On V_n-Arithmetic Graph

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ABSTRACT

V_n-Arithmetic graph has been introduced by Vasumathi and Vangipuram [9]. In this paper some properties of V_n-Arithmetic graph, maximum degree, minimum degree, number of edges, diameter, radius, Hamiltonian and Eulerian are studied. Also, we introduce m-Arithmetical graphs. Some properties and interesting results for m-Arithmetical graphs are established.

Keywords

V_n-Arithmetic graph, m-Arithmetic graph

1. INTRODUCTION

Number Theory is one of the oldest branches of mathematics, which inherited rich contributions from almost all greatest mathematicians, ancient and modern. Nathanson [5] was the pioneer in introducing the concepts of Number Theory, particularly, the Theory of Congruences in Graph Theory, and paved the way for the emergence of a new class of graphs, namely Arithmetic Graphs. Inspired by the interplay between Number Theory and Graph Theory several researchers in recent times are carrying out extensive studies on various Arithmetic graphs in which adjacency between vertices is defined through various arithmetic functions.

All the graphs considered here are finite and undirected with no loops and multiple edges. Let $G = (V, E)$ be a graph. As usual $|V|$ and $|E|$ denote the number of vertices and edges of a graph G, respectively. In general, we use $\langle X \rangle$ to denote the subgraph induced by the set of vertices X and $N(v)$ and $N[v]$ denote the open neighbourhood and closed neighbourhood of a vertex v, respectively. The degree of a vertex v, in a graph G, is denoted $deg(v)$, and is defined to be the number of edges incident with v. In simple graphs, $deg(u) = |N(u)|$. The minimum degree of a graph G is denoted by δ, and the maximum degree is denoted by Δ. If $\delta = \Delta = r$ for any graph G, we say G is a regular graph of degree r.

The distance $d(u, v)$ between any two vertices $u, v \in G$ is the minimum length of a u-v path, and the eccentricity of a vertex v of a connected graph G is $e(v) = \max\{d(u, v), v \in V\}$. The diameter of G is $diam(v) = \max\{e(v), v \in V\}$ and the radius of G is $rad(v) = \min\{e(v), v \in V\}$. A cycle passing through all the vertices of a graph is called a Hamiltonian cycle. A graph containing a Hamiltonian cycle is called a Hamiltonian graph. Also, A closed walk in a graph G containing all the edges of G is called an Euler line in G. A graph containing an Euler line is called an Euler graph.

All the definitions in this paper are referenced by [1].

Vasumathi and Vangipuram [9] introduced the concept of V_n-Arithmetic graphs and studied some of its properties. Let n be a positive integer such that $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots {p_k}^{\alpha_k}$. Then the V_n-Arithmetic graph is defined as the graph whose vertex set consists of the divisors of n and two vertices u, v are adjacent in V_n graph if and only if $\gcd(u, v) = p_i$ for some prime divisor p_i of n. In this graph vertex 1 becomes an isolated vertex. Hence we consider V_n-Arithmetic graph without vertex 1 as the contribution of this isolated vertex is nothing when the properties of these graphs and enumeration of some domination parameters are studied. Clearly, V_n graph is a connected graph. Because if n is a prime, then V_n graph consists of a single vertex. Hence it is a connected graph. In other cases, by the definition of adjacency in V_n there exist edges between prime number vertices and their prime power vertices and also to their prime product vertices. Therefore each vertex of V_n is connected to some vertex in V_n. Some domination parameters and domination parameters of direct product graphs of Cayley graphs with Arithmetic graphs are presented in [6, 7, 4, 8, 3, 2].

In this paper, we obtain some properties of V_n-Arithmetic graph, maximum degree, minimum degree, number of edges, diameter, radius, Hamiltonian and Eulerian. Also, we introduce m-Arithmetical graphs. Some properties and interesting results for m-Arithmetical graphs are established.

2. SOME PROPERTIES OF V_N-ARITHMETIC GRAPH

In this section some properties of V_n-Arithmetic graph are obtained.

PROPOSITION 1. *Let G be a V_n-Arithmetic graph, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots {p_k}^{\alpha_k}$. Then the number of vertices of G is*

$$|V| = \prod_{i=1}^{k} (\alpha_i + 1) - 1.$$

PROOF. Straightforward by Fundamental Theorem of Arithmetic, the number of positive divisors $d(n)$ of any natural number $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots {p_k}^{\alpha_k}$ is

$$d(n) = \prod_{i=1}^{k} (\alpha_i + 1),$$

then, by the definition of V_n-Arithmetic graph

$$|V| = \prod_{i=1}^{k}(\alpha_i + 1) - 1.$$

□

THEOREM 1. *Let G be a V_n-Arithmetic graph, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$. For any vertex $u = \prod_{i\in B} p_i^{a_i}$, where $B \subseteq \{1, 2, \dots, k\}$, $1 \le a_i \le \alpha_i$, $\forall i \in B$,*

(1) If $u = p_j$, where $j \in \{1, 2, \dots, k\}$, then

$$deg(u) = \alpha_j \prod_{\substack{i=1\\ i\neq j}}^{k}(\alpha_i + 1) - 1.$$

(2) If $u = \prod_{i\in B} p_i^{a_i}$, $1 < a_i \le \alpha_i$, $\forall i \in B$, then

$$deg(u) = |B| \prod_{\substack{i=1\\ i\notin B}}^{k}(\alpha_i + 1).$$

(3) If $u = \prod_{i\in B} p_i^{a_i}$, $a_i = 1$ for some $i \in B' \subseteq B$, then

$$deg(u) = (|B - B'| + \sum_{i\in B'} \alpha_i) \prod_{\substack{i=1\\ i\notin B}}^{k}(\alpha_i + 1).$$

