Strong Non-Split Geodetic Number of a Line Graph

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ABSTRACT
A Set $S \sqsubseteq V[L(G)]$ is a strong non split geodetic set of $L(G)$, if 'S' is a geodetic set and $\langle V - S\rangle$ is complete. The strong non split geodetic number of a line graph $L(G)$, is denoted by $g_{sns}[L(G)]$, is the minimum cardinality of a strong non split geodetic set of $L(G)$. In this paper we obtain the strong non split geodetic number of line graph of some special graph and many bounds on strong non split geodetic numbers in terms of elements of G.

Keywords
Tadpole graph, Banana tree graph, Helm graph, Line graph, strong non split geodetic number of a line graph.

1. INTRODUCTION
In this paper we follow notations of [1]. As usual $n = |V|$ and $m = |E|$ denote the number of vertices and edges of a graph G respectively.

The graph considered here have at least one component which is not complete or at least two non-trivial components.

For any graph G(V, E) the line graph $L(G)$ whose vertices correspond to the edges of G and two vertices in $L(G)$ are adjacent if and only if the corresponding angles in G are adjacent. The distance $d(u, v)$ between two vertices u and v in a connected graph G in the length of a shortest $u - v$ path in G. It is well known that this distance is a metric on the vertex set $V(G)$. For a vertex v of G, the eccentricity $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is radius, rad G, and the minimum eccentricity is the diameter, diam G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodetic. We define $I[u, v]$ to the set (interval) of all vertices lying on some $u - v$ geodesic of G and for a nonempty subset S of $V(G)$, $I(S) = \cup_{u,v\in S} I[u, v]$

A set S of vertices of G is called a geodetic set in G if $I(S) = V(G)$, and a geodetic set of minimum cardinality is a minimum geodetic set. The cardinality of a minimum geodetic set in G is called the geodetic number of G, and we denote it by $g(G)$.

Strong non split geodetic number of a graph was studied in [5]. A geodetic set S of a graph $G = V, E$ is a non split geodetic set if the induced sub graph $\langle V - S\rangle$ is connected. The non-split geodetic number $g_{ns}(G)$ of G is the minimum cardinality of a non-split geodetic set. A set S' of vertices of $G = (V, E)$ is called the strong non split geodetic set if the induced sub graph $\langle V - S'\rangle$ is complete and a strong non split geodetic number is denoted by $g_{sns}(G)$. Geodetic number of a line graph was studied by in [3]. Geodetic number of a line graph $L(G)$ of G is a set S' of vertices of $L(G) = H$ is called the geodetic set in H if $I(S') = V(H)$ and a geodetic set of minimum cardinality is the geodetic number of $L(G)$ and is denoted by $g[L(G)]$. Now we define strong non split geodetic number of a line graph. A set S' of vertices of $L(G) = H$ is called the strong non split geodetic set in H if the induced subgraph $\langle V(H) - S'\rangle$ is complete and a strong non split geodetic set of minimum cardinality is the strong non split geodetic number of $L(G)$ and is denoted by $g_{sns}[L(G)]$.

Tadpole Graph: The (m, n) tadpole graph is a special type of graph consisting of a cycle graph on m (at least 3) vertices and a path graph on n vertices connected with a bridge preliminaries geodetic number of Tadpole graph denoted by $(T_{m,n})$.

A helm graph, denoted by H_n is a graph obtained by attaching a single edge and vertex to each vertex of the C_{n-1} of a wheel graph W_n.

Banana tree as defined by chen et al(1997) is a graph obtained by connecting one leaf of each of copies of an star graph with a single root vertex that is distinct from all the stars.

For any undefined terms in this paper, see [1] and [2].

2. PRELIMINARY NOTES
We need the following results to prove further results

Theorem 2.1 [4] Every geodetic set of a graph contains its external vertices.

Theorem 2.2 [4] For any path P_n, with n vertices, $g[L(P_n)] = 2$.

