P_{2k} −Factorization Induced Network Flow

U.S. Rajput
Department of mathematics and astronomy,
Lucknow University, Lucknow (India).

Bal Govind Shukla
Department of mathematics and astronomy,
Lucknow University, Lucknow (India).

ABSTRACT
P_k −factorizations of complete bipartite graph have been studied by several researchers. For even value of k, the spectrum problem is completely solved [6, 7]. Here in this paper we will obtain a feasible network flow of P_{2k} −factorization of a complete bipartite graph satisfying the conditions of P_{2k} −factorization. In this paper we construct the disjoint flow paths in P_{2k} −factorization of complete bipartite graph $K_{m,n}$ (for $k = 1$ and 2). We deduce that P_{2k} −factorization of complete bipartite graph is helpful in finding the disjoint flow paths in a compete bipartite graph $K_{m,n}$ $(m = n)$. The result can be generalized for any value of k with $m = n$.

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Complete bipartite Graph, Factorization of Graph, Network Flow

1. INTRODUCTION
Let $K_{m,n}$ be the complete bipartite graph with two partite sets having m and n vertices, respectively. A spanning subgraph F of $K_{m,n}$ is called a path factor if each component of F is a path of order at least two. For any positive integer k, a path on $2k$ vertices is denoted by P_{2k}. In particular a spanning sub graph F of $K_{m,n}$ is called a P_{2k} −factor of $K_{m,n}$ if each component of F is isomorphic to P_{2k}. If $K_{m,n}$ is expressed as an arc disjoint sum of P_{2k} −factors, then this sum is called a P_{2k} −factorization of $K_{m,n}$.

A Network flow is called feasible flow with source s and sink t if it satisfies the following two condition as given by [3, 4].

(i) $0 \le f(u,v) \le c(u,v) \quad \forall (u,v) \in E$. These are the capacity constraints. (If a capacity is ∞, then there is no upper bound on the flow value on that edge.)

(ii) For all $v \in V - \{s,t\}$, the total flow into v is same as the total flow out of v:

$$\sum_{u:(u,v)\in E} f(u,v) = \sum_{w:(v,w)\in E} f(v,w).$$

Constraint (ii) is called flow conservation law for network flow graph. The conservation law holds at all vertices other than the source and the sink. The value of flow denoted by $val(f)$, is the net flow out of the source:

$$val(f) = \sum_{u:(s,u)\in E} f(s,u).$$

In P_{2k} −factorization induced network flow, each path factor of complete bipartite graph $K_{m,n}$ will contribute to a flow path, and hence this network flow will be a collection of flow paths, each of which is an undirected path factor. We assume capacity of each edge as one which is not shown in the figures drawn.

2. MATHEMATICAL ANALYSIS
In the study of $P_{2k} -$ path factorization of complete bipartite graph $K_{m,n}$, we find different mutually vertex/edge disjoint paths. Each path will be a flow path. For each value of k(even/odd) flow path can be developed. In this paper we consider particular cases of even and odd values of k. For simplicity, we are considering only P_2 and P_4 factorization.

Network Flow in P_{2k} −factorization: Let P_{2k} be a path on even vertices and $K_{m,n}$ be a complete bipartite graph with partite sets having m and n vertices. Wang [9] gave the necessary and sufficient conditions for the existence of a P_{2k} −factorization of $K_{m,n}$ when $m = n$. Which are given in theorem 2.1 below.

Theorem 2.1: $K_{m,n}$ has P_{2k} −factorization if and only if $m = n$ and $m \equiv 0\big(\text{mod } k(2k-1)\big)$.

To show the feasible network flow in P_{2k} −factorization of $K_{m,n}$, we consider $k = 1, m = 4$ and $n = 4$, Fig. 1.

Here let t be the number of copies in a P_2 −factor graph, then in this case

$$t = \frac{m+n}{2}$$

$$= \frac{4+4}{2} = 4$$

And r be the number of disjoint P_2 −factors in graph, *i.e.*

$$r = \frac{2mn}{m+n}$$

$$= \frac{2 \times 4 \times 4}{8} = 4.$$

To make it disjoint network flow graph we add source and sink in P_2 −factorization of $K_{4,4}$.