PROOF. Let $u = \prod_{i\in B} p_i^{a_i}$ be a vertex of G, where $B \subseteq \{1, 2, \dots, k\}$, $1 \le a_i \le \alpha_i$, $\forall i \in B$. Since, $|V| = \prod_{i=1}^{k}(\alpha_i + 1) - 1$, then the degree of $u \in V(G)$ is the number of vertices in G minus the number of the vertices which are not adjacent to u, so,
Case 1. Let $u = p_j$, $j \in \{1, 2, \dots, k\}$. Then

$$deg(u) = \left(\prod_{i=1}^{k}(\alpha_i + 1) - 1\right) - \left(\prod_{\substack{i=1\\ i\neq j}}^{k}(\alpha_i + 1) - 1\right) - 1$$

$$= \alpha_j \prod_{\substack{i=1\\ i\neq j}}^{k}(\alpha_i + 1) - 1.$$

Case 2. Let $u = \prod_{i\in B} p_i^{a_i}$, $1 < a_i \le \alpha_i$, $\forall i \in B$. Suppose that, $B = \{1, 2, \dots, r\}$, $r \le k$ (elements of B need not be ordered). Then

$$deg(u) = \left(\prod_{i=1}^{k}(\alpha_i + 1) - 1\right) - \left(\prod_{\substack{i=1\\ i\notin B}}^{k}(\alpha_i + 1) - 1\right)$$

$$-\left(\sum_{i=1}^{r}(\alpha_i - 1)\right) \prod_{\substack{i=1\\ i\notin B}}^{k}(\alpha_i + 1) - \left(\sum_{i=1}^{r-1}\alpha_i \sum_{j=i+1}^{r}\alpha_j\right)$$

$$\prod_{\substack{i=1\\ i\notin B}}^{k}(\alpha_i + 1) - \left(\sum_{i=1}^{r-2}\alpha_i \sum_{j=i+1}^{r-1}\alpha_j \sum_{l=j+1}^{r}\alpha_l\right) \prod_{\substack{i=1\\ i\notin B}}^{k}(\alpha_i + 1)$$

$$- \dots - \left(\prod_{i=1}^{r}\alpha_i\right) \prod_{\substack{i=1\\ i\notin B}}^{k}(\alpha_i + 1) = |B| \prod_{\substack{i=1\\ i\notin B}}^{k}(\alpha_i + 1).$$

Because,

$$\prod_{i=1}^{r}(\alpha_i + 1) = 1 + \sum_{i=1}^{r}\alpha_i + \sum_{i=1}^{r-1}\alpha_i \sum_{j=i+1}^{r}\alpha_j + \sum_{i=1}^{r-2}\alpha_i$$

$$\sum_{j=i+1}^{r-1}\alpha_j \sum_{l=j+1}^{r}\alpha_l + \dots + \prod_{i=1}^{r}\alpha_i.$$

Case 3. Let $u = \prod_{i\in B} p_i^{a_i}$, where $a_i = 1$ for some $i \in B' \subseteq B$. Then, all the vertices of the form $p_j^{a} \prod_{\substack{i=1\\ i\notin B}} p_i^{a_i}$, where $1 \le a \le \alpha_j$, $\forall j \in B$ are adjacent to u, then

$$deg(u) = \prod_{i=1}^{k}(\alpha_i + 1) - \prod_{\substack{i=1\\ i\notin B}}^{k}(\alpha_i + 1) - \left(\sum_{i\in B-B'}(\alpha_i - 1)\right)$$

$$\prod_{\substack{i=1\\ i\notin B}}^{k}(\alpha_i + 1) - \left(\sum_{i=1}^{r-1}\alpha_i \sum_{j=i+1}^{r}\alpha_j\right) \prod_{\substack{i=1\\ i\notin B}}^{k}(\alpha_i + 1)$$

$$-\left(\sum_{i=1}^{r-2}\alpha_i \sum_{j=i+1}^{r-1}\alpha_j \sum_{l=j+1}^{r}\alpha_l\right) \prod_{\substack{i=1\\ i\notin B}}^{k}(\alpha_i + 1)$$

$$- \dots - \left(\prod_{i=1}^{r}\alpha_i\right) \prod_{\substack{i=1\\ i\notin B}}^{k}(\alpha_i + 1)$$

$$= \left(|B - B'| + \sum_{i\in B'}\alpha_i\right) \prod_{\substack{i=1\\ i\notin B}}^{k}(\alpha_i + 1).$$

□

From Theorem 1. it is easy to see that for any two vertices u, v of G such that $u = \prod_{i=1}^{j} p_i^{a_i}$, $v = \prod_{i=1}^{j} p_i^{b_i}$, $j \le k$ (the elements of $\{1, 2, \dots, j\}$ need not be ordered), then $deg(u) = deg(v)$, where $1 < a_i \le \alpha_i$, $1 < b_i \le \alpha_i$, $\forall i \in \{1, 2, \dots, j\}$. Let G be a V_n-Arithmetic graph, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k{}^{\alpha_k}$. Then,

$$|V| = \begin{cases} deg(p_j) + deg(p_j^{a}), & \text{if at least } \alpha_j \neq 1 \text{ and } 1 < a \le \alpha_j; \\ deg(p_j) + deg(p_r p_s), & \text{if } \alpha_i = 1, \forall i \in \{1, 2, \dots, k\}. \end{cases}$$

