τ^*- gλ– Closed Sets in Topological Spaces

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ABSTRACT
In this paper, we introduce the new notions τ^*- gλ– closed sets and τ^*- gλ– open sets τ^*- gλ– continuous maps in Topological spaces. Also we introduce a new space called $T_{\tau^*\text{-}g\lambda}$- space. We study some of its properties in Topological spaces.

General Terms
2000 Mathematics Subject Classification: 54A05

Keywords
τ^*- gλ– closed sets , τ^*- gλ– open sets , τ^*- gλ– continuous maps, $T_{\tau^*\text{-}g\lambda}$- space.

1. INTRODUCTION
In 1970, Levine[9] introduced the concept of generalized closed sets as a generalization of closed sets in topological spaces. Using generalized closed sets, Dunham[8] introduced the concept of the closure operator cl* and a new topology $\boldsymbol{\tau}^*$ and studied some of their properties. P.Arya[3],P.Bhattacharyya and B.K.Lahiri[5], J.Dontchev[7], H.Maki, R.Devi and K.Balachandran[12], [13], P.Sundaram and A.Pushpalatha[19] introduced and investigated generalized semi closed sets, semi generalized closed sets, generalized semi preclosed sets, α- generalized closed sets, generalized-α closed sets, strongly generalized closed sets, gλ– closed sets respectively.

In this paper, we obtain a new generalization of closed sets in the weaker topological space (X, τ *). Throughout this paper X and Y are topological spaces on which no separation axioms are assumed unless otherwise explicitly stated. For a subset A of a topological space X, int(A), cl(A), cl*(A), scl(A), spcl(A), clα(A) and A^c denote the interior, closure, closure*, semi-closure, semipreclosure, α-closure and complement of A respectively.

2. PRELIMINARIES
Definition 2.1
A subset A of X is called

(a) semi open [8] if there exists an open set G such that $G \subseteq A \subseteq cl(G)$ and semi closed if there exists an closed set F such that $int(F) \subseteq A \subseteq F$.

Equivalently, a subset A of X is called semi-open if $A \subseteq cl(int(A))$ and semi closed if $A \supseteq int(cl(A))$

(b) α-closed if $cl(int(cl(A))) \subseteq A$ and an α-open if $A \subseteq int(cl(int(A)))$ [17].

(c) pre-closed if $cl(int(A)) \subseteq A$ and pre-open if $A \subseteq int(cl(A))$ [14].

Definition 2.2
A subset A of X is called a

(a) generalized closed(g-closed) set if $cl(A) \subseteq G$ whenever $A \subseteq G$ and G is open in X and generalized open if A^c is g-closed set in X [9].

(b) generalized semi closed (gs-closed) if $scl(A) \subseteq G$ whenever $A \subseteq G$ and G is open in X and generalized semi open if A^c is gs-closed set in X[3].

(c) generalized α-closed (gα-closed) if $\alpha cl(A) \subseteq G$ whenever $A \subseteq G$ and G is α-open in X and generalized α-open if A^c is gα-closed set in X[13].

(d) generalized semi-preclosed set (gsp-closed) if $spcl(A) \subseteq G$ whenever $A \subseteq G$ and G is open in X [7].

(e) b-closed if $int(cl(A)) \cap cl(int(A)) \subseteq A$[2]

(f) g^* α-closed set if $\alpha cl(A) \subseteq$ □ U whenever $A \subseteq$ □ U and U is gα-open in X [16]

(g) strongly g^* α-closed set if $\alpha cl(A) \subseteq$ □ U whenever $A \subseteq$ □ U and U isg^* α-open in X. [15]

(h) $\hat{g}$-closed if $cl(A) \subseteq G$ whenever $A \subseteq G$ and G is semi-open in X [20]

Definition 2.3
A subset A of a topological space X is called a λ-closed set in X if $A \supseteq cl(G)$ whenever $A \supseteq G$ and G is open in X. [14]

Remark 2.4
(i) Every closed set is $\hat{g}$-closed [20]

(ii) Every $\hat{g}$-closed is g-closed [20]

Definition 2.5
A function f: X → Y is called

(i) semi continuous (s-continuous) if $f^{-1}(V)$ is semi-closed in X for each closed set V in Y [10].

(ii) generalized continuous (g-continuous) if $f^{-1}(V)$ is g-open in X for each open set V in Y [18].

(iii) generalized semi continuous (gs-continuous) if $f^{-1}(V)$ is gs-closed in X for each closed set V in Y [3].

