is identical with the concept of relative stability (5).

Definition 7.1. *The set A is daid to be*

(1) ***Conditionally equistable*** *for the set M with respect to a g.d.s. in E if there exists a function $a \in K^*$ such that*

$$d(F(t, t_0, X_0), A) \le a(t_0, d(X_0, A)) \quad (7.1)$$

(2) ***Conditionally uniformly stable*** *for the set M with respect to a g.d.s. in E if there exists a function $a \in K$ such that*

$$d(F(t, t_0, X_0), A) \le a(d(X_0, A)) \quad (7.2)$$

(3) ***Conditionally equiasymptotically stable*** *for the set M with respect to a g.d.s. in E if there exist functions $a \in K^*$ and $b \in L^*$ such that*

$$d(F(t, t_0, X_0), A) \le a(t_0, d(X_0, A)) b(t_0, t - t_0) \quad (7.3)$$

(4) ***Conditionally uniformly asymptotically stable*** *for the set M with respect to a g.d.s. in E if there exist functions $a \in K$ and $b \in L$ such that*

$$d(F(t, t_0, X_0), A) \le a(d(X_0, A)) b(t - t_0) \quad (7.4)$$

WHENEVER (IN THE DEFINITIONS (1) TO (4) ABOVE)

$X_0 \subset M \cap \bar{S}(A, P)$, for some $p > 0$ and for all $t \ge t_0$.

Note:

(1) If $M = E$, the above definitions reduce to S_1, S_3, S_5 and S_7 (of section 2).

(2) These definitions are similar to the ones given in (7). They are expressed here in terms of monotonic functions belonging to the classes: K, K^*, L and L^*.

(3) If M is a neighbourhood of A, then also note (1) above holds.

To obtain sufficient conditions for the conditional stability properties of the set A, we use the comparison techniques based on Vector Lyapunov function.

Let $W = W(t, r)$ be a continuous vector function with components $w_1, w_2, w_3, \ldots, w_n$ in $r = (r_1, r_2, \ldots, r_n)$ so that we write $W \in C(I\ X\ R^n, R^n)$.

W is said to possess quasi-monotone property in r for each fixed $t \in I$, if for each $i = 1, 2, \ldots, n$, the $i-$th component $w_1(t, r)$ is monotonic non-decreasing in r_j, $j \ne i$ for each j.

If W has the quasi-monotone property in r, then the differential system:

$$r' = W(t, r), (' = d/dt) \quad (7.5)$$

has the maximal (in the sense of component-wise majorisation) solution existing to the right of t_0.

W is assumed to be smooth enough that the maximal solution exists for all $t \in [t_0, \infty]$.

Let V be a $n-$vector and $V \in C(I\ X\ A(E), R_+^n)$, where R_+^N the set of $n-$tuples with all components non-negative. Interpreting the vector inequality as being satisfied component-wise,

$$\text{let } V^+(t, X(t)) = \lim_{h \to 0^+} \left[\frac{1}{h} \Big\{ V(t+h, X(t+h)) - V(t, X(t)) \Big\} \right] \quad (7.6)$$

for $(t, X(t)) \in I\ X\ A(E)$.

Theorem 7.1. *Let there exist a vector function V defined above so that V^+ defined above in (7.6) satisfy the vector inequality:*

$$V^+(t, X(t)) \le W(t, V(t, X(t))), \quad t \ge t_0 \quad (7.7)$$

where W is a smooth function having the quasi-monotone property.

Let $r(t, t_0, r_0)$ be the maximal solution of the differential system (7.5), existing to the right of t_0. Then

$$V(t_0, X((t_0)) \le r(t_0, t_0, r_0) = r_0 \quad \text{(i)}$$

$$\text{implies} \quad V(t, X(t)) \le r(t, t_0, r_0) \quad \text{(ii)}$$

(7.8)

We will henceforth (unless otherwise stated) use V and W in **two-dimensions** only. Thys $V = (V_1, V_2)$, $W = (W_1, W_2)$ and the quasi-monotone property of W is now equivalent to W_1 being non-decreasing in r_2 and W_2 being non-decreasing in r_1.

Let $(r_1, r_2) R_+^2$.

Define $|V| = V_1 + V_2$ and $|r| = r_1 + r_2$.