PROOF. Let G be a V_n-Arithmetic graph, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k{}^{\alpha_k}$.
Case1. Suppose that, at least $\alpha_j \neq 1$ and $1 < a \le \alpha_j$. Then

$$deg(p_j) + deg(p_j^{a}) = \alpha_j \prod_{\substack{i=1\\ i\neq j}}^{k}(\alpha_i + 1) - 1 + \prod_{\substack{i=1\\ i\neq j}}^{k}(\alpha_i + 1) = |V|.$$

Case2. Suppose that, $\alpha_i = 1, \forall i \in \{1, 2, \dots, k\}$. Then the number of vertices of G in this case is $|V| = 2^k - 1$ and so,

$$deg(p_j) + deg(p_r p_s) = 2^{k-1} - 1 + 2^{k-1} = |V|.$$

□

THEOREM 2. *Let G be a V_n-Arithmetic graph, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k{}^{\alpha_k}$, such that at least one of α_i, $i \in \{1, 2, \dots, k\}$ does not equal one. Then,*

(1) $\Delta(G) = \alpha_j \prod_{\substack{i=1\\ i\neq j}}^{k} (\alpha_i + 1) - 1$,
where, α_j is the maximum exponent of p_i, $i \in \{1, 2, \dots, k\}$;

(2) $\delta(G) = k$.

PROOF. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k{}^{\alpha_k}$ such that at least one of α_i, $i \in \{1, 2, \dots, k\}$ does not equal one. Then, we have the following cases:

(1) Suppose that, $u = p_j$ for some $j \in \{1, 2, \dots, k\}$) be a vertex of G such that α_j is the maximum exponent of p_i, $i \in \{1, 2, \dots, k\}$. The vertex $u = p_j$ has the maximum degree of G which is

$$deg(u) = \alpha_j \prod_{\substack{i=1\\ i\neq j}}^{k} (\alpha_i + 1) - 1.$$

By Theorem 1 the vertex $u = p_j$ has the greatest degree among the primes and the power primes vertices of G. Furthermore Theorem 1 tell us the vertices of the form $p_i p_j$, $i \neq j$, $(i, j \in \{1, 2, \dots, k\})$ have degrees greater than the degrees of all the primes product vertices, which is smaller than or equal the degree of $u = p_j$, because,
either,

$$deg(p_j) - deg(p_j p_l) = \Bigg(\alpha_j \prod_{\substack{i=1\\ i\neq j}}^{k} (\alpha_i + 1) - 1\Bigg)$$

$$-\big(\alpha_j + \alpha_l\big) \prod_{\substack{i=1\\ i\neq j,l}}^{k} (\alpha_i + 1)$$

$$= -1 + \alpha_l\big(\alpha_j - 1\big) \prod_{\substack{i=1\\ i\neq j,l}}^{k} (\alpha_i + 1) \geq 0$$

since, $\alpha_j > 1$.
or,

$$deg(p_j) - deg(p_r p_s) = \Bigg(\alpha_j \prod_{\substack{i=1\\ i\neq j}}^{k} (\alpha_i + 1) - 1\Bigg)$$

$$-\big(\alpha_r + \alpha_s\big) \prod_{\substack{i=1\\ i\neq r,s}}^{k} (\alpha_i + 1)$$

$$= -1 + \Bigg((\alpha_r \alpha_s + 1)(\alpha_j + 1) - (\alpha_r + 1)(\alpha_s + 1)\Bigg) \prod_{\substack{i=1\\ i\neq j,r,s}}^{k} (\alpha_i + 1)$$

since, all of α_i, $i \in \{1, 2, \dots, k\}$ are positive integers and $\alpha_j > 1$ is the greatest value of them then,

$$(\alpha_r \alpha_s + 1)(\alpha_j + 1) - (\alpha_r + 1)(\alpha_s + 1) \geq 1$$

so,

$$deg(p_j) - deg(p_r p_s) \geq 0.$$

Which leads to the required result.

(2) On the other hand, the vertex $t = \prod_{i=1}^{k} p_i^{\alpha_i}$ has the minimum degree of G which is $deg(t) = k$ (Theorem 1). Suppose that, $u = \prod_{i=1}^{j} p_i^{a_i}$, $j < k$ (elements of $\{1, 2, \dots, j\}$ need not be ordered) be a vertex of G, where $1 \leq a_i \leq \alpha_i$, $\forall i \in \{1, 2, \dots, j\}$. Then,

$$deg(u) > j + \binom{j}{1}\Bigg[\binom{k-j}{1} + \binom{k-j}{2} + \dots$$

$$+ \binom{k-j}{k-j}\Bigg] > j2^{k-j} \geq k.$$

Also, if $j = k$, then by Theorem 1 $deg(u) \geq k$.

□

THEOREM 3. *Let G be a V_n-Arithmetic graph, where $n = p_1 p_2 \dots p_k$ ($\alpha_i = 1, \forall i \in \{1, 2, \dots, k\}$). Then,*

(1) $\Delta(G) = 2^{k-1}$

(2) $\delta(G) = \begin{cases} k, & k \geq 3; \\ 1, & k = 2. \end{cases}$

PROOF. Let $n = p_1 p_2 \dots p_k$. By substitute $\alpha_i = 1$, $\forall i \in \{1, 2, \dots, k\}$ in Theorem 1 we get for any $u \in G$,

$$deg(u) = \begin{cases} 2^{k-1} - 1, & \text{if } u = p_j, \text{ for some } j \in \{1, 2, \dots, k\}; \\ |B| 2^{k-|B|}, & \text{if } u = \prod_{i \in B} p_i. \end{cases}$$

We observe that when $|B| = 2$, then the vertices $u = p_i p_j$, $i, j \in \{1, 2, \dots, k\}$ have the maximum degree of G which is

$$\Delta(G) = deg(u) = 2^{k-1}$$

Also, the minimum degree of G,

$$\delta(G) = deg(\prod_{i=1}^{k} p_i) = k,$$

where $k \geq 3$. But, if $k = 2$ (i.e., $n = p_1 p_2$), then $V = \{p_1, p_2, p_1 p_2\}$. So, $deg(p_1) = 1$, $deg(p_2) = 1$ and $deg(p_1 p_2) = 2$, hence, $\delta(G) = 1$. □

The connectivity $\kappa = \kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph. For $\kappa \geq k$, we say that G is k-connected. By other words, a k-connected graph is the graph that the removal of fewer than k vertices will not disconnect it.

