# TM-Algebra - An Introduction

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## ABSTRACT

In this paper, a new notion, called TM-algebra, which is a generalization of the idea of Q/BCH/BCI/BCK/BCC-algebra, is introduced. Some theorems discussed in Q-and BCK algebras are generalized. Definition of TM-algebra along with various propositions are stipulated and presented with their respective proofs. The relation between TM-algebra and other algebra has been investigated and detailed in the paper. Also ideal, p-radical, p-semi simple are discussed .

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## Keywords

TM-Algebra, ideal, p-radical, p-semi simple.

## 1. INTRODUCTION

J.Neggers and Kim (see [1] ) introduced the notion of d-algebras which is a generalization of BCK- algebras. Also Joseph Neggers, Sun Shin Ahn, and Hee Sik Kim introduced Q-algebras[2], which is a generalization of BCH/BCI/BCK-algebras, and generalized some theorems from the theory of BCI-algebras. Imai and Iseki introduced two classes of abstract algebras BCI-algebras and BCK-algebras [3, 4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI- algebra. In [5, 6] Hu and Li introduced a wide class of abstract algebra namely BCH- algebras., At the same time, Jun, Roh, and Kim [7] introduced a new notion called BH-algebra, which is a generalization of BCH / BCI / BCK-algebra. In this paper a new notion, called TM-algebra, which is a generalization of BCH /BCI/BCK/Q-algebras, is introduced and some theorems of BCI/ BCK/Q-algebras are generalized..

## 2. TM-ALGEBRA

### 2.1 Definition

A TM-algebra $(X, *, 0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms :

(i) $x * 0 = x$ for $x \in X$

(ii) $(x * y) * (x * z) = z * y$ for $x, y, z \in X$

In $X$ we can define a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 0$

### 2.2 Proposition

*If $(X, *, 0)$ is a TM-algebra, then i) $x * x = 0$ ;*

*ii) $(x * y) * x = 0 * y$*

*iii) $x * (x * y) = y$ for any $x, y \in X$*

**Proof**

i) $x * x = (x * 0) * (x * 0)$ by (i) of definition 2.1

$= 0 * 0,$ by (ii) of definition 2.1

$= 0$

ii) $(x * y) * x = (x * y) * (x * 0)$ by (i) of definition 2.1

$= 0 * y$ by (ii) of definition 2.1

iii) $x * (x * y) = (x * 0) * (x * y)$ by (i) of definition 2.1

$= y * 0$ by (ii) of definition 2.1

$= y$ by (i) of definition 2.1

### 2.3 Proposition

*Let $(X, *, 0)$ be a TM-algebra. Then*

*$(x * y) * z = (x * z) * y$ for any $x, y, z \in X$*

**Proof**

Given $(X, *, 0)$ is a TM- algebra. Then

$(x * y) * (x * z) = z * y$. ( 2.3.1 )

Put $z = x * y$ and $y = z$ in ( 2.3.1 ). Then

$(x * y) * z = (x * z) * (x * (x * y))$

$= (x * z) * y$, by (iii) of proposition 2.2

## 2.4 Proposition

*Let $(X, *, 0)$ be a TM-algebra. Then for any $x, y, z \in X$*

*i)* $x * 0 = 0 \Longrightarrow x = 0$;

*ii)* $(x * z) * (y * z) \leq x * y$

*iii)* $x \leq y \Longrightarrow x * z \leq y * z$ *and* $z * y \leq z * x$

*iv)* $x * (x * (x * y)) = x * y$

*v)* $0 * (x * y) = y * x = (0 * x) * (0 * y)$

*vi)* $(x * (x * y)) * y = 0$ ;

*vi )* *If* $x * y = 0$, $y * x = 0$ *then* $x = y$ .

**Proof**

i) $x * 0 = x$ , by (i) of definition 2.1.