(iv) $\hat{g}$ − continuous if $f^{-1}(V)$ is $\hat{g}$- closed in X for each closed set V in Y [21].

Definition 2.6
For the subset A of a topological X, the generalized closure operator cl*[5] is defined by the intersection of all g-closed sets containing A.[8]

Definition 2.7
For the subset A of a topological X, the topology τ* is defined by τ* = {G : cl*(G^C) = G^C }[8]

Definition 2.8
A subset A of a topological space X is called a τ^*- g - closed set in X if $cl^*(A) \subseteq G$ whenever $A \subseteq G$ and G is τ^*- open in X. [18]

Remark 2.9
(i) Every closed set in X is g −closed [9]

(ii) Every closed set in X is τ *-g-closed.[18]

(iii) Every g-closed set in X is a τ*-g-closed set but not conversely.[18]

3. τ^*- g λ – CLOSED SET IN TOPOLOGICAL SPACES
In this section, we introduce the concept of τ^*- gλ–closed sets in topological spaces and study some of its properties.

Definition 3.1
A subset A of a topological space X is called a τ^*- gλ– closed set in X if $A \supseteq cl^*(G)$ whenever $A \supseteq G$ and G is τ^*- open in X.

Theorem 3.2
Every closed in X is τ^*- gλ–closed in X but not conversely.

Proof
Assume that A be a closed set in X. Let G be an τ^*- open set in X such that A⊇G. Then $cl^*(A) \supseteq cl^*(G)$. But cl*(A) = A. Therefore $A \supseteq cl^*(G)$ and hence A is τ^*- gλ–closed in X.

Example 3.3
Let X = {a, b, c} be a topological space with topology τ = {ϕ, X,{a }}. The sets {a},{b},{c},{a,b}and {a,c} are τ^*- gλ - closed, but it is not closed in X.

Theorem 3.4
Every τ*- closed set in X is τ*- gλ– closed set in X but not conversely.

Proof: Assume that A be a τ*– closed set in X. Let G be an τ*- open set in X such that A⊇G. Then $cl^*(A) \supseteq cl^*(G)$. Since A is τ*- closed set , cl*(A) = A.Hence A is τ*- gλ– closed in X.

Example 3.5
Let X = {a, b, c} be a topological space with topology τ = { ϕ, X, {a,b }}. The sets {a} and {b} are τ^*- gλ– closed, but it is not τ*- closed in X.

Theorem 3.6
Every τ*- g- closed set in X is τ*- gλ– closed set in X but not conversely.

Proof: Assume that A be a τ*– g-closed set in X. Let G be an τ*- open set in X such that A⊇G. Then $cl^*(A) \supseteq cl^*(G)$. Since A is τ*-g - closed set , cl*(A) ⊆ □ G.Hence A is τ*- g– closed in X.

Example 3.7
Let X = {a, b, c} be a topological space with topology τ = { ϕ, X, { b,c }}. The sets {c} and {b} are τ^*- gλ - closed, but it is not τ*-g - closed in X.

Theorem 3.8
If A and B are τ^*- gλ - closed sets in a topological space X, then A ∩ B is τ^*- gλ - closed in X.

Proof: Let A ∩ B ⊇ G, where G is τ^*- open.

⇒ A ⊇ G, B ⊇ G

⇒ A ⊇ cl*(G), B ⊇ cl*(G), since A and B are τ^*- gλ - closed.

⇒ A ∩ B ⊇ cl*(G)

⇒ A ∩ B is τ^*- gλ - closed.

Remark 3.9
If A and B are τ^*- gλ - closed in X, then their union need not be τ^*- gλ– closed in X as seen from the following example.

Example 3.10
Let X = {a, b, c} be a topological space with topology τ = { ϕ, X, {a, b}}. The sets {a} and {b} are τ^*- gλ - closed, but their union {a, b} is not τ^*- gλ– closed in X.

Theorem 3.11: Every g − closed set in X is τ^*- gλ– closed set in X but not conversely

Proof: Assume that A be a g– closed set in X. Let G be an τ^*- open set in X such that A⊇G. Then $cl^*(A) \supseteq cl^*(G)$. Since A is g - closed set, $cl^*(A)$ = A . Therefore $A \supseteq cl^*(G)$ and hence A is τ^*- gλ - closed in X.

Example 3.12
Let X = {a, b, c} be a topological space with topology τ = { ϕ, X, {a }}. The set {a} τ^*- gλ - closed, but it is not g − closed in X.