THEOREM 4. *Let G be a V_n-Arithmetic graph, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ such that $n \neq p_1 p_2$. Then G is a k-connected graph.*

PROOF. Let G be a V_n-Arithmetic graph, where $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$ such that $n \neq p_1p_2$. Then from Theorem 2 and Theorem 3 the minimum degree of G is $\delta(G) = k$, which is the degree of the vertices of the form $t = \prod_{i=1}^{k} p_i^{a_i}$, where $1 < a_i \leq \alpha_i$, $\forall i \in \{1, 2, \dots, k\}$ (or $t = \prod_{i=1}^{k} p_i$ if $\alpha_i = 1, \forall i \in \{1, 2, \dots, k\}$) because, those vertices adjacent only to the prime vertices p_i, $i \in \{1, 2, \dots, k\}$ and each the other vertices have degrees greater than or equal k. So, the minimum number of vertices whose removal disconnect the graph G is k. Hence, G is a k-connected graph. □

By using the Euler Theorem, the number of edges of any graph $G(V, E)$ is given by:

$$|E| = \frac{1}{2}\sum_{i=1}^{n} deg(u_i), \qquad \forall u_i \in V.$$

PROPOSITION 2. *Let G be a V_n-Arithmetic graph, where $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$. Then*

$$|E| = \frac{1}{2}\Big(\sum_{i=1}^{k} deg(p_i) + \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} deg(p_ip_j) +$$

$$\sum_{i=1}^{k-2}\sum_{j=i+1}^{k-1}\sum_{l=j+1}^{k} deg(p_ip_jp_l) + \dots + deg(\prod_{i=1}^{k} p_i) +$$

$$k\sum_{j=1}^{k}(\alpha_j - 1)deg(p_j^a) + \sum_{j=1}^{k}(\alpha_j - 1)\sum_{\substack{l=1\\ l\neq j}}\sum_{\substack{m=1\\ m\neq j,l}}^{k} deg(p_j^ap_lp_m)$$

$$+ \dots + \sum_{j=1}^{k}(\alpha_j - 1)deg(p_j^a\prod_{\substack{i=1\\ i\neq j}}^{k}) + \sum_{j=1}^{k-1}(\alpha_j - 1)\sum_{l=j+1}^{k}(\alpha_l - 1)$$

$$deg(p_j^ap_l^b) + \sum_{j=1}^{k-1}(\alpha_j - 1)\sum_{l=j+1}^{k}(\alpha_l - 1)\sum_{\substack{m=1\\ m\neq j,l}}^{k} deg(p_j^ap_l^bp_m)$$

$$+\sum_{j=1}^{k-1}(\alpha_j - 1)\sum_{l=j+1}^{k}(\alpha_l - 1)\sum_{\substack{m=1\\ m\neq j,l}}^{k}\sum_{\substack{r=1\\ m\neq j,l,m}}^{k} deg(p_j^ap_l^bp_mp_r)$$

$$+ \dots + \sum_{j=1}^{k-1}(\alpha_j - 1)\sum_{l=j+1}^{k}(\alpha_l - 1)deg(p_j^ap_l^b\prod_{\substack{i=1\\ i\neq j,l}}^{k} p_i)$$

$$+ \dots + (\prod_{i=1}^{k}(\alpha_i - 1))deg(\prod_{i=1}^{k} p_i^{a_i})\Big)$$

such that, $1 < a \leq \alpha_j$, $1 < b \leq \alpha_l$, ...

It's clearly that, this formula is very long and difficult to use, and we cannot reduce it to a short formula because the degrees of the vertices of G are depending on the powers α_i's and the options of their products. But we can make a short formula for special cases of G.

THEOREM 5. *Let G be a V_n-Arithmetic graph, where $n = p_1^{\alpha}p_2^{\alpha}\dots p_k^{\alpha}$ ($\alpha_i = \alpha$, $\forall i \in \{1, 2, \dots, k\}$). Then the number of edges of G is given by*

$$|E| = \frac{1}{2}\Big[-k + \sum_{j=0}^{k}\binom{k}{j}(\alpha-1)^j\sum_{i=0}^{k-j}\binom{k-j}{i}(i\alpha+j)(\alpha+1)^{k-i-j}\Big].$$