These make sense since V_1, V_2, r_1, r_2 are all non-negative, by definition.

Let

$$r_0 = (r_1, 0) \quad (7.9)$$

Then

$$|r_0| = r_1$$

Let $r(t, t_0, r_0)$ be the maximal solution of (7.5) with r_0 defined in (7.9). Corresponding to the definitions (7.1) (1) to (4) we state the following properties –

Properties 7.1.

(1) – s: *There exists $a \in K^*$, for a given $p > 0$ such that $|r_0| \leq p$ implies*

$$|r(t, t_0, r_0)| \leq a(t_0, |r_0|)t \geq t_0 \tag{7.10}$$

(2) – s: *There exists a function $a \in K$ for a given $p > 0$ such that $|r_0| \leq p$ implies*

$$|r(t, t_0, r_0)| \leq a(|r_0|), t \geq t_0 \tag{7.11}$$

(3) – s: *There exist functions $a \in K^*$ and $b \in L^*$ for a given $p > 0$ such that $|r_0| \leq p$ implies*

$$|r(t, t_0, r_0)| \leq a(t, |r_0|)b(t_0, t - t_0), t \geq t_0 \tag{7.12}$$

(4) – s: *There exist functions $a \in K$ and $b \in L$ for a given $p > 0$ such that $|r_0| \leq p$ implies*

$$|r(t, t_0, r_0)| \leq a(|r_0|)b(t - t_0), t \geq t_0 \tag{7.13}$$

Theorem 7.2. *Let*

$$M = \{X \in A(E) : V_2(t, X) = 0\} \tag{7.14}$$

and V (where (V_1, V_2)) satisfy –

$$b(d(X, A)) \leq V(t, X) \leq a(t, d(X, A)) \tag{7.15}$$

for all $(t, X) \in I\ X\ A(E)$. Further, let the conditions of theorem (7.1) be satisfied.

Then (i) property (7.1): (1) – s implies the conditional equistability of A, and

(ii) property (7.1): (3) – s implies the conditional equiasymptotic stability of A, for the set M with respect to the g.d.s. in E (where M is given by (7.14)*).*

Proof.

(i) By the property of a in (7.15), there exists $p_1 = p_1(t_0, p) > 0$ such that $d(X_0, A) \leq p_1$ and $a(t_0, d(X_0, A)) \leq p$ hold simultaneously.

Choose $r_0 = (r_1, 0)$ with $r_1 = V_1(t_0, X_0)$.

Let $X_0 \in M$. Then $V_2(t_0, x_0) = 0$ by (7.14). If $X_0 \in \bar{S}(A, p_1)$, then $d(X_0, A) \leq p_1$ and the choice of p_1 and r_0 show that

$$r_0 \leq a(t_0, d(X_0, A)) \leq p, a \in K^* \tag{i}$$

Thus (7.8)–(i) of theorem (7.1) is satisfied so that

$$V(t, F(t, t_0, X_0)) \leq r(t, t_0, r_0), t \in t_0 \tag{ii}$$

for $X_0 \in M \cap \bar{S}(A, p_1)$.

This inequality, component-wise would mean

$$V_1(t, F(t, t_0, X_0)) \leq r_1(t, t_0, r_0) \tag{iii}$$

and
$$V_2(t, F(t, t_0, X_0)) \leq r_2(t, t_0, r_0) \tag{iv}$$

From these two and the definition of the norm, we have

$$|V(t, F(t, t_0, X_0))| \leq |r(t, t_0, r_0)| \tag{v}$$

Now

$$\begin{aligned} b(d(F(t, t_0, X_0), A)) &\leq |V(t, F(t, t_0, X_0))| \\ &\leq |r(t, t_0, X_0)| \\ &\leq a_1(t_0, |r_0|) \\ &\leq a_1(t_0, a(t_0, d(X_0, A))) \\ &= a_2(t_0, d(X_0, A)) \end{aligned}$$

so that $d(F(t, t_0, X_0), A) \leq b^{-1-} a_2(t_0, d(X_0, A)) = a_3(t_0, d(X_0, A))$

where $a_3 \in K^*, \quad t \geq t_0.$

Therefore A is conditionally equistable for the set M with respect to a g.d.s. in E.