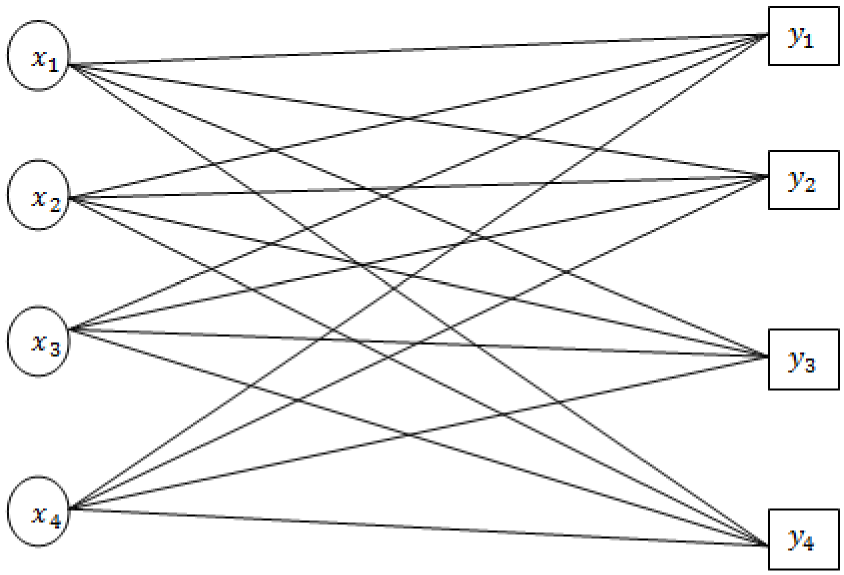

Fig. 1: Complete bipartite graph $K_{4,4}$

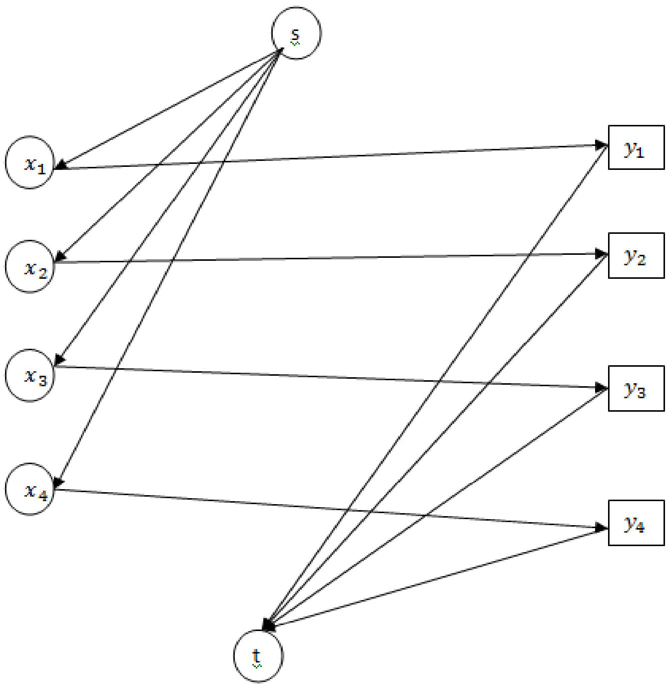

Fig. 2: Flow path $sx_1y_1t, sx_2y_2t, sx_3y_3t, sx_4y_4t$.

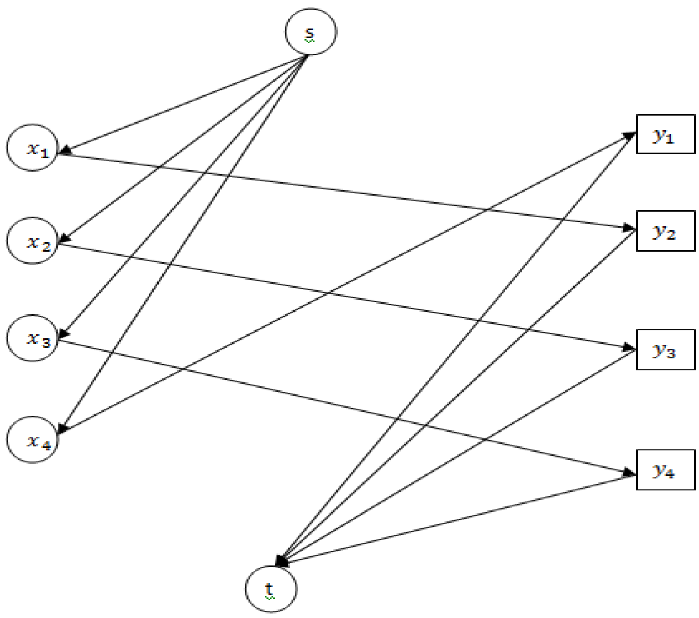

Fig. 3: Flow path $sx_1y_2t, sx_2y_3t, sx_3y_4t,\ sx_4y_1t$.