Theorem 2.3 [4] For the wheel $W_n = K_1 + C_{n-1}, (n \geq 6)$

$$g[L(W_n)] = \begin{cases} \frac{n}{2} & if\ n\ is\ even \\ \frac{n+1}{2} & if\ n\ is\ odd \end{cases}$$

Theorem 2.4 [4] For any cycle C_n of order $n \geq 3$ $g[(C_n)] = \begin{cases} 2 & if\ n\ is\ even \\ 3 & if\ n\ is\ odd \end{cases}$

Proposition 1 Line graph of a cycle is again a cycle.

3. MAIN RESULTS
Theorem 3.1. For complete bipartite graph

$$g_{sns}[L(K_{2,n})] = \begin{cases} \frac{3n}{2} & if\ n\ is\ even \\ \frac{3n-1}{2} & if\ n\ is\ odd \end{cases}$$

Proof: for $n > 1$ we have the following cases

Let $U = \{u_1, u_2, \dots\dots u_i\}$ are the vertices of $V[L(k_{2,n})]$ formed from edges of one set of vertices of $k_{2,n}$ i.e., $U \sqsubseteq V[L(K_{2,n})]$ and $W = \{w_1, w_2, \dots w_i\}$ are the vertices of $V[L(k_{2,n})]$ formed from edges of other set of vertices of $K_{2,n}$ i.e., $W \sqsubseteq V[L(K_{2,n})]$

Case 1: Let n be even; let $S = \{u_1, u_2, \dots u_k, w_1, w_2, \dots w_l\}$ be the geodetic set consisting of $\frac{n}{2}$ vertices from the set U and $\frac{n}{2}$ vertices from the set W such that $\langle V[L(K_{2,n})] - S\rangle$ is connected. Further $S' = S \cup X$ where $X \sqsubseteq U\ or\ W$. clearly $\langle V[L(K_{2,n})] - S'\rangle$ is complete graph. Thus S' is the minimum strong non split geodetic set of $L(K_{2,n})$.

$\Rightarrow |S'| = |S \cup X|$

$\Rightarrow |S'| = |S| + |X|$

$\Rightarrow |S'| = n + \frac{n}{2}$

$\Rightarrow g_{sns}[L(K_{2,n})] = \frac{3n}{2}$

Case 2: Let n be odd; let $S = \{u_1, u_2, \dots u_k, w_1, w_2, \dots w_l\}$ where $l < k$ be the geodetic set consisting of $n - l$ vertices from the set U and $n - k$ vertices from the set W such that $\langle V[L(K_{2,n})] - S\rangle$ is connected. Further $S' = S \cup X$ where $X \sqsubseteq W$. clearly $\langle V[L(K_{2,n})] - S'\rangle$ is complete graph. Thus S' is the minimum strong non split geodetic set of $L(K_{2,n})$.

$\Rightarrow |S'| = |S \cup X|$

$\Rightarrow |S'| = |S| + |X|$

$\Rightarrow |S'| = n + \frac{n-1}{2}$

$\Rightarrow g_{sns}[L(K_{2,n})] = \frac{3n-1}{2}$

Theorem 3.2 For any path of order $\boldsymbol{n \geq 5}$ $\boldsymbol{g_{sns}[L(P_n)] = n - 3}$.

Proof: Let $V = \{v_1, v_2, \dots \dots . . v_n\}$ be the set of vertices in a path P_n consider a geodetic set $S = \{v_1, v_n\}$ of P_n such that $\langle V[L(P_n)] - S\rangle$ is connected and also we have diam $(v_1, v_n) = d, thus$ 'S' is not a strong non split geodetic set of P_n. Further we consider a set $S' = S \cup U$ where $U \subseteq V[L(P_n)] - S$ having $n - 5$ vertices. Thus S' is a minimum set of vertices such that $V[L(P_n)] = I[S']$ and the set of vertices of subgraph $\langle V[L(P_n)] - S'\rangle$ is complete. Hence S' is a strong non split geodetic set of P_n. Clearly it follows that

$|S'| = |S \cup H|$

$\Rightarrow |S'| = |S| + |S \cup H| = 2 + n - 5 = n - 3$

$\Rightarrow g_{sns}(P_n) = n - 3.$

Theorem 3.3 For any cycle $\boldsymbol{C_n}$ of order $\boldsymbol{n > 5}$, $\boldsymbol{g_{sns}[L(C_n)] = n - 2}$

Proof: For $n > 5$, we have the following cases.