(ii) Proceeding on the same lines, as above, because of property (7.1):(3)–s, we get

$$\begin{aligned} b(d(F(t, t_0, X_0), A)) &\leq |V(t, F(t, t_0, X_0))| \\ &\leq |r(t, t_0, X_0)| \\ &\leq a_1(t_0, |r_0| b_1(t_0, t - t_0)) \\ &\leq a_1(t_0, a(t_0, d(X_0, A)))b_1(t_0, t - t_0) \\ &= a_2(t_0, d(X_0, A))b_1(t_0, t - t_0) \end{aligned}$$

implying

$$\begin{aligned} d(F(t, t_0, X_0), A) &\leq b^{-1}(a_2(t_0, d(X_0, A))b_1(t_0, t - t_0)) \\ &\leq a_3(t_0, d(X_0, A)b_3(t_0, t - t_0)) \end{aligned}$$

(Here $p_1 = p_1(t_0, p)$ is such that $d(X_0, A) \leq p_1$ and $a(t_0, d(X_0, A)) \leq p$ hold simultaneously).

Thus A is conditionally equiasymptotic stable for M with respect to the g.d.s. in E. □

Theorem 7.3. *Let M be the set defined in* (7.14) *and V satisfy –*

$$b(d(X, A)) \leq V(t, X) \leq a(d(X, A)), t \geq t_0 \tag{7.16}$$

for all $(t, X) \in I\ X\ A(E)$ where $b \in K$ and $a \in K$.

Let the conditions of theorem (7.1) be satisfied.

(i) Property (7.1): (2)–s implies the conditional uniform stability of A, for the set M with respect to the g.d.s. in E and

(ii) Property (7.1): (4)–s implies the conditional uniform asymptotic stability of A, for the set M with respect to the g.d.s. in E.

Proof. The properties of a in (7.16) imply that there exists $p_1 > 0$, $p_1 = p_1(p)$ such that $d(X_0, A) \leq p_1$, and $a(d(X_0, A)) \leq p$ hold simultaneously. Choose $r_0 = (r_1, 0)$ with $r_1 = V(t_0, X_0)$. Let X_0 M so that $V_2(t_0, X_0) = 0$.

If $X_0 \in \bar{S}(A, p_1)$ then $d(X_0, A) \leq p_1$ and the choice of p_1 and r_0 show that

$$r_0 \leq a(d(X_0, A)) \leq p \tag{i}$$

Thus (7.8) – (i) of theorem (7.1) is satisfied so that

$$V(t, F(t, t_0, X_0)) \leq r(t, t_0, r_0), t \geq t_0 \tag{ii}$$

for $X_0 \in M \cap \bar{S}(A, p_1)$.

Component-wise, this implies

$$V_1(t, F(t, t_0, X_0)) \leq r_1(t, t_0, r_0) \quad \text{(iii)}$$

and $$V_2(t, F(t, t_0, X_0)) \leq r_2(t, t_0, r_0) \quad \text{(iv)}$$

which means $$|V(t, F(t, t_0, X_0))| \leq |r(t, t_0, r_0)| \quad \text{(v)}$$

Therefore

$$\begin{aligned} b(d(F(t, t_0, X_0), A)) &\leq |V(t, F(t, t_0, X_0))| \\ &\leq |r(t, t_0, r_0)| \\ &\leq a_1(|r_0|) \\ &\leq a_1(a(d(X_0, A))) = a_2(d(X_0, A)) \end{aligned}$$

where $a_2 = a_1 a \in K$ implying,

$$d(F(t, t_0, X_0), A) \leq b^{-1} a_2(d(X_0, A)) = a_3(d(X_0, A))$$

where $a_3 = b^{-1} a_2 \in K$.

Hence the conditional uniform stability of A for M with respect to the g.d.s. in E.

(ii) The proof of this part is just the same as above, but for the following deviation –

$$b(d(F(t, t_0, X_0), A)) \leq a_1(a(d(X_0, A)))b_1(t - t_0)$$

so that $$\begin{aligned} d(F(t, t_0, X_0), A) &\leq b^{-1}(a_1(a(d(X_0, A))b_1(t - t_0))) \\ &\leq a_3(d(X_0, A))b_3(t - t_0) \end{aligned}$$

Here $p_1 = p_1(p) > 0$. Such that $d(X_0, A) \leq p_1$ and $a(d(X_0, A)) \leq p$ hold simultaneously. Further $a_3 \in K$ and $b_3 \in L$.