Theorem 3.13
Every $\hat{g}$ − closed set in X is τ^*- gλ– closed set in X but not conversely.

Proof: By Theorem 3.11 and by Remark 2.4, it follows the proof

Example 3.14

Let X = {a, b, c} be a topological space with topology τ = { ϕ, X, {b }}. The set {b} is τ^*- gλ - closed, but it is not $\hat{g}-$ closed in X.

Example 3.15

Let X={a,b,c}

(i) The concepts of gs-closed sets and τ^*- gλ - closed are independent as seen from the following examples. Consider the topology $\tau = \{\emptyset, X, \{a\}\}$. Then the set {a} is τ^*- gλ - closed but not gs-closed In the topology σ = $\{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$, the sets {a}and{b} are gs-closed but not τ^*- gλ– closed.

(ii) The concepts of λ-closed sets and τ^*- gλ– closed are independent as seen from the following examples. Consider the topology $\tau = \{\emptyset, X, \{a\}\}$. Then the set {a}, {a,b},{b,c} are τ^*- gλ– closed but not λ-closed . In the topology σ = $\{\emptyset, X, \{a\}, \{a,b\}\}$, the sets {b} and {c} are λ-closed but not τ^*- gλ– closed.

(iii) The concepts of wλ-closed sets and τ^*- gλ - closed are independent as seen from the following examples. Consider the topology $\tau = \{\emptyset, X, \{a\}, \{a,b\}\}$. Then the set {c}, is τ^*- gλ - closed but not wλ-closed . In the topology σ = $\{\emptyset, X, \{a\}, \{a,b\}\}$, the set {a,c} are wλ-closed but not τ^*- gλ– closed.

(iv) The concepts of $g^*\alpha$-closed sets and τ^*- gλ– closed are independent as seen from the following examples. Consider the topology $\tau = \{\emptyset, X, \{a\}\}$. Then the set {a}, {a,b},{b,c} are τ^*- gλ– closed but not $g^*\alpha$ - closed . In the topology σ = $\{\emptyset, X, \{a\}, \{a,b\}\}$, the set {b} is $g^*\alpha$ - closed but not τ^*- gλ– closed.

(v) The concepts of strongly $g^*\alpha$-closed sets and τ^*- gλ– closed are independent as seen from the following examples. Consider the topology $\tau = \{\emptyset, X, \{a\}\}$. Then the set {a}, {a,b},{a,c} are τ^*- gλ-closed but not strongly $g^*\alpha$- closed . In the topology σ = $\{\emptyset, X, \{b\}, \{a,b\}\}$, the sets {a} is strongly $g^*\alpha$- closed but not τ^*- gλ– closed.

(vi) The concepts of semi-closed sets and τ^*- gλ– closed are independent as seen from the following examples. Consider the topology $\tau = \{\emptyset, X, \{a\}\}$. Then the set {a}, {a,b},{a,c} are τ^*- gλ– closed but not semi- closed . In the topology σ = $\{\emptyset, X, \{a\}, \{a,b\}\}$, the set {b} is semi - closed but not τ^*- gλ– closed.

(vii) The concepts of pre - closed sets and τ^*- gλ– closed are independent as seen from the following examples. Consider the topology $\tau = \{\emptyset, X, \{a\}\}$. Then the set {a}, {a,b},{a,c} are τ^*- gλ– closed but not pre - closed . In the topology σ = $\{\emptyset, X, \{a\}, \{a,c\}\}$, the set {b} is pre - closed but not τ^*- gλ– closed.

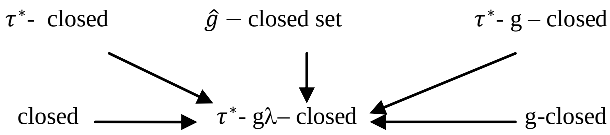
(viii) The concepts of α - closed sets and τ^*- gλ– closed are independent as seen from the following examples. Consider the topology $\tau = \{\emptyset, X, \{a\}\}$. Then the set {a}, {b},{c},{a,b},{a,c} are τ^*- gλ– closed but not α-closed. In the to σ = $\{\emptyset, X, \{c\}, \{a,c\}\}$, the set {a} is α- closed but not τ^*- gλ– closed.