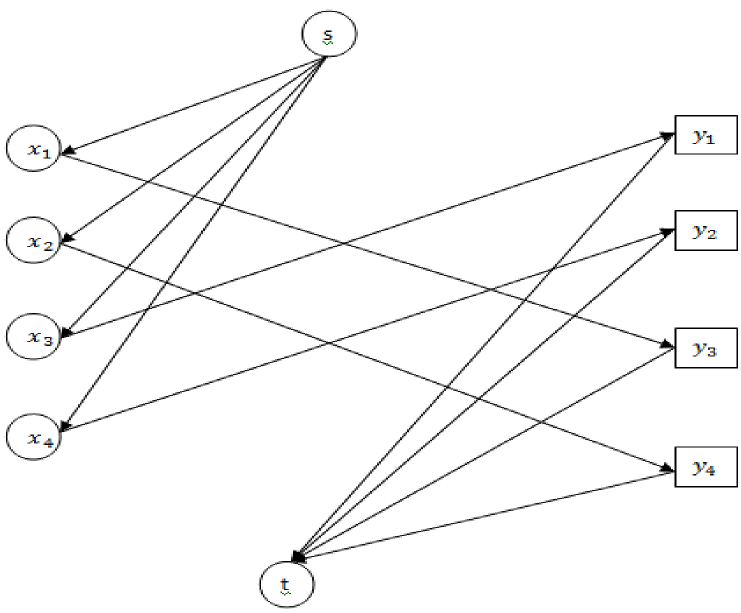

Fig. 4: Flow path $sx_1y_3t, sx_2y_4t, sx_3y_1t,\ sx_4y_2t$.

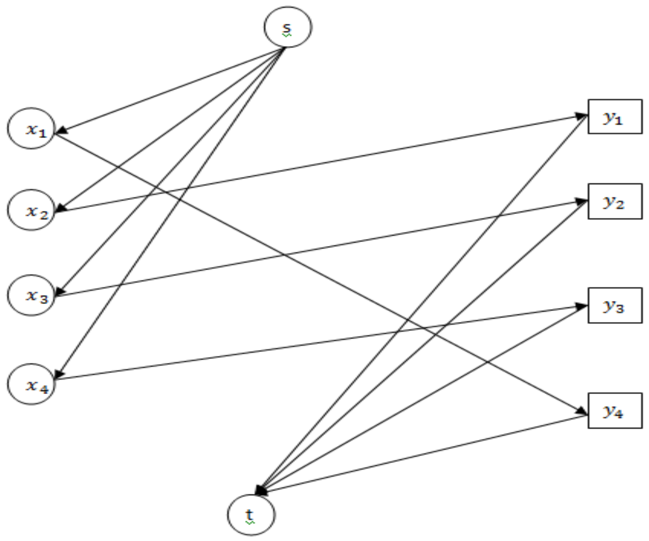

Fig. 5: Flow path $sx_1y_4t, sx_2y_1t, sx_3y_2t,\ sx_4y_3t$.

As shown above in Fig. 1 to Fig. 5, we find the disjoint feasible network flow path between source s and sink t in P_2 −factorization of $K_{2,2}$.

Similarly, for P_4 −factorization of $K_{m,n}$, we consider $k = 2, m = 6$ and $n = 6$(Theorem -2.1), Fig. 4.

