Classification of TM-Algebra

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ABSTRACT

In this paper, the notion of TM-algebra is introduced which is a generalization of Q/BCI/BCH-algebras. The concepts of positive implicative, implicative, 1-weakly positive implicative, 2-weakly positive implicative, right translation, left translation, weak right translation, weak left translation and the related properties are duly characterized.

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TM-Algebra, implicatives, weakly and positive implicatives, right and left translation.

1.INTRODUCTION

In 1966, Imai and Iseki introduced two classes of abstract algebras, BCK-algebras and BCI-algebras. These algebras have been extensively studied since their introduction. In 1983, Hu and Li introduced the notion of a BCH-algebra which is a generalization of the notion of BCK and BCI-algebras and studied a few properties of these algebras. After that J.Neggers, S.S.Ahn and H.S.Kim introduced Q-algebras which is a generalization of BCK/BCI-algebras and obtained several results.

Here, a new notion of TM-algebra is introduced, which is a generalization of Q/BCK/BCI/BCH-algebras. The oncepts of weakly commutative, positive implicative, implicative, weakly positive implicative and weakly implicative were introduced in BCH-algebra. In this note, the characterization of positive implicative, implicative, 1-weakly positive implicative, 2-weakly positive implicative, weak right translation, weak left translation are introduced.

2. PRELIMINARIES

In this section, certain definitions, known results

and examples that will be used in the sequel are described..

2.1 Definition

A BCI-algebra is an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following conditions:

i) $(x * y) * (x * z) \leq z * y$

ii) $x * (x * y) \leq y$

iii) $x \leq x$,

iv) $x \leq y$ and $y \leq x$ imply $x = y$

v) $x \leq 0$ implies $x = 0$, where $x \leq y$ is defined by $x * y = 0$ for all $x, y, z \in X$.

2.2 Definition

A BCK-algebra is an algebra (X,*, 0) of type (2 , 0) satisfying the following conditions:

i) $(x * y) * (x * z) \leq z * y$

ii) $x * (x * y) \leq y$

iii) $x \leq x$,

iv) $x \leq y$ and $y \leq x$ imply $x = y$,

v) $0 \leq x$ implies $x = 0$, where $x \leq y$ is defined by $x * y = 0$ for all $x, y, z \in X$.

2.3 Definition

A BCH-algebra is an algebra (X, *, 0) of type (2 , 0) satisfying the following conditions :

i) $x * x = 0$

ii) $(x * y) * z = (x * z) * y$.

iii) If $x * y = 0$ and $y * x = 0$ imply $x = y$, for all $x, y, z \in X$.

2.4 Definition

A Q-algebra is an algebra (X *, 0) of type (2 , 0) satisfying the following axioms :

i) $x * x = 0$

ii) $x * 0 = x$

iii) $(x * y) * z = (x * z) * y$, for all $x , y , z \in X$.

Every BCK-algebra is a BCI-algebra but not conversely.

Every BCI-algebra is a BCH-algebra but not conversely

Every BCH-algebra is a Q-algebra but not conversely.

Every Q-algebra satisfying the conditions

$(x * y) * (x * z) = z * y$ and $x * y = 0, y * x = 0$ imply $x = y$

is a BCI-algebra.

3. TM-ALGEBRA

3.1 Definition

A **TM-algebra** $(X, *, 0)$ is a non-empty set X with a constant "0" and a binary operation "$*$" satisfying the following axioms :

i) $x * 0 = x$

ii) $(x * y) * (x * z) = z * y$ for any $x, y, z \in X$

In X we can define a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 0$

In any TM-algebra $(X, *, 0)$, the following holds good for all $x, y, z \in X$

iii) $x * x = 0$

iv) $(x * y) * x = 0 * y$

v) $x * (x * y) = y$

vi) $(x * z) * (y * z) \leq x * y$

vii) $(x * y) * z = (x * z) * y$

viii) $x * 0 = 0 \implies x = 0$

xi) $x \leq y \implies x * z \leq y * z$ and $z * y \leq z * x$

x) $x * (x * (x * y)) = x * y$

xi) $0 * (x * y) = y * x = (0 * x) * (0 * y)$

xii) $(x * (x * y)) * y = 0$

xiii) If $x * y = 0, y * x = 0$ then $x = y$.

A QS-algebra is obviously a TM –algebra, But a TM– algebra is said to be a QS-algebra if it satisfies the additional relations $(x * y) * z = (x * z) * y$ and $y * z = z * y$ for all $x, y, z \in X$.

3.2 Example

Let X = {0,1, 2, 3} be a set with cayley table (Table 1)

Table 1

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X, *, 0)$ is a TM- algebra .

The relations between TM-algebra and other algebras are investigated and presented below.

3.3 Theorem

Every BCK- algebra is a TM-algebra but the converse is not true.

The above example 3.2 is a TM - algebra but not BCK-algebra, since $0 * x \neq 0$ for all $x = 1, 2, 3$.