PROOF. Let G be a V_n-Arithmetic graph, where $n = p_1^{\alpha}p_2^{\alpha}\dots p_k^{\alpha}$. Then

$$|E| = \frac{1}{2}\sum_{i=1}^{(\alpha+1)^k-1} deg(u_i), \qquad u_i \in V.$$

So,

$$\sum_{i=1}^{(\alpha+1)^k-1} deg(u_i) = \binom{k}{1}deg(p_j) + \binom{k}{2}deg(p_jp_l) + \dots + \binom{k}{k}$$

$$deg(\prod_{i=1}^{k} p_i) + \binom{k}{1}\binom{\alpha-1}{1}deg(p_j^a) + \binom{k}{1}\binom{\alpha-1}{1}\binom{k-1}{1}$$

$$deg(p_j^ap_l) + \binom{k}{1}\binom{\alpha-1}{1}\binom{k-1}{2}deg(p_j^ap_lp_m) + \dots + \binom{k}{1}$$

$$\binom{\alpha-1}{1}\binom{k-1}{k-1}deg\big(p_j^a\prod_{\substack{i=1\\ i\neq j}}^{k} p_i) + \binom{k}{2}\Big[\binom{\alpha-1}{1}\Big]^2 deg(p_j^ap_l^b)$$

$$+\binom{k}{2}\Big[\binom{\alpha-1}{1}\Big]^2\binom{k-2}{1}deg(p_j^ap_l^bp_m) + \dots + \binom{k}{2}\Big[\binom{\alpha-1}{1}\Big]^2$$

$$\binom{k-2}{k-2}deg\big(p_j^ap_l^b\prod_{\substack{i=1\\ i\neq j,l}}^{k} p_i) + \dots + \binom{k}{k-1}\Big[\binom{\alpha-1}{1}\Big]^{k-1}$$

$$deg(\prod_{\substack{i=1\\ i\neq j}}^{k} p_i^{a_i}) + \binom{k}{k-1}\Big[\binom{\alpha-1}{1}\Big]^{k-1}$$

$$deg\big(p_j\prod_{\substack{i=1\\ i\neq j}}^{k} p_i^{a_i}) + \binom{k}{k}\Big[\binom{\alpha-1}{1}\Big]^{k} deg(\prod_{i=1}^{k} p_i^{a_i})$$

where, $1 < a \leq \alpha_j$, $1 < b \leq \alpha_l$, ...
By substitute $\alpha_i = \alpha$, $\forall i \in \{1, 2, \dots, k\}$ in Theorem 1 we get for any vertex u of G

$$(1)\quad deg(u) = \begin{cases} \alpha(\alpha+1)^{k-1} - 1, & \text{if } u = p_i, i \in \{1, 2, \dots, k\}; \\ |B|\big(\alpha+1\big)^{k-|B|}, & \text{if } u = \prod_{i\in B} p_i^a, 1 < a \leq \alpha, \forall i \in B. \end{cases}$$

(2) $deg(u) = \big(|B - B'| + |B'|\alpha\big)\big(\alpha+1\big)^{k-|B|}$, if $u = \prod_{i\in B} p_i^a$, $a = 1$ for some $i \in B' \subseteq B$.

Then

$$\sum_{i=1}^{(\alpha+1)^k-1} deg(u_i) = -k + \sum_{i=0}^{k} i\alpha\binom{k}{i}(\alpha+1)^{k-i} + k(\alpha-1)$$

$$\sum_{i=0}^{k-1}\binom{k-1}{i}(i\alpha+1)(\alpha+1)^{k-i-1} +$$

$$\binom{k}{2}\big(\alpha-1\big)^2\sum_{i=0}^{k-2}\binom{k-2}{i}\big(i\alpha+2\big)\big(\alpha+1\big)^{k-i-2} + \dots$$

$$+\binom{k}{k-1}\big(\alpha-1\big)^{k-1}\big(i\alpha+k-1\big)\big(\alpha+1\big) + k\big(\alpha-1\big)^k$$

$$= -k + \sum_{j=0}^{k}\binom{k}{j}\big(\alpha-1\big)^j\sum_{i=0}^{k-j}\binom{k-j}{i}\big(i\alpha+j\big)\big(\alpha+1\big)^{k-i-j}.$$

□

Let G be a V_n-Arithmetic graph, where $n = p_1p_2\dots p_k$ ($\alpha_i = 1$, $\forall i \in \{1,2,\dots,k\}$). Then the number of edges of G is given by

$$|E| = \frac{1}{2}\left(\sum_{i=1}^{k-1} i\binom{k}{i}2^{k-i}\right).$$

PROPOSITION 3. *Let G be a V_n-Arithmetic graph, where $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$. Then, the diameter of G at most equal 2.*

PROOF. Let G be a V_n-Arithmetic graph, such that $V(G) = X_1 \cup X_2 \cup X_3$, where $X_1 = \{p_i : i = 1,2,\dots,k\}$, $X_2 = \{p_i^a : 1 < a \le \alpha_i, i = 1,2,\dots,k\}$ and $X_3 = V - \big(X_1 \cup X_2\big)$. we have two cases:

Case I. Suppose that, $k > 1$ and $n \ne p_1p_2$. Then, we have the following subcases:

Subcase 1. Assume that, $u, v \in X_1$, where $u = p_i$, $v = p_j$, $i \ne j$. Then there is a vertex $t \in X_3$, $t = p_ip_j$ which adjacent to both u and v. So, $d(u,v) = 2$.

Subcase 2. Assume that, $u, v \in X_2$. Then we have two possibilities:

(1) Let u and v are different powers of the same prime p_i. Then they are adjacent to the vertex $p_i \in X_1$. So, $d(u,v) = 2$.
(2) Let $u = p_i^a$ and $v = p_j^b$, $i \ne j$. Then there is a vertex $t \in X_3$, $t = p_ip_j$ which adjacent to both u and v. So, $d(u,v) = 2$.