Here in this case let t be the number of copies in a P_4 −factor graph, then

$$t = \frac{m+n}{4}$$

$$= \frac{6+6}{4} = 3$$

and r be the number of disjoint P_4 −factors in graph, *i.e.*

$$r = \frac{2mn}{3(m+n)}$$

$$= \frac{4 \times 6 \times 6}{3 \times 12} = 4.$$

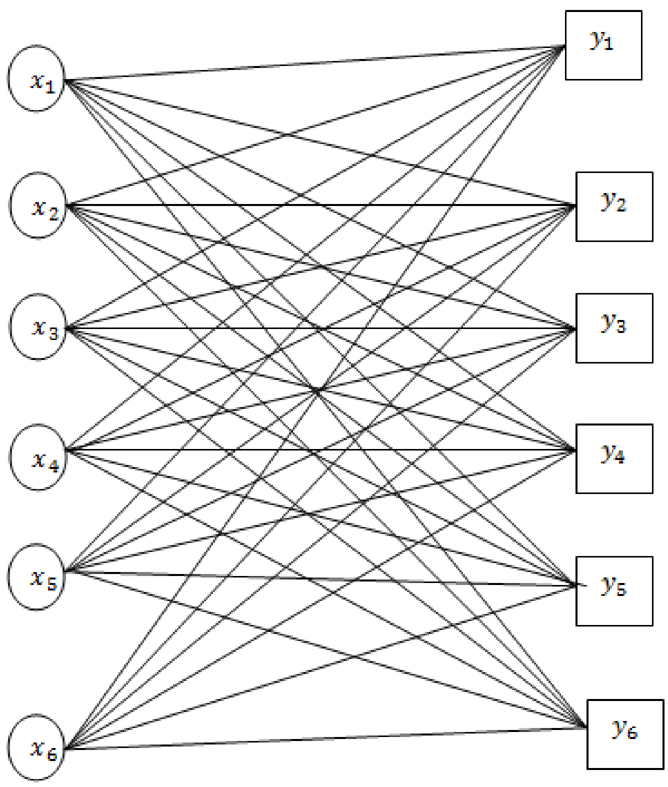

Fig. 6: Complete bipartite graph $K_{6,6}$

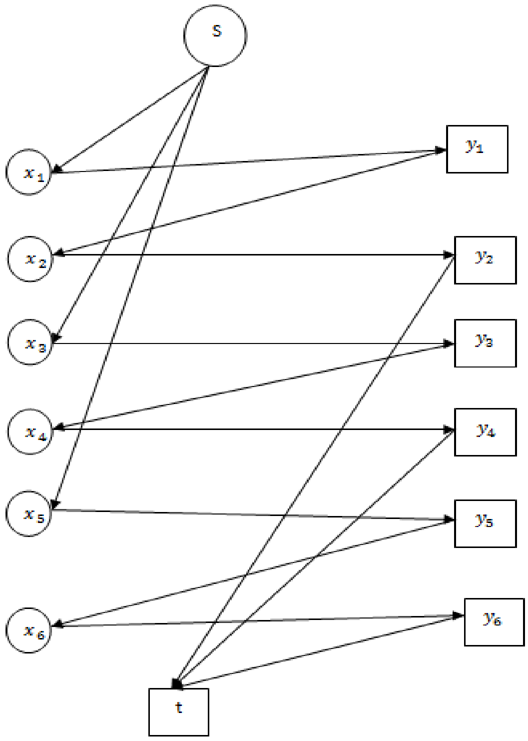

Fig. 7: Flow paths $sx_1y_1x_2y_2t, sx_3y_3x_4y_4t, sx_5y_5x_6y_6t$.

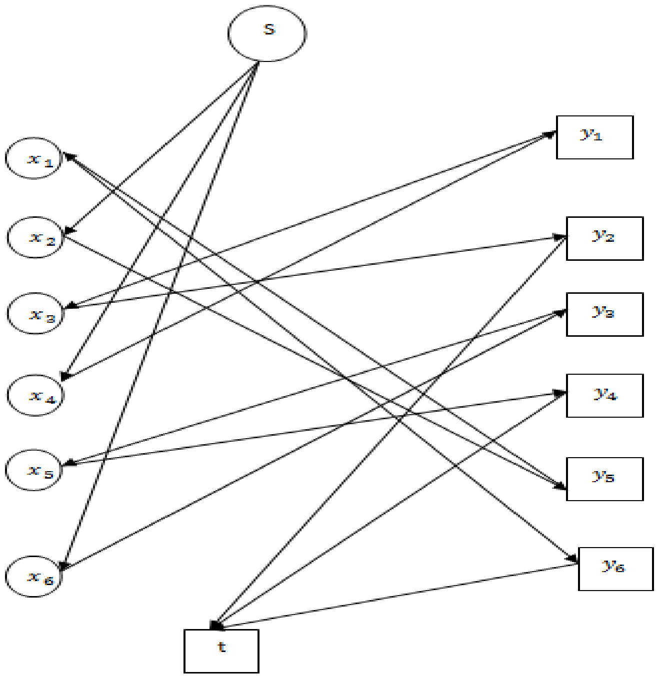

Fig. 8: Flow paths $sx_4y_1x_3y_2t, sx_6y_3x_5y_4t, sx_2y_5x_1y_6t$.