If $x * 0 = 0$, then $x = 0$ , proves the result.

ii) $((x * z) * (y * z)) * (x * y)$

$= ((x * z) * (x * y)) * (y * z)$ by proposition 2.3

$= (y * z) * (y * z)$ by (ii) of definition 2.1

$= 0$ by proposition 2.2

Hence $(x * z) * (y * z) \leq x * y$.

iii) $x \leq y \Longrightarrow x * y = 0$.

To prove $x * z \leq y * z$ .

That is to prove $(x * z) * (y * z) = 0$.

By (ii) of this proposition , $(x * z) * (y * z) \leq x * y$

Since $x * y = 0$, $(x * z) * (y * z) = 0$.

Similarly, $(z * y) * (z * x) = 0$.

Hence (iii) follows.

iv) $x * (x * (x * y)) = (x * 0) * (x * (x * y))$

by (i) of definition 2.1

$= (x * y) * 0$ by (ii) of definition 2.1

$= x * y$ by (i) of definition 2.1

v) $0 * (x * y) = (x * x) * (x * y)$ by proposition 2.2

$= y * x$ by (ii) of definition 2.1

$= (0 * x) * (0 * y)$ by (ii) of definition 2.1

vi) $(x * (x * y)) * y = (x * y) * (x * y)$ by proposition 2.3

$= 0$ by ( iii ) of proposition 2.2

vii) $x = x * 0 = x * (x * y) = y$ (or)

$y = y * 0 = y * (y * x) = x$ .

Recently, Ahn & Kim introduced the notion of **QS-algebras.**

A **QS-algebra** is obviously a **TM–algebra**, But a **TM- algebra** is said to be a **QS-algebra** if it satisfies the additional relations,

$(x * y) * z = (x * z) * y$ and $y * z = z * y$ for all $x, y, z \in X$.

## 2.5 Example

Let $Z$ be the set of all integers, and let $nZ = \{nx: x \in Z\}$, $n \in Z$. Then $(Z, -, 0)$ and $(nZ, -, 0)$ are TM-algebras (where "–" is the usual subtraction).

**Solution**

$x - 0 = x$ for all $x \in Z$, and

$(x - y) - (x - z) = z - y$, for all $x, y, z \in Z$.

Hence $(Z, -, 0)$ is a TM –algebra.

Similarly we prove $(nZ, -, 0)$ is a TM–algebra.

## 2.6 Example

Let $X = \{0,1, 2, 3\}$ be a set with cayley table (Table 1).

**Table 1**

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 2 | 1 | 0 |

Then $(X, *, 0)$ is a TM-algebra.

The relations between TM-Algebra and other algebras are investigated and presented below.

## 2.7 Theorem

*Every BCK-algebra is a TM-algebra but the converse is not true.*

The above example in 2.6 is a TM-algebra but not BCK - algebra since $0 * x \neq 0$ for all $x = 1, 2, 3$.

## 2.8 Theorem

*Every TM -algebra is a BH - algebra, but the converse is not true. Similarly, every TM- algebra is a Q-algebra, but the converse is not true, as shown in Table 2.*

Let, $X = \{0, 1, 2, 3\}$

**Table 2**

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| 2 | 2 | 0 | 0 | 0 |
| 3 | 3 | 3 | 3 | 0 |

Note that $(X, *, 0)$ is a Q-algebra.

The condition $(x * y) * (x * z) = z * y$ is not satisfied as

$(1 * 2) * (1 * 3) = 0 * 0 = 0 \neq 3 = 3 * 2$.

## 2.9 Theorem

*Every TM-algebra is a BCH-algebra. Every BCH-algebra satisfying $(x * y) * (x * z) = z * y$ is a TM-algebra .*

## 2.10 Theorem

*Every TM- algebra is a BCI-algebra.*

## 2.11 Theorem

*Every TM-algebra $X$ satisfying $x * z = z$ is a trivial algebra.*

## Proof

Put $x = z$ in $x * z = z$, then $z * z = z$ which implies $0 = z$

Hence $X$ is a trivial algebra.