Subcase 3. Assume that, $u, v \in X_3$. Then there are two possibilities:

(1) Let $\gcd(u,v) = \prod_{i\in B} p_i^{a_i}$ where $B \subseteq \{1,2,\dots,k\}$, $1 \le a_i \le \alpha_i, \forall i \in B$. Then u and v are either adjacent (if $|B| = 1$ and $a = 1$) or choose $r \in B$ such that the vertex $p_r \in X_1$ is adjacent to both u and v. So, $d(u,v) = 1$ or $d(u,v) = 2$.
(2) Let $\gcd(u,v) = 1$. Then there is a vertex $t \in X_3$, $t = p_rp_s$ such that, p_r is a prime divisor of u and p_s is a prime divisor of v which is adjacent to both u and v. So, $d(u,v) = 2$.

Subcase 4. Assume that, $u \in X_1$ and $v \in X_2$. Then there are two possibilities:

(1) Let $\gcd(u,v) = p_i$, $i \in \{1,2,\dots,k\}$. Then $d(u,v) = 1$.
(2) Let $\gcd(u,v) = 1$, i.e. $u = p_r, v = p_s^a$, where $r \ne s$. Then there exists a vertex $t \in X_3$, $t = p_rp_s$ which is adjacent to both u and v. So, $d(u,v) = 2$.

Subcase 5. Assume that, $u \in X_1$ and $v \in X_3$. Then there are two possibilities:

(1) Let $\gcd(u,v) = p_i$, $i \in \{1,2,\dots,k\}$. Then $d(u,v) = 1$.
(2) Let $\gcd(u,v) = 1$, i.e. $u = p_r$, $v = \prod_{\substack{i\in B\\ r\notin B}} p_i^{a_i}$, $B \subseteq \{1,2,\dots,k\}$, $1 \le a_i \le \alpha_i, \forall i \in B$. Then there exists a vertex $t \in X_3$, $t = p_rp_s$ such that p_s is a prime divisor of v which is adjacent to both u and v. So, $d(u,v) = 2$.

Subcase 6. Assume that, $u \in X_2$ and $v \in X_3$. Then we characterize two possibilities:

(1) Let $\gcd(u,v) = p_r^b$, $1 \le b \le a \le \alpha_r$. Then u and v are either adjacent (if $b = 1$) or there is a vertex $p_r \in X_1$ which is adjacent to both u and v. So, $d(u,v) = 1$ or $d(u,v) = 2$.
(2) Let $\gcd(u,v) = 1$, i.e. $u = p_r^a$, $v = \prod_{\substack{i\in B\\ r\notin B}} p_i^{a_i}$, $B \subseteq \{1,2,\dots,k\}$, $1 \le a_i \le \alpha_i, \forall i \in B$. Then there exists a vertex $t \in X_3$, $t = p_rp_s$ such that p_s is a prime divisor of v which is adjacent to both u and v. So, $d(u,v) = 2$.

Hence, if $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$, $k > 1$ then, $diam(G) = 2$.

Case II. Suppose that, $k = 1$ (i.e. $n = p^\alpha$ where p is prime) or $n = p_1p_2$. Then we characterize three subcases:

Subcase 1. Let $\alpha = 2 \Rightarrow n = p^2$. Then G has only two vertices p, p^2 and one edge joining p and p^2. So, $diam(G) = 1$.

Subcase 2. Let $\alpha > 2$. Then $X_1 = \{p\}$, $X_2 = \{p_2,\dots,p_\alpha\}$ and $X_3 = \phi$, so,

(1) For $u, v \in X_2$, the vertex $p \in X_1$ is adjacent to both u and v. So, $d(u,v) = 2$.
(2) For $u = p \in X_1$ and $v \in X_2$, then $\gcd(u,v) = p$ and hence, $d(u,v) = 1$.

This means that, $e(u) = 2$ for $u \in X_2$ and $e(u) = 1$ for $u \in X_1$.

Subcase 3. Let $n = p_1p_2$. Then $X_1 = \{p_1,p_2\}$, $X_3 = \{p_1p_2\}$ and $X_2 = \phi$. So, as in subcase2, $e(u) = 2$ for $u \in X_1$ and $e(u) = 1$ for $u \in X_3$. □

Let G be a V_n-Arithmetic graph, where $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k{}^{\alpha_k}$ such that, $k > 1$ and $n \ne p_1p_2$. Then

$$diam(G) = 2 = rad(G).$$

THEOREM 6. *Let G be a V_n-Arithmetic graph, where $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$. Then G is not an Eulerian graph.*

PROOF. Suppose that G is a V_n-Arithmetic graph, where $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$. it is well known that the graph G is an Eulerian graph if and only if every vertex of G has an even degree. So,

(1) Let all α_i, $i \in \{1,2,\dots,k\}$ are odd. Then the degrees of the prime vertices p_i, $i \in \{1,2,\dots,k\}$ are odd.
(2) Suppose that, at least one of α_i, $i \in \{1,2,\dots,k\}$ is even call it α_r then the vertex p_r has an odd degree.

□

The Hamiltonian closure of a graph G, denoted $Cl(G)$, is the simple graph obtained from G by repeatedly adding edges joining pairs of nonadjacent vertices with degree sum at least $|V(G)|$ until no such pair remains. A graph G is Hamiltonian if and only if its closure is Hamiltonian.

THEOREM 7. *Let G be a V_n-Arithmetic graph, where $n = p_1p_2\dots p_k$, $3 \le k \le 6$. Then G is a Hamiltonian graph.*

PROOF. Suppose that G is a V_n-Arithmetic graph, where $n = p_1p_2\dots p_k, 3 \le k \le 6$. We show that the Hamiltonian closure of G is a complete graph i.e. $Cl(G) \cong K_{2^k-1}$. and hence, G is Hamiltonian. Let $X_q = \{u \in V : u = \prod_{i\in B_q} p_i,\ B_q \subseteq \{1,2,\dots,k\},\ |B_q| = q \le k\}$ i.e. $X_1 = \{u = p_j : j = 1,2,\dots,k\}$, $X_2 = \{u \in V : u = p_rp_s,\ r,s = 1,2,\dots,k\}$, $X_3 = \{u \in V : u = \prod_{i\in B_3} p_i, |B_3| = 3\}$ and so on.