Case1: Let n be even: consider $\{v_1, v_2, \dots \dots . v_n, v_1\}$ be a cycle with $'n'$ vertices. Let S be the geodetic set of $L(C_n)$ therefore by theorem 2.4 $g(C_n) = g[L(C_n)] = 2$. Further we consider a set $S' = S \cup H$ where H having $n - 4$ vertices. Clearly, $V[L(C_n)] - S'$ is complete. Thus S' is a minimum strong non split geodetic set of $L(C_n)$. It follows that $|S'| = |S \cup H|$

$\Rightarrow |S'| = |S| + |H|$

$\Rightarrow |S'| = 2 + n - 4$

$\Rightarrow |S'| = n - 2$

$\Rightarrow g_{sns}[L(C_n)] = n - 2$

Theorem 3.4 For Tadpole graph $\boldsymbol{m \geq 3}$, $\boldsymbol{n \geq 2}$ $\boldsymbol{g_{sns}[L(T_{m,n})] = m + n - 3}$

Proof: Let $T_{m,n} = C_m + P_n$ connected by a bridge and let $U = \{u_1, u_2, u_3, \dots \dots u_m\}$ are the vertices of $L[T_{m,n}]$ forward from edges of C_m i.e., $U \sqsubseteq V[L(T_{m,n})]$ and $W = \{w_1, w_2, w_3, \dots \dots w_n\}$ are the vertices of $L[T_{m,n}]$ formed from edges of P_n of $T_{m,n}$ i.e., $W \sqsubseteq V[L(T_{m,n})]$ for $m \geq 3$ & $n \geq 2$. We have the following cases.

Case 1: For m is odd

Let $S = \{u_j\, w_i\}$ be the set of geodetic set of $L[T_{m,n}]$ such that $\langle V[L(T_{m,n})] - S\rangle$ is connected.

Further consider set $S' = S \cup A \cup B$ where $A \subseteq U$ having $m - 3$ vertices and $B \subseteq W$ having $n - 2$ vertices, clearly $\langle V[L(T_{m,n})] - S'\rangle$ is complete. Thus S' is a minimum strong non split geodetic set of $L(T_{m,n})$ it follows that

$|S'| = |S \cup A \cup B|$

$\Rightarrow |S'| = |S| + |A| + |B|$

$\Rightarrow |S'| = 2 + m - 3 + n - 2$

$\Rightarrow g_{sns}[L(C_{m,n})] = m + n - 3$

Case 2: For m is even

Let $S = \{u_1, u_2, w_1\}$ be the set of geodetic set of $L[T_{m,n}]$ such that $\langle V[L(T_{m,n})] - S\rangle$ is connected.

Further consider the set $S' = S \cup A \cup B$ where $A \subseteq U$ having $m - 4$ vertices and $B \subseteq W$ having $n - 2$ vertices, clearly $\langle V[L(T_{m,n})] - S'\rangle$ is complete. Thus S' is a minimum strong non split geodetic set of $L(T_{m,n})$. It follows that $|S'| = |S \cup A \cup B|$

$\Rightarrow |S'| = |S| + |A| + |B|$

$\Rightarrow |S'| = 2 + m - 3 + n - 2$

$\Rightarrow g_{sns}[L(T_{m,n})] = m + n - 3.$

Theorem 3.5. For any Banana Tree for $n \geq 2$ $g_{sns}[L(B_{n,k})] = m + n - 3.$

Proof: Let $B_{n,k}$ is a banana tree connecting one leaf of each of n-copies of an k-star graph with a single root vertex that is distinction from all the stars.

$U = \{v_1, v_2, \dots \dots . v_i\}$ are the vertices of $L(B_{n,k})$ formed from pendent edges of $B_{n,k}$ i.e., $U \sqsubseteq V[L(B_{n,k})]$, $w = \{w_1, w_2, \dots . w_j\}$are the vertices of $L(B_{n,k})$ formed from internal edges of $B_{n,k}$ that are connected to n-copies, i.e., W⊑ $V[L(B_{n,k})]$ and $X = \{x_1, x_2, \dots . x_t\}$ are the vertices of $L(B_{n,k})$ formed from internal edges of $B_{n,k}$ that are connected to single root vertex i.e., $X \sqsubseteq V[L(B_{n,k})]$.