Hence the conditional uniform asymptotic stability of A for M with respect to the g.d.s. in E. □

8. CONVERSE THEOREMS FOR THE EXISTENCE OF VECTOR LYAPUNOV FUNCTIONS FOR CONDITIONAL STABILITY.

Let $g = g(t, u)$, $u = (u_1, u_2)$ so that $g = g(t, u_1, u_2)$ be defined and continuous on $I\ X\ R_+^2$ into R^2 and satisfy the quasimonotone decreasing condition in u. g is assumed to be smooth enough to ensure the existence, uniqueness and continuous dependence of the solution on the initial conditions of the equation.

$$u' = g(t, u), \text{ for all } t \in \big[0, \infty\big], t_0 \in I. \quad (8.1)$$

Let

$$\begin{aligned} g(t, u) &= (g_1(t, u), g_2(t, u)) \\ &= (g_1(t, u_1, u_2), g_2(t, u_1, u_2)). \end{aligned}$$

Define $g^*(t, u) = (g_1(t, u_1, u_2), g_2(t, 0, u_2))$.

As $u_1, u_2 \geq 0$, by the quasi-monotone property, it follows that

$$g^*(t, u) \leq g(t, u) \quad (8.2)$$

Suppose $u^*(t, 0, u_0)$ and $u(t, 0, u_0)$ are solutions of

$$u' = g^*(t, u) \quad (8.3)$$

and (8.1) respectively through the same point $(0, u_0), u_0 \in R_+^2$.

Then we have

$$u^*(t, 0, u_0) \leq u(t, 0, u_0), t \in I \quad (8.4)$$

Suppose $p_1 = (u_{10}, 0) \in R_+^2$ and $p_2 = (u_{10}, u_{20}) \in R_+^2$.

Then let the solutions of (8.3) through $(0, p_1)$ and $(0, p_2)$, be denoted $u_1^*(t, 0, p_1)$ and $u_2^*(t, 0, p_2)$ respectively.

Writing these equation component-wise -

$$u_1^*(t, 0, p_1) = (u_{11}^*(t, 0, p_1), u_{12}^*(t, 0, p_1))$$

and $$u_2^*(t, 0, p_2) = (t_{12}^*(t, 0, p_2), u_{22}^*(t, 0, p_2)).$$

Then we have

$$\text{i.e.,} \quad \left.\begin{aligned} u_1^*(t, 0, p_1)\ u_2^*(t, 0, p_2) \\ u_{11}^*(t, 0, p_1)\ u_{21}^*(t, 0, p_2) \\ u_{12}^*(t, 0, p_1)\ u_{22}^*(t, 0, p_2) \end{aligned}\right\} \quad (8.5)$$

Theorem 8.1.

(1) *Let the g.d.s. be r.d.s. and the flow $F(t, t_0, X_0)$, $X_0 \in A(E)$, be Hausdorff continuous in the triplet of its arguments.*

(2) *Let there exist functions $a, b \in K$ such that*

$$b(d(X_0, A)) \leq d(F(t, t_0, X_0), A) \leq a(d(X_0, A)) \quad (8.6)$$

for $X_0 \in M$

(3) *Let $g \in C(I\ X\ R_+^2, R^2)$, $g(t, 0) = 0$ and g has the properties mentioned earlier (viz., existence, uniqueness and continuous depenbdence of solutions (on the initial conditions) of the equation* (8.1)*)*

(4) *The solution $u(t, 0, u_0)$ of* (8.1) *satisfy*

$$u(t, 0, u_0) \leq r_2(|u_0|) \quad (8.7)$$

where $u_0 = u_{20}$ as $u_{10} = 0$, when $u_0 = (u_{10}, u_{20})$.

(5) *The component $u_2^*(t, 0, u_0)$ of the solution $u^*(t, 0, u_0)$ of* (8.3) *has the property:*

$$u_2(t, 0, u_0) \geq r_1(|u_0|) = r_1(u_{20}) \quad (8.8)$$

where u_0 satisfies the definition given in (4) above.