Step 1. In this step:

(1) The subset of the prime vertices X_1 will be adjacent to all the vertices of the subset X_2 because, $deg(p_j)+deg(p_rp_s) = |V|$ (Corollary 1).

(2) The vertices of X_2 will be adjacent one to each others because, $deg(p_j) \le deg(p_rp_s)$.

So, in the end of this step the degrees of the vertices of X_1 and X_2 will become:

(1) $deg(p_j) = 2^{k-1} - 1 + \binom{k-1}{2}$.

(2) $deg(u)_{u\in X_2} = 2^{k-1} + \binom{k-2}{1} + \binom{k-2}{2}$.

Step 2. In this step:

(1) the vertices of X_1 will be adjacent one to each others because in this case, $deg(p_j) + deg(p_r) \ge |V|$. Also, the vertices of X_1 will be adjacent to all the vertices of X_3 because, $deg(p_j) + deg(u)_{u\in X_3} = 7(2^{k-3}) + \binom{k-1}{2} - 1 \ge |V|$, where $3 \le k \le 6$.

(2) The vertices of X_2 will be adjacent to all the vertices of X_3 because, $deg(u)_{u\in X_2}+deg(u)_{u\in X_3} = 7(2^{k-3})+\binom{k-2}{1}+\binom{k-2}{2} \ge |V|$, where $3 \le k \le 6$.

So, in the end of this step the degrees of the vertices of X_1, X_2 and X_3 will become:

(1) $deg(p_j) = 2^{k-1} - 1 + \binom{k-1}{1} + \binom{k-1}{2} + \binom{k-1}{3}$.

(2) $deg(u)_{u\in X_2} = 2^{k-1} + 2\binom{k-2}{1} + \binom{k-2}{2} + \binom{k-2}{3}$.

(3) $deg(u)_{u\in X_3} = 3(2^{k-3}) + \binom{k-3}{1} + \binom{k-3}{2} + \binom{3}{2}$.

Step3. In this step:

(1) the vertices of X_1 will be adjacent to all the vertices of X_4 because, $deg(p_j)+deg(u)_{u\in X_4} = 3(2^{k-2})+\binom{k-1}{1}+\binom{k-1}{2}+\binom{k-1}{3} - 1 \ge |V|$, where $4 \le k \le 6$.

(2) The vertices of X_2 will be adjacent to all the vertices of X_4 because, $deg(u)_{u\in X_2} + deg(u)_{u\in X_4} = 3(2^{k-2}) + 2\binom{k-2}{1} + \binom{k-2}{2} + \binom{k-2}{3} \ge |V|$, where $4 \le k \le 6$.

So, in the end of this step the degrees of the vertices of X_1, X_2, X_3 and X_4 will become:

(1) $deg(p_j) = 2^{k-1} - 1 + \binom{k-1}{1} + \binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4}$.

(2) $deg(u)_{u\in X_2} = 2^{k-1} + 2\binom{k-2}{1} + 2\binom{k-2}{2} + \binom{k-2}{3} + \binom{k-2}{4}$.

(3) $deg(u)_{u\in X_3} = 3(2^{k-3}) + \binom{k-3}{1} + \binom{k-3}{2} + \binom{3}{2}$.

(4) $deg(u)_{u\in X_4} = 2^{k-2} + \binom{k-4}{1} + \binom{k-4}{2} + \binom{4}{2}$.

and so on. We observe that:

(1) If $k = 3$ then in step 2, $Cl(G) \cong K_{2^3} - 1$.

(2) If $k = 4$ then in step 4, $Cl(G) \cong K_{2^4} - 1$.

(3) If $k = 5$ then in step 5, $Cl(G) \cong K_{2^5} - 1$.

(4) $k = 6$ then in step 6, $Cl(G) \cong K_{2^6} - 1$.

Hence, G is Hamiltonian. □

PROPOSITION 4. *Let G be a V_n-Arithmetic graph, where $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$, such that $\alpha_i > 1$, $\forall i \in B$ where $B \subseteq \{1,2,\dots,k\}$. If $\prod_{i\in B}(\alpha_i - 1) \ge k$, then G is not Hamiltonian.*

PROOF. Let G be a V_n-Arithmetic graph, where $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$, such that $\alpha_i > 1$, $\forall i \in B$ where $B \subseteq \{1,2,\dots,k\}$ and let $V_0 \subset V$ be the subset of vertices of V whose have degree k i.e. $u \in V_0 \Leftrightarrow u$ is adjacent only to the prime vertices p_i, $i \in \{1,2,\dots,k\}$. It is clearly that $|V_0| = \prod_{i\in B}(\alpha_i - 1)$. Suppose $|V_0| \ge k$. Since V_0 is an independent set and all p_i, $i \in \{1,2,\dots,k\}$ are not adjacent one to each others, then G has no Hamiltonian cycle because if there is a spanning cycle of G then it should pass on at least one vertex twice. So, G is not Hamiltonian. □

Let G be a V_n-Arithmetic graph, where $n = p_1^{\alpha_1}p_2^{\alpha_2}$, and $n \ne p_1p_2$. Then G is a Hamiltonian graph if and only if $1 \le \alpha_1, \alpha_2 \le 2$.