## 2.12 Lemma

*Let $(X, * 0)$ be a TM-algebra. Then $a * b = a * c$ for all $a,b,c \in X$ implies $0 * b = 0 * c$ .*

## Proof

Since $(x * y) * z = (x * z) * y$

$(a * b) * a = (a * a) * b = 0 * b$

and $(a * c) * a = (a * a) * c = 0 * c$.

Therefore $a * b = a * c \implies 0 * b = 0 * c$.

## 2.13 Definition

Define $\boldsymbol{G(X)} = \{ x \in X : 0 * x = x \}$.

For any TM- algebra $(X, *, 0)$ the set $\boldsymbol{B(X)} = \{ x \in X : 0 \leq x \}$ is called the ***p*-radical** of $X$. If $B(X) = \{0\}$, then $X$ is said to be a ***p*-semisimple** TM-algebra.

Note that $G(X) \cap B(X) = \{0\}$

## 2.14 Proposition.

*$X$ is p-semisimple if and only if $0 * (0 * x) = x$ for all $x \in X$.*

## Proof

Let $X$ be $p$- semisimple. Then $B(X) = \{ 0 \}$.

That is $0 * 0 = 0$ and $0 * x \neq 0$ for all $x \neq 0$.

Now, $0 * (0 * x) = (0 * 0) * (0 * x) = x * 0 = x$.

Conversely, let $0 * (0 * x) = x$.

Then $0 * (0 * x) = x$ implies $0 * x \neq 0$.

If $0 * x = 0$ for $x \neq 0$, then $0 * 0 = x$ which implies $x = 0$.

Hence $X$ is $p$–semisimple.

## 2.15 Proposition

*If $(x, *, 0)$ is a TM-algebra and $x, y \in X$ then $y \in B(X) \Leftrightarrow (x * y) * x = 0$.*

## Proof

Let $y \in B(X)$. Then $0 * y = 0$.

Now $(x * y) * x = (x * y) * (x * 0)$

$= 0 * y$ by (ii) of definition 2.1

$= 0$, since $y \in B(X)$.

Conversely,

let $(x * y) * x = 0$. Then

$0 = (x * y) * x$

$= (x * y) * (x * 0)$

$= 0 * y$

$\Rightarrow y \in B(X)$.

## 2.16 Proposition.

*Let $(X, *, 0)$, be a TM-Algebra. If $G(X) = X$, then $X$ is p- semi-simple.*

## Proof

Let $G(X) = X$. Also $G(X) \cap B(X) = \{0\}$.

So $X \cap B(X) = \{0\}$. That is $B(X) = \{0\}$.

Hence $X$ is P-semi simple.

Note that a TM-Algebra in which $G(X) = X$ is a **d-algebra.**

## 2.17 Definition

Let $(X, *, 0)$ be a TM–algebra. A non-empty subset $I$ of $X$ is called an **ideal** of $X$ if it satisfies

i) $0 \in I$

ii) $x * y \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in X$.

Any ideal $I$ has the property that $y \in I$ and $x \leq y$ imply $x \in I$.

**Table 3**

| * | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | c | b |
| a | a | 0 | c | b |
| b | b | b | 0 | c |
| c | c | c | b | 0 |

Then the set $I = \{0, a\}$ is an ideal of $X$.

## 2.18 Proposition

*Let $(X, *, 0)$ be a TM-algebra. Then $B(X)$ is an ideal of $X$.*

## Proof

Since $0 * (0 * 0) = 0$, $0 \in B(X)$.

Let $x * y \in B(X)$ and $y \in B(X)$, then

$0 * (x * y) = 0$ and $0 * y = 0$

Now, by proposition (2.14) $y \in B(X)$ implies $(x * y) * x = 0$

and $x * y \in B(X)$ implies $(x * (x * y)) * x = 0$.

$\Rightarrow (x * (x * y)) * (x * 0) = 0$

$\Rightarrow 0 * (x * y) = 0$

$\Rightarrow (0 * x) * (0 * y) = 0$

$\Rightarrow (0 * x) * 0 = 0$

$\Rightarrow (0 * 0) * x = 0$

$\Rightarrow 0 * x = 0$

$\Rightarrow x \in B(X)$.