Then there exists function $V = Vt, X)$ with the following properties:

(i) $V \in C(I\ X\ A(E), r_+^2)$

(ii) $V^+(t, X) \leq g(t, V(t, X))$ for the flows X of r.d.s.

(iii) If $X \in M$, then $V_1(t, X) = 0$

(iv) $b_1(bd(X, A)) \leq |V(t, X)| \leq a_1(d(X, A))$

where $a_1, b_1 \in K$ and $(t, X) \in I\ X\ A(E)$.

Proof. The g.d.s. is r.d.s. Therefore $X = F(t, 0, X_0)$ implies $X_0 = F(0, t, X)$

Choosing a function $\mu \in C(A(E), R_+^2)$ such that

$$\alpha_1(d(X, A)) \leq \mu(X) \leq \alpha_2(d(X, A)) \quad \text{(i)}$$

and $$\mu_1(X) = 0 \text{ if } X \in M \quad \text{(ii)}$$

$$\mu(X) = (\mu_1(X), \mu_2(X)).$$

Let $u_1^*(t,0,(\mu_1(X),0))$ and $u_2^*(t,0(\mu_1(X),\mu_2(X)))$ be the solutions of the equation (8.2).

Define

$$\begin{aligned} V_1(t,X) &= u_{11}^*(t,0,(\mu_1(F(0,t,X),0))) \\ &= u_{11}^*(t,0,(\mu_1(X_0),0)) \\ \text{and} \quad V_2(t,X) &= u_{22}^*(t,0,(\mu_1(0,t,X))),\mu_2(F(0,t,X)) \\ &= u_{22}^*(t,0,(\mu_1(X_0),\mu_2(X_0))) \end{aligned}$$

where u_{11}^* and u_{22}^* are first and second components of u_1^* and u_2^* respectively. The continuity of V_1 and V_2 follow from the continuity of u_{11}^* and u_{22}^* with respect to the initial conditions together with the continuity properties of μ and F with respect to their arguments.

Let $X(t) = F(t,t_0,X(t_0))$, so that $X(t+h) = F(t+h,t_0,X(t_0))$.

Also $F(0,t+h,X(t+h)) = F(0,t,x(t)) = X_0$, by the reversibility property. Hence

$$\begin{aligned} V_1^+(t,X(t)) &= \lim_{h\to 0+} \frac{1}{h}\left[u_{11}^*(t+h,0,(\mu_1(X_0),0)) - u_{11}^*(t,0,(\mu_1(X_0),0))\right] \\ &= u_{11}^{\prime *}(t,0,(\mu_1(X_0),0)) \\ &= g_1^*(t,u_{11}(t,0,(\mu_1(X_0),0)),u_{12}^*(t,0,(\mu_1(X_0),0))) \end{aligned}$$

Similarly,

$$v_2^+(t,X(t)) = g_2^*(t,0,u_{22^*}(t,0,(\mu_1(X_0),\mu_2(X_0)))).$$

With the definitions of V_1 and V_2

$$V_1^*(t,X(t)) = g_1^*(t,V_1(t,X(t)),u_{12}^*(t,0,(\mu_1(X_0),0))) \tag{iii}$$

$$\text{and} \quad V_2^+(t,X(t)) = g_2^*(t,0,V_2(t,X(t))) \tag{iv}$$

Now from the inequalities (8.5)

$$\begin{aligned} u_{12}(t,0,(\mu_1(X_0),0)) &\le u_{12}^*(t,0,(\mu_1(X_0),\mu_2(X_0))) \\ &\le V_2(t,X(t)) \end{aligned} \tag{v}$$

$$\text{also trivially,} \quad 0 \le V_1(t,X(t)) \tag{vi}$$

Hence by the quasimonotonicity of g_1 and g_2 we have

$$\begin{aligned} V_1^+(t,X(t)) &\le g_1^*(t,v_1(t,X(t)),V_2(t,X(t))) \\ &\le g_1(t,V_1(t,X(t)),V_2(t,X(t))) \end{aligned} \tag{vii}$$

and

$$\begin{aligned} V_2^+(t,X(t)) &\le g_2^*(t,0,V_2(t,X(t))) \\ &\le g_2(t,0,V_2(t,X(t))) \\ &\le g_2(t,V_1(t,X(t)),V_2(t,X(t))) \end{aligned} \tag{viii}$$

(v) and (vi) verify property (ii).