Let $S = \{u_1, u_2, \dots . u_i\}$ be the geodetic set of $L(B_{n,k})$ such that $\langle V[L(B_{n,k})] - S\rangle$ is connected. Further consider $S' = S \cup W$, clearly $\langle V[L(B_{n,k})] - S'\rangle$ is complete, we obtain n-complete graph. Thus S' is a minimum strong non split geodetic set of $L[B_{n,k}]$

$\Rightarrow |S'| = |S \cup W|$

$\Rightarrow |S'| = |S| + |W|$

$\Rightarrow |S'| = n(k - 2) + n$

$\Rightarrow |S'| = nk - 2n + n$

$\Rightarrow |S'| = nk - n$

$\Rightarrow g_{sns}\,[L(B_{n,k})] = n(k-1).$

Theorem 3.6. For the wheel $W_n = K_1 + C_{n-1}\ (n > 3)$

$$g_{sns}\,[L(W_n)] = \begin{cases} n & if\ n\ is\ even \\ n+1 & if\ n\ is\ odd \end{cases}$$

Let $W_n = K_1 + C_{n-1}\ (n > 3)$ and let $V(W_n) = \{x, v_1, v_2, \dots v_{n-1}\}$, where $\deg(x) = n-1 > 3$ and $\deg(v_i) = 3$ for each $i \in \{1,2, \dots n-1\}$. Now $U = \{u_1, u_2, \dots u_j\}$ are the vertices of $L(W_n)$ formed from the edges of C_{n-1} $i.e.,\ U \sqsubseteq V[L(W_n)]$ and $W = \{w_1, w_2, \dots w_j\}$ are the vertices of $L(W_n)$ formed from internal edges of $W_n\ i.e., W \leq V[L(W_n)]$. we have the following cases.

Case 1: For n is even:

Let $S = \{u_1, u_2, \dots u_k, w_j\}$ where $1 \leq k \leq j$ forms the minimum geodetic set of $L(W_n)$ such that $\langle V[L(W_n)] - S\rangle$ is connected. Further $S' = S \cup X$ where $X \sqsubseteq U$. Clearly $\langle V[L(W_n)] - S'\rangle$ is complete graph. Thus S' is the minimum strong non split geodetic set of $L(W_n)$.

$\Rightarrow |S'| = |S \cup X|$

$\Rightarrow |S'| = |S| + |X|$

$\Rightarrow |S'| = \frac{n}{2} + \frac{n}{2}$

$\Rightarrow g_{sns}\,[L(W_n)] = n$

Case 1: For n is odd:

Let $S = \{u_1, u_2, \dots u_k, w_{j-1}, w_j\}$ where $1 \leq k \leq j$ forms the minimum geodetic set of $L(W_n)$, such that $\langle V[L(W_n)] - S\rangle$ is connected. Further $S' = S \cup X$ where $X \sqsubseteq U$. Clearly $\langle V[L(W_n)] - S'\rangle$ is complete graph. Thus S' is the minimum strong non split geodetic set of $L(W_n)$.

$\Rightarrow |S'| = |S \cup X|$

$\Rightarrow |S'| = |S| + |X|$

$\Rightarrow |S'| = \frac{n+1}{2} + \frac{n+1}{2}$

$\Rightarrow g_{sns}\,[L(W_n)] = n+1.$

Theorem 3.7 For the wheel $W_n = K_1 + C_{n-1}\ (n > 3)$

$$g_{sns}\,[L(W_n)] = \begin{cases} \Delta + \delta - 2 & if\ n\ is\ even \\ \Delta + \delta - 1 & if\ n\ is\ odd \end{cases}$$

Proof: Let $W_n = K_1 + C_{n-1}\ (n > 3)$ and let $V(W_n) = \{x, v_1, v_2, \dots v_{n-1}\}$, where $\deg(x) = n-1 > 3$ and $\deg(v_i) = 3$ for each $i \in \{1,2, \dots n-1\}$. Maximum degree(Δ) of W_n is $n-1$ and minimum degree () is 3. We have the following cases.