3.4 Theorem

Every TM -algebra is a BH - algebra, but the converse is not true. Similarly, every TM- algebra is a Q-algebra (Table 2), but the converse is not true.

3.5 Example

Let $X = \{0, 1, 2, 3\}$

Table 2

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

Note that $(X, *, 0)$ is a Q-algebra .

The condition $(x * y) * (x * z) = z * y$ is not satisfied as

$(1 * 2) * (1 * 3) = 0 * 0 = 0 \neq 3 = 3 * 2$

3.6 Theorem

Every TM-algebra is a BCH-algebra. Every BCH-algebra satisfying $(x * y) * (x * z) = z * y$ is a TM-algebra .

3.7 Theorem

Every TM- algebra X satisfying $x * z = z$ is a trivial algebra.

3.8 Definition

Let $(X, *, 0)$ be a TM –algebra. A non-empty subset I of X is called an ideal of X if it satisfies

(i) $0 \in I$

ii) $x * y \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in X$.

Any ideal I has the property that $y \in I$ and $x \leq y$ imply $x \in I$

3.9 Example

Table-3

*	0	a	b	c
0	0	0	c	b
a	a	o	c	b
b	b	b	o	c
c	c	c	b	0

Then the set $I = \{0, a\}$ is an ideal of X.

4. CLASSIFICATION OF TM-ALGEBRA.

4.1 Definition

A BCK-algebra $(X, *, 0)$ is called **positive implicative**

if $(x * y) * z = (x * z) * (y * z)$, for all $x, y, z \in X$..

It is called **implicative** if $x * (y * x) = x$. It is **commutative** if $x * (x * y) = y * (y * x)$, for all $x, y \in X$.

4.2 Definition

A BCI-algebra $(X, *, 0)$ is called **weakly positive implicative** if $(x * y) * z = ((x * z) * z) * (y * z)$ for all $x, y, z \in X$.

It is called **weakly implicative** if

$(x * (y * x)) * (0 * (y * x)) = x$. for all $x, y \in X$.

It is **weakly commutative** if

$(x * (x * y)) * (0 * (x * y)) = y * (y * x)$ for all $x, y \in X$.

4.3 Theorem

A BCI-algebra $(X, *, 0)$ is called weakly positive implicative if and only if $x * y = ((x * y * y) * (0 * y)$.

4.4 Definition

A BCH-algebra $(X, *, 0)$ is called **weakly positive implicative** if

$x * y = ((x * y) * y) * (0 * y)$ for all $x, y \in X$.

It is called **weakly implicative** if

$(x * (y * x)) * (0 * (y * x)) = x$ for all $x, y \in X$.

It is **weakly commutative** if

$(x * (x * y)) * (0 * (x * y)) = y * (y * x)$,for all $x, y \in X$.

Every weakly implicative BCI-algebra X is a weakly positive implicative BCI-algebra.

Every implicative BCK-algebra X is a positive implicative BCK-algebra.

4.5 Definition

A TM-algebra $(X, *, 0)$ is called **positive implicative** if $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in X$.

It is called **implicative if** $x * (y * x) = x$.

4.6 Definition

An **ideal** I of a TM-algebra $(X, *, 0)$ is said to be implicative if $(x * y) * z \in I$ and $y * z \in I$ imply $x * z \in I$ for any $x, y, z \in X$

4.7 Theorem

Let $(X, *, 0)$ be a TM-algebra and let I be an implicative ideal of X. Then $G(X) \subseteq I$.

Proof

Let $x \in G(X)$. Then $0 * x = x$.

Now $0 = x * x = (0 * x) * x \in I$ and $x * x = 0 \in I$.

Since I is an implicative ideal $0 * x \in I$.

That is $x \in I$.

Hence $G(X) \subseteq I$.

4.8 Definition

A TM-algebra $(X, *, 0)$ is said to be 1-weakely positive implicative if

$$((x * y) * y) * (0 * y) = x * y, \text{ for all } x, y \in X.$$

A TM-algebra $(X, *, 0)$ is said to be 2-weakely positive implicative if

$$(x * (x * y)) * (0 * (x * y)) = x * y, \text{ for all } x, y \in X.$$

4.9 Definition

Let $(X, *, 0)$ be a TM-algebra. For a fixed $x \in X$, the map $R_x : X \rightarrow X$ given by $R_x(y) = y * x$, for all $y \in X$ is called a right translation of X.

Similarly, the map $R_x' : X \rightarrow X$ given by $R_x'(y) = x * y$, for all $y \in X$, is called a left translation of X.

4.10 Definition

Let $(X, *, 0)$ be a TM-algebra. For a fixed $x \in X$ the map $T_x : X \rightarrow X$ given by $T_x(y) = (y * x) * (0 * x)$, for all $y \in X$, is called a weak right translation of X.