Property (iii) follows from the proepety (8.7) and the fact that $V_1(t,X) = 0$ if $\mu_1(X) = 0$.

Now

$$\begin{aligned} |V(t,X)| &= V_1(t,X) + V_2(t,X) \\ &= u_{11}^*(t,0,(\mu_1(X_0),0)) + u_{22}^*(t,0,(\mu_1(X_0),\mu_2(X_0))) \\ &\le u_{21}(t,0,(u_1(X_0),\mu_2(X_0))) + u_{22}(t,0,(\mu_1(X_0),\mu_2(X_0))) \\ &\quad r_2(|\mu(X_0)|), \text{ by hupothesis (4) and } \mu(X_0) \\ &\qquad = 0 \text{ implies } X_0 \in M. \\ &= r_2(\alpha_2(d(X_0,A))) = r_2(\alpha_2 b^{-1}(d(X,A))) \\ &= a_1(d(X,A)), a_1 \in K \end{aligned} \tag{ix}$$

$$\begin{aligned} V(t,X) &= V_1(t,X) + V_2(t,X) \\ &\ge V_2(t,X) \\ &= u_{22}(t,0),(\mu_1(X_0),\mu_2(X_0)) \\ &\ge r_1(\mu(X_0)) \text{ by hypothesis (5) and } \mu(X_0) = 0 \text{ implies } X_0 \in M. \\ &\ge r_1(\alpha_1(d(X_0,A))) \\ &\ge r_1(\alpha_1 a^{-1}(d(X,A))) = b_1(d(X,A)) \text{ where } b_1 \in K. \end{aligned} \tag{x}$$

(ix) and (x) together verify the property (iv). Hence the theorem. □

Note:

(1) It is to be noted that the theorem just proved is not strictly a converse for either of the theorems (7.1) and (7.2). We find that the hypothesis (2) on the estimates for $d(F(t,0,X_0),A)$ imply strict conditional stability for the set A with respect to the set M. Similarly the condition (4) corresponds to property (7.1) - (2) -s for (8.1), but we also require condition (5), which is compatible with the property (7.1)-(2)-s. Similar remarks hold for the theorem (8.2) stated below.

(2) Using the notion of mini-max solutions for a system, we can obtain theorems that will give strict conditional stability for the set A.

(3) Theorem (8.1) can be considered as the extension of theorem 4.5.1 of (32) on conditional stability of ordinary differential system to reversible dynamical system. The results are special cases fo theorems (4.5.2), (4.5.3) and (4.5.4) from the reference.

we can also prove the following extension of theorem (4.5.2) to reversible dynamical system and a simply state the theorem without proof.

Theorem 8.2. *Let the assumptions (1) and (3) of theorem (8.1) hold. Assume further that*

(a) *there exist functions $b_1, b_2 \in K$, $c_1, c_2 \in L$ such that, for $X_0 \in M$*

$$b_1(d(X_0,A))c_1(t) \le d(F(t,0,X_0),A) \le b_2(d(X_0,A))c_2(t) \, for \, t \ge 0$$

(b) *the solution $u(t,0,u_0)$ of (8.1) satisfy the condition*

$$u(t,0,u_0) \le r_2(|u_0|)s_2(t), t \ge 0, r_2 \in K, s_2 \in L$$

where $u_0 = (u_{10},u_{20})$ and $u_{10} = 0$.

(c) *the component $u_{22}^*(t,0,u_0)$ of the solution of equation (8.3) satisfy the condition*

$$u_{22}^*(t,0,u_0) \ge r_1(u_0)s_1(t),$$

with u_0 satisfying conditions in (b), and $r_1 \in K$ and $s_1 \in L$

(d) *$r_1(r)$ is differentiable and $r_1'(r) \ge \lambda > 0$*

(e) s_1 *and* c_2 *are such that* $s_1(t) \geq \lambda_1 c_2(t)$, $t \geq t_0$, $\lambda_1 > 0$

Then there exists a function $V(t, X)$ with properties (i) (ii) and (iii) of theorem (8.1) and

$$b(d(X, A)) \leq V(t, X) \leq a(t, d(X, A))$$

where $b \in K$, and $a \in K^*$. This theorem shows the existence of a Lyapunov function for asymptotic conditional stability.

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