Therefore $B(X)$ is an ideal of $X$.

## 2.19 Definition

A non-empty subset $S$ of a TM-algebra $(X, *, 0)$ is said to be a sub-algebra of $X$ if $x * y \in S$ whenever $x, y \in S$.

## 2.20 Definition

An ideal $A$ of a TM-algebra $(X, *, 0)$ is said to be closed if $0 * x \in A$ for all $x \in A$.

## 2.21 Proposition

*Every closed ideal of a TM-algebra is a TM- sub-algebra.*

## Proof

Let $A$ be a closed ideal of a TM-algebra $(X *, 0)$.

Let $x, y \in A$ Then

$0 * x,\ 0 * y \in A$.

As $0 * (x * y) = (0 * x) * (0 * y)$, $0 * (x * y) \in A$.

Hence $x * y \in A$. So $A$ is a sub-algebra of TM-algebra.

Note that the converse of the above proposition is not true.

## 2.22 Definition

An ideal $A$ of a TM-algebra $(X, *, 0)$ is said to be **translational ideal** of $X$ if whenever $x * y \in A$, $y * x \in A$, then $(x * z) * (y * z) \in A$ and $(z * x) * (z * y) \in A$ for all $x, y, z \in X$.

## 2.23 Proposition

*If $S$ is a sub algebra of a TM-algebra $(X *, 0)$ then $G(X) \cap S = G(S)$.*

## Proof

Obviously $G(X) \cap S \subseteq G(S)$.

We know $G(S) = \{x \in S \subseteq X: 0 * x = x\}$.

Let $x \in G(S)$. Then $0 * x = x$ and $x \in S \subseteq X$ which implies $x \in G(X) \cap S$.

Hence $G(S) \subseteq G(X) \cap S$, which proves the result.

## 2.24 Theorem

*If $(X, *, 0)$ is a TM- algebra of order 3, then*

*i) $G(X) \neq X$*

*ii) $G(X)$ is an ideal of $X$ if $|G(X)| = 1$.*

## Proof

i) Let $X = \{0, a, b\}$ be a TM-algebra.

Assume $G(X) = X$.

Then $0 * 0 = 0$, $0 * a = a$, $0 * b = b$.

Also we know $x * x = 0$ and $x * 0 = x$, for all $x \in X$.

Therefore $a * 0 = a$, $a * a = 0$ and $b * b = 0$.

Let $a * b = 0$. It is argued for $b * a = 0, a, b$.

Now if $b * a = 0$, then $a * b = 0 = b * a$ and

$(a * b) * a = (a * b) * (a * 0) = 0 * b = b$ and

$(b * a) * a = 0 * a = a$.

Since $(a * b) * a = (b * a) * a$, it follows that $a = b$, a contradiction.

So $b * a \neq 0$.

If $b * a = a$, then

$a = b * a = (0 * b) * a$

$= (0 * a) * b$

$= a * b$

$= 0$, a contradiction.

If $b * a = b$, then

$b = b * a = (0 * \mathbf{b}) * a$

$= (0 * a) * b$

$= a * b$

$= 0$, a contradiction.

Next if $a * b = a$, then

$(a * (a * b)) * b = (a * a) * b = 0 * b = b \neq 0$,

contradicting (*vi*) of proposition (2.4)

Finally, let $a * b = b$.

It is argued for $b * a = 0, a, b$.

If $b * a = 0$, then $b = a * b$

$= (0 * a) * b$

$= (0 * b) * a$

$= b * a$

$= 0$, a contradiction.

If $b * a = a$, then

$b = a * b$

$= (0 * a) * b$

$= (0 * b) * a$

$= b * a$

$= a$, a contradiction.

If $b * a = b$, then

$a = 0 * a$

$= (b * b) * a$

$= (b * a) * b$

$= b * b$

0, a contradiction.

Thus it is concluded that there exist some other elements in $G(X)$, which is not in $X$.