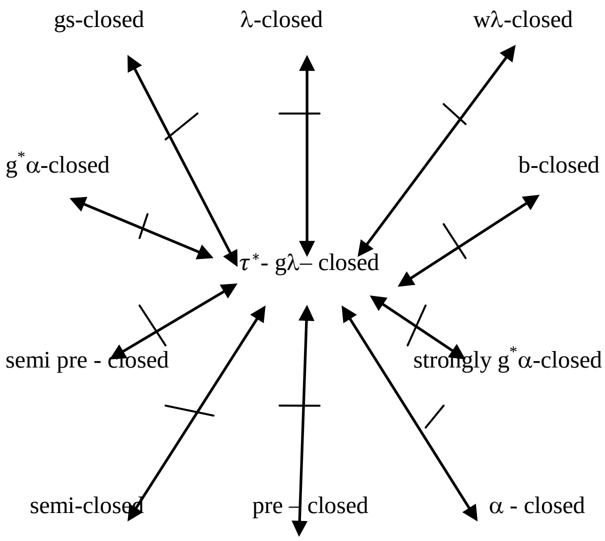
(ix) The concepts of b - closed sets and τ^*- gλ– closed are independent as seen from the following examples. Consider the topology $\tau = \{\emptyset, X, \{b\}\}$. Then the set {b},{a,b},{b,c} are τ^*- gλ– closed but not b - closed . In the topology σ = $\{\emptyset, X, \{b\}, \{c\}, \{b,c\}\}$, the sets {b} and {c} are b- closed but not τ^*- gλ– closed

(x) The concepts of semi pre - closed sets and τ^*- gλ– closed are independent as seen from the following examples. Consider the topology $\tau = \{\emptyset, X, \{b\}\}$. Then the sets {b},{a,b}and {b,c} are τ^*- gλ - closed but not semi pre-closed . In the topology σ = $\{\emptyset, X, \{b\}, \{c\}, \{b,c\}\}$, the sets {b} and {c} are semi pre - closed but not τ^*- gλ– closed.

Remark 3.16

From the above discussions, we obtain the following implications





4. τ^*- g λ– CONTINUOUS FUNCTION IN TOPOLOGICAL SPACES

In this section, we introduce the concept of τ^* - gλ - continuous maps in topological spaces and study some of its properties.

Definition 4.1

A function f: X → Y is called τ^* - gλ - continuous if $f^{-1}(V)$ is τ^* - gλ - closed in X for every closed set V of Y.

Theorem 4.2

Let f: (X, τ) → (Y, σ) be g-continuous. Then f is τ^* - gλ - continuous but not conversely.

Proof: Let V be a closed set in Y . then $f^{-1}(V)$ is g-closed set in (X, τ^*) since f is g-continuous . By theorem " Every g- closed set is τ^* - gλ - closed", $f^{-1}(V)$ is τ^* - gλ - closed set in X. Then f is τ^* - gλ - continuous .

Example 4.3

Let (X, τ) and (Y, σ) be topological spaces where X=Y={a, b, c} with topology τ = {ϕ, X, {a}} and σ = {ϕ, X, {a}, {b,c}}. Let f: X → Y be the identity function Then f is τ^* - gλ - continuous. But f is not g-continuous. Since the closed set {a} in Y, f^{-1}({a}) is not g-closed in X.

Theorem 4.4

Let f: (X, τ) → (Y, σ) be $\hat{g}$-continuous. Then f is τ^* - gλ– continuous but not conversely.

Proof: Let V be a closed set in (Y, σ) . then $f^{-1}(V)$ is g-closed set in (X, τ) since f is $\hat{g}$-continuous . By theorem " Every $\hat{g}$- closed set is τ^* - gλ– closed", $f^{-1}(V)$ is τ^* - gλ - closed set in (X, τ). Then f is τ^* - gλ - continuous .

Example 4.5

Let (X, τ) and (Y, σ) be topological spaces where X=Y={a, b, c} with topology τ = {ϕ, X, {b}} and σ = {ϕ, X, {b}, {a,c}}. Let f: X → Y be the identity function. Then f is τ^* - gλ - continuous. But f is not $\hat{g}$-continuous. Since the closed set {b} in Y, f^{-1}({b}) is not $\hat{g}$-closed in X.

Theorem 4.6

Let f: (X, τ) → (Y, σ) and g: (Y, σ) → (Z, γ) be two functions. Then g . f in τ^* - gλ - continuous if g is continuous and f is τ^* - gλ - continuous

Proof: Let V be the closed set in Z. Then $g^{-1}(V)$ is closed in Y. Since g is continuous. since f is τ^* - gλ - continuous $f^{-1}(g^{-1}(V))$ is τ^* - gλ *- closed in X. But $f^{-1}(g^{-1}(V))$ = (g . f $)^{-1}(V)$. Thus g . f is τ^* - gλ - continuous.