Similarly the map $T_x' : X \rightarrow X$ given by $T_x'(y) = (x * y) * (0 * y)$, for all $y \in x$ is called a weak left translation of X.

The following theorem characterizes the weakly positive implicative TM-algebra

4.11 Theorem

A TM-algebra $(X, *, 0)$ is 1-weakly positive implicative if and only *if* $R_z = T_z \circ R_z$ for all $z \in X$, and "o" is the composition of functions.

Proof

Let X be a TM-algebra and let $R_z = T_z \circ R_z$ for $z \in X$. Then

$$\begin{aligned} y * z = R_z(y) &= (T_z \circ R_z)(y) \\ &= T_z \circ (R_z(y)) \\ &= T_z(y * z) \\ &= ((y * z) * z) * (0 * z), \text{ for all } y, z \in X. \end{aligned}$$

Hence X is 1-weakly positive implicative TM-algebra.

Conversely, if X is a 1-weakly positive implicative TM-algebra then

$(y * z) = ((y * z) * z) * (0 * z)$.

$$\begin{aligned} \text{So} \quad R_z(y) &= y * z \\ &= ((y * z) * z) * (0 * z) \\ &= ((R_z(y) * z) * (0 * z) \\ &= T_z(R_z(y)) \\ &= (T_z \circ R_z)(y), \text{ for all } y, z \in X. \end{aligned}$$

Hence $R_z = T_z \circ R_z$.

4.12 Theorem

A TM-algebra X is 2 weakly positive implicative iff

$R_z' = T_z' \circ R_z'$ for all $z \in X$

Proof

Let X be a TM-algebra and let $R_z{}' = T_z{}' \circ R_z{}'$ for $z \in X$.

Then

$z * y = R_z{}'(y)$

$= (T_z{}' \circ R_z{}')(y)$

$= T_z{}'(R_z{}'(y))$

$= T_z{}'(z * y)$

$= (z * (z * y) * (0 * (z * y))$ for all $y, z \in X$.

Hence X is 2-weakly positive implicative TM-algebra.

Conversely,

if X is 2-weakly positive implicative TM-algebra, then

$x * y = (x * (x * y)) * (0 * (x * y))$.

Now,

$R_x{}'(y) = x * y$

$= (x * (x * y)) * (0 * (x * y))$

$= (x * R_x{}'(y)) * (0 * R_x{}'(y))$

$= T_x{}'(R_x{}'(y))$

$= (T_x{}' \circ R_x{}')(y)$ for all $x, y, \in X$.

Hence $R_x{}' = T_x{}' \circ R_x{}'$.

4.13 Theorem

Let X be a 1-weakly positive implicative TM-algebra. Then $T_y = T_y \circ T_y = T_y{}^2$

Proof

Let X be a 1-weakly positive implicative TM-algebra.

Then

$x * y = ((x * y) * y) * (0 * y)$. -------------- (1)

Now

$(x * y) * (0 * y) = ((x * y) * y) * (0 * y)) * (0 * y)$

$= (((x * y) * (0 * y)) * y) * (0 * y)$. -------------(2)

$T_y(x) = (x * y) * (0 * y)$

$T_y{}^2(x) = T_y((x * y) * (0 * y))$

$= (((x * y) * (0 * y)) * y) * (0 * y)$

$= (x * y) * (0 * y)$

$= T_y(x)$, for all $x, y, \in X$.

The converse of the above theorem is not true, which can be shown by the following example (Table 3).

Let $X = \{0, a, b, c\}$ in which $*$ is defined by the following table.

Table 3

*	0	a	b	c
0	0	0	b	b
a	a	o	b	b
b	b	b	o	0
c	c	b	a	0

Then $(X, *, 0)$ is a BCI algebra,which in turn is a TM-algebra.

As $a = c * b \neq ((c * b) * b) * (0 * b) = (a * b) * (0 * b) = b * b = 0$, X is not 1-weakly positive implicative .

But $T_0{}^2 = T_0$, $T_a{}^2 = T_a$, $T_b{}^2 = T_b$, $T_c{}^2 = T_c$.

4.14 Theorem

Let X be a 2- weakly positive implicative TM-algebra. Let $T_x{}'(y) \in G(X)$ and $x \leq T_x{}'(y)$. Then $T_x{}'^2 = T_x{}'^2$ for all $x, y, \in X$.

Proof

Let X be a 2-weakly positive implicative TM-algebra.

Then $(x * (x * y)) * (0 * (x * y)) = x * y$

Now

$T_x{}'^2(y) = T_x{}'(T_x{}'(y))$

$= T_x{}'((x * y) * (0 * y))$

$= (x * ((x * y) * (0 * y))) * (0 * ((x * y) * (0 * y)))$

$= 0 * ((x * y) * (0 * y))$ since $x \leq T_x{}'(y)$

$= (x * y) * (0 * y)$, since $T_x{}'(y) \in G(X)$

$= T_x{}'(y)$

Converse need not be true.

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