(ii) Let $X = \{0, a, b\}$ be a TM-algebra of order 3.

If the order of $G(X)$ is 1, then

$G(X) = \{0\}$, is the trivial ideal of $X$.

Conversely,

assume $G(X)$ is an ideal of $X$.

By (i) of this proposition

$|G(X)| = 1$ or $|G(X)| = 2$.

Suppose $|G(X)| = 2$. Then

$G(X) = \{0, a\}$ or $G(X) = \{0, b\}$.

If $G(X) = \{0, a\}$, since $G(X)$ is an ideal of $X$,

$b * a \notin G(X)$ so $b * a = b$.

Now, $a = 0 * a$

$= (b * b) * a = (b * a) * b$

$= b * b$

$= 0$, a contradiction .

Hence $|G(X)| \neq 2$ and so $|G(X)| = 1$.

## 2.25 Definition

Let $(X, *, 0)$ and $(Y, \Delta, 0^1)$ be TM-algebras. A mapping $f : X \rightarrow Y$ is called a **homomorphism** if

$f(x * y) = f(x) \Delta f(y)$ for all $x, y \in X$.

Note that $f^{-1}(Y) = \{ x \in X : f(x) = y$ for some $y \in Y \}$ and $f(X) = \{ f(x) : x \in X \}$ is called the image of $f$.

## 2.26 Proposition

*Suppose $f : X \rightarrow Y$ is a homomorphism of TM-algebras. Then*

*(i) $f(0) = 0^1$*

*(ii) If $x * y = 0$, for all $x, y \in X$ then $f(x) \Delta f(y) = 0^1$.*

## Proof

$f(0) = f(0 * 0) = f(0) \Delta f(0) = 0^1$.

ii) Let $x, y \in X$ and $x * y = 0$. Then

$f(x) \Delta f(y) = f(x * y) = f(0) = 0^1$.

## 2.27 Theorem

*Let $(X, *, 0)$, $(Y, \Delta, 0')$ be TM-algebras and let $B$ be an ideal of $Y$. Let $f : X \rightarrow Y$ be a homomorphism. Then $f^{-1}(B)$ is an ideal of $X$.*

## Proof

We know $f^{-1}(B) = \{ x \in X : f(x) = y$ for $y \in B \}$.

Since $0^1 \in B$ and $f(0) = 0^1$, $0 \in f^{-1}(B)$.

Assume $x * y \in f^{-1}(B)$ and $y \in f^{-1}(B)$, then $f(x * y) \in B$ and $f(y) \in B$.

Since $f$ is a homomorphism, $f(x * y) = f(x) \Delta f(y) \in B$

Since $B$ is an ideal of $Y$, $f(x) \in B$, so $x \in f^{-1}(B)$.

Hence $f^{-1}(B)$ is an ideal of $X$.

## 2.28 Definition

Let $(X, *, 0)$, $(Y, \Delta, 0')$ be TM-algebras. Let $f : X \rightarrow Y$

be a homomorphism. Then the set $\{x \in X : f(x) = 0^1\}$ is called the ***kernel*** of ***f*** and is denoted by ***ker f***.

### 2.29 Theorem.

*Let $f : X \rightarrow Y$ is a homomorphism of TM-algebras. Then ker f is an ideal of X.*

### Proof

Obviously $0 \in ker f$, since $f(0) = 0^1$.

Let $x * y \in ker f$, and $y \in ker f$.

So $f(x * y) = 0^1$ and $f(y) = 0^1$.

That is $f(x) \Delta f(y) = 0^1$

$\Rightarrow f(x) \Delta 0^1 = 0^1$

$\Rightarrow f(x) = 0^1$

Hence $x \in ker f$,

So $ker f$ is an ideal of $X$.

## CONCLUSION

Concept of TM-algebra proposed, in the present work, has been evaluated against well established algebraic theorems. It has been observed that the TM-algebra satisfy the various conditions stated in the Q/BCH/BCI/BCK algebras and can be considered as the generalization of all these algebras.

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