Theorem 4.7

Let f: (X, τ) → (Y, σ) be a map. The following statement are equivalent

(a) f is τ^* - gλ– continuous

(b) The inverse image of each open- set in Y is τ^* - gλ– open in X.

Proof: (a)⇒(b) . Let G be a open set in Y. Then G^c is closed in Y. Since f is τ^* - gλ - continuous $f^{-1}(G^c)$ is τ^* - gλ– continuous in X. But $f^{-1}(G^c)$ = $[f^{-1}(G)]^c$. Thus $f^{-1}(G)$ is τ^* - gλ–open in X.

(b) ⇒ (a) Let V be a closed set of Y. Then V^c is open in Y. By assumption (b) $f^{-1}(V^c)$ is τ^* - gλ - open in X. Thus $f^{-1}(V)$ is τ^* - gλ - closed in X. Therefore f is τ^* - gλ - continuous.

Remark 4.8

The concepts of τ^* - gλ- continuity and semi – continutity are independent, as seen from the following examples.

Example 4.9

Let X = Y = {a, b, c}. Let f: (X, τ) → (Y, σ) be the identity map where τ = {ϕ, X, {a, b}} and σ = {ϕ, Y, {a}, {b}, {a,b}}. Then f is τ^* - gλ - continuous but not semi- continuous. Since the inverse image of the closed set {a, c} and {b,c} in Y are not semi- closed in X.

Example 4.10

Let X = Y = {a, b, c}, τ = {ϕ, X, {a}, {b}, {a,b}} and σ = {ϕ, X, {b,c}}. Let f: (X, τ) → (Y, σ) be the identity map. Then f is semi – continuous but not τ^* - gλ - continuous, since the inverse image of the closed set {a} in Y is not τ^* - gλ - continuous .

Remark 4.11

The concepts of τ^* - gλ - continuity and gs – continutity are independent, as seen from the following examples.

Example 4.12

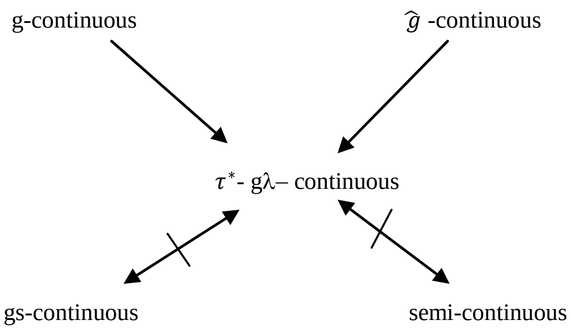
Let X = Y = {a, b, c}, τ = {ϕ, X, {a, b}} and σ = {ϕ, Y, {b, c}}. Let f: (X, τ) → (Y, σ) be the identity map. Then f is τ^* - gλ - continuous but not gs- continuous, since the inverse image of the closed set {a} in Y is not gs-closed in X.

Example 4.13

Let X = Y = {a, b, c}, τ = {ϕ, X, {a}, {b}, {a, b}} and σ = {ϕ, Y, {b, c}}. Let f: (X, τ) → (Y, σ) be the identity map. Then f is gs-continuous but not τ^* - gλ - continuous, since the inverse image of the closed set {a} in Y is not λ-closed in X.

Remark 4.14

From the above discussions, we obtain the following implications



Definition 4.15

A topological space (X, τ) is called $T_{\tau^* - g\lambda}$ - space if every τ^* - gλ - closed set is closed in X.

Theorem 4.16

If X is $T_{\tau^* - g\lambda}$ - space and f: (X, τ) → (Y, σ) is τ^* - gλ - continuous , then f is continuous.

Proof: Let f: (X, τ) → (Y, σ) be τ^* - gλ - continuous and let V be any closed set in (Y, σ^*). Then $f^{-1}(V)$ is τ^* - gλ - closed set in X, since f is τ^* - gλ - continuous. Since X is $T_{\tau^* - g\lambda}$ - space $f^{-1}(V)$ is closed in X. Hence f is continuous.

5. CONCLUSION

In this paper we have introduced τ^*- gλ– closed sets , τ^*- gλ– open sets and studied some of its basic properties.Also we have studied the relationship between τ^*- gλ– continuous maps and some of the mappings already exist.

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