Case 1: Let n be even:

We have from case 1 of Theorem 3.6

$\Rightarrow g_{sns}\,[L(W_n)] = n$

$\Rightarrow g_{sns}\,[L(W_n)] = (n-1) + 3 - 2$

$\Rightarrow g_{sns}\,[L(W_n)] = n + \delta - 2$

Case 2: Let n be odd:

We have from case 2 of Theorem 3.6

$\Rightarrow g_{sns}\,[L(W_n)] = n+1$

$\Rightarrow g_{sns}\,[L(W_n)] = (n-1) + 3 - 1$

$\Rightarrow g_{sns}\,[L(W_n)] = n + \delta - 1.$

Theorem 3.8 For Helm graph, $n > 4$, $g_{sns}[L(H_n)] = 2(n-1)$.

Proof: Helm graph is obtained by attaching a single edge and vertex to each vertex of the C_{n-1} of a wheel graph $W_n = K_1 + C_{n-1}$. Let $U = \{u_1, u_2, \dots u_{n-1}\}$ are the vertices of $L(H_n)$ formed from pendent edges i.e $U \sqsubseteq V[L(H_n)]$, $W = \{w_1, w_2, \dots w_{n-1}\}$ are the vertices of $L(H_n)$ formed from edges of C_{n-1} i.e $W \sqsubseteq V[L(H_n)]$ and $X = \{x_1, x_2, \dots x_l\}$ are the vertices of $L(H_n)$ formed from internal edges of W_n i.e $X \sqsubseteq V[L(H_n)]$. Let $S = \{u_1, u_2, \dots, u_{n-1}\} = U$ forms minimum geodetic set of $L(H_n)$, such that $\langle V[L(H_n)] - S\rangle$ is connected. Further $S' = S \cup W$, clearly $\langle V[L(H_n)] - S'\rangle$ is complete graph. Thus S' is the minimum strong non split geodetic set of $L(H_n)$.

$\Rightarrow |S'| = |S \cup W|$

$\Rightarrow |S'| = |S| + |W|$

$\Rightarrow |S'| = n - 1 + n - 1$

$\Rightarrow g_{sns}\,[L(W_n)] = 2(n-1).$

4. ADDING AN END-EDGE

Definition: For an edge $e = \{u, v\}$ of a graph G with deg(u)=1 and deg(v)>1, we call e an end edge and u an end vertex. Let G' be the graph obtained by adding an end -edge {u, v} to a cycle $C_n = G$ of order n>3, with $u \in G\ and\ v \notin G$, we have the following results.

Theorem 4.1 Let G' be the graph obtained by adding end-edges $\{u_i, u_j\}$, $i = 1,2,3 \dots, n$, $j = 1,2,3, \dots k$ to each vertex of $G = C_n$ of order n>3 such that $u_i \in G\ and\ u_j \notin G$ then $g_{sns}[L(G')] = k + n - 2$.

Proof: Let $U = \{u_1, u_2, \dots u_k\}$ are the vertices of $L(G')$ formed from the pendent edges of G' i.e $U \subseteq V[L(G')]$ and $W = \{w_1, w_2, \dots w_n\}$ are the vertices of $L(G')$ formed from the edges of C_n, clearly $S = \{u_1, u_2, \dots u_k\} = U$ be the minimum geodetic set. Further $S' = S \cup X$ where $X \sqsubseteq W$, clearly $\langle V[L(G')] - S'\rangle$ is complete graph. Thus S' is the minimum strong non split geodetic set of $L(G')$.

$\Rightarrow |S'| = |S \cup X|$

$\Rightarrow |S'| = |S| + |X|$

$\Rightarrow |S'| = k + n - 2$

$\Rightarrow g_{sns}\,[L(G')] = k + n - 2.$

5. CONCLUSION

In this paper I have established many results on strong non split geodetic number of some special graph and some observations.

6. ACKNOWLEDGMENTS

Our thanks to the referees who helped to improve the research paper, experts who have contributed towards development of the template

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