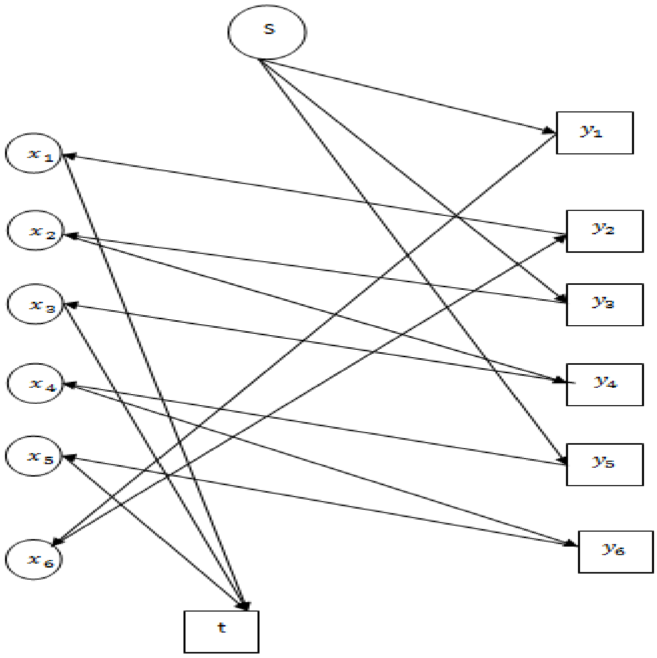

Fig. 9: Flow paths $sy_1x_6y_2x_1t, sy_3x_2y_4x_3t, sy_5x_4y_6x_5t$.

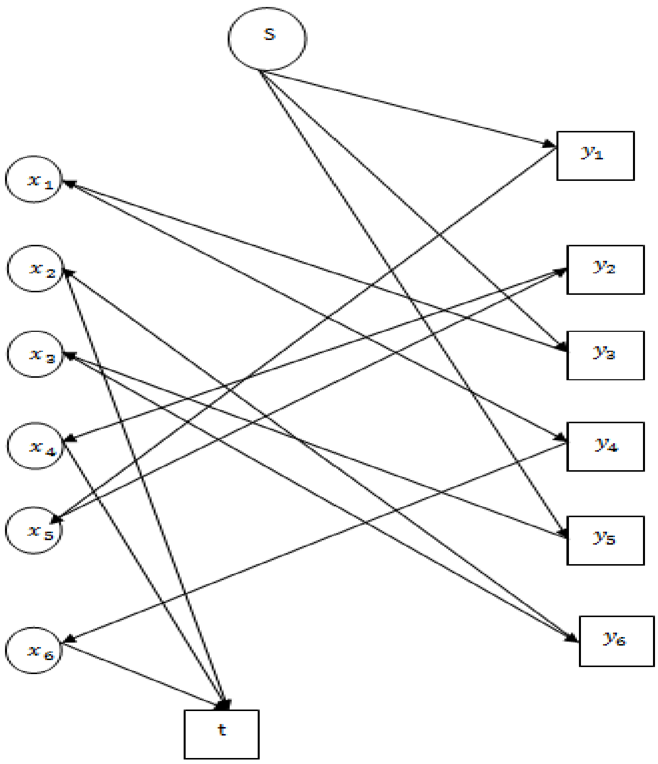

Fig. 10: Flow Paths $sy_1x_5y_2x_4t, sy_3x_1y_4x_6t, sy_5x_3y_6x_2t$.

3. DISCUSSION

In this paper it is analyzed that P_{2k} −factorization of complete bipartite graph $K_{m,n}$ (for $k = 1$ and 2), will give the disjoint flow paths. Hence we deduce that P_{2k} −factorization of complete bipartite graph is helpful in finding the disjoint flow paths in a complete bipartite graph $K_{m,n}$ when $m = n$. The result can be generalized for any value of k with $m = n$.

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