PROOF. Let G is a V_n-Arithmetic graph, where $n = p_1^{\alpha_1}p_2^{\alpha_2}$, and $n \ne p_1p_2$. Suppose that G is Hamiltonian and at least $\alpha_1 \ge 3$. Then there exist at least two vertices $t_1, t_2 \in V$ which are adjacent only to both p_1 and p_2. Since p_1, p_2 are not adjacent and t_1, t_2 also are not adjacent, then G has no Hamiltonian cycle. Contradiction. On the other hand if $1 \le \alpha_1, \alpha_2 \le 2$, then we have two cases:
Case 1. If $n = p_1^2p_2^2$, then the cycle $\left(p_1, p_1^2p_2^2, p_2, p_1p_2^2, p_1^2, p_1p_2, p_2^2, p_1^2p_2, p_1\right)$ in G is a Hamiltonian cycle.
Case 2. If $n = p_1^2p_2$ or $n = p_1p_2^2$, then G has the Hamiltonian cycles $\left(p_1, p_1^2p_2, p_2, p_1p_2, p_1^2, p_1\right)$, $\left(p_2, p_1p_2^2, p_1, p_1p_2, p_2^2, p_2\right)$ respectively.
So G is Hamiltonian. □

3. M-ARITHMETICAL GRAPHS

DEFINITION 8. *Let $G(V, E)$ be a connected graph with n vertices. Then G is called m-Arithmetical graph for some integer $m \ge 1$ if and only if there exists at least one Arithmetic graph $V_m \cong G$.*

EXAMPLE 1. *:*

—*P_3 is 6-Arithmetical.*

—*K_2 is 4-Arithmetical.*

—*S_n is 2^n-Arithmetical. (where S_n is the star with n vertices).*

Of course there are infinite Arithmetic graphs V_m which is isomorphic to G, so, we convince to select the V_m where m is the minimum in all cases.
Note that. Since, V_m is a connected graph we conclude that all disconnected graphs are not m-Arithmetical.

THEOREM 9. *Let G be a regular graph. Then G is m-Arithmetical graph if and only if $G \cong P_2$.*

PROOF. Let G be a regular graph. Suppose, G is m-Arithmetical graph. Then there is an Arithmetic graph V_m such that $V_m \cong G$. Then, we have two cases:
Case1. Suppose, $m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k{}^{\alpha_k}$, $k \geq 2$. From the proof of Theorem 2.(1) we have

$$deg(p_j) - deg(p_j p_l) = -1 + \alpha_l \big(\alpha_j - 1\big) \prod_{\substack{i=1 \\ i \neq j,l}}^{k} (\alpha_i + 1),$$

where, $j, l \in \{1, 2, \dots, k\}$, $j \neq l$.
Let α_j be the minimum degree of p_i, $i \in \{1, 2, \dots, k\}$. Then

(1) if $\alpha_j = 1 \Rightarrow deg(p_j) - deg(p_j p_l) = -1$ which implies that V_m is not regular.
(2) if $\alpha_j > 1$, so, $\alpha_l \geq \alpha_j > 1, \forall l \in \{1, 2, \dots, k\}, l \neq j$, then,

$$deg(p_j) - deg(p_j p_l) > 0.$$

Hence, V_m, where $m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k{}^{\alpha_k}$, $k \geq 2$ is not a regular graph.
Case2. Let $m = p^\alpha$. Then, by Theorem 3.

$$\Delta(V_m) = \alpha - 1, \qquad \delta(V_m) = 1.$$

Since, a graph G is a regular graph if and only if $\Delta(G) = \delta(G)$. Then,

$$\alpha - 1 = 1 \Rightarrow \alpha = 2.$$

So, V_m, where $m = p^2$ is a regular graph and hence, $G \cong V_m \cong P_2$.
The converse is clear. □

Let G be a complete graph. Then G is m-Arithmetical graph if and only if $G \cong K_2$.

THEOREM 10. *Let G be a bipartite graph. Then G is m-Arithmetical graph if and only if G is a star.*

PROOF. Let G be a bipartite graph. Suppose, G is m-Arithmetical graph. Then there exists an Arithmetic graph $V_m \cong G$. We prove that V_m is a star. So, we characterize the following cases:
Case1. Suppose, $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k{}^{\alpha_k}$, $k \geq 2$, where at least one of $\alpha_i \neq 1$, $i \in \{1, 2, \dots, k\}$. Then the graph V_m has a triangle (an odd cycle subgraph) $\langle X_1 \rangle$ where $X_1 = \{p_j, p_j^2, p_j p_l\}$, such that $j, l \in \{1, 2, \dots, k\}$, $j \neq l$. Hence, V_m is not a bipartite graph and so not a star.
Case2. Suppose, $m = p_1 p_2 \dots p_k$, $k \geq 3$ ($\alpha_i = 1$, $\forall i \in \{1, 2, \dots, k\}$). In this case the subgraph $\langle X_2 \rangle$ of V_m where $X_2 = \{p_r, p_r p_s, p_r p_t\}$ is an odd cycle subgraph of V_m. So, V_m is not a star.
Case3. Now it remains that $m = p^\alpha$ or $m = p_1 p_2$,

(1) Let $m = p^\alpha$. Then all the vertices of V_m have a form p^a, $a \in \{1, \dots, \alpha\}$. So, the vertex p is adjacent to all the other vertices. Also, the vertices p^a, $a \in \{2, \dots, \alpha\}$ are not adjacent one to each others because, $\gcd(p^a, p^b) = p^a$ where $a < b$, $a, b \in \{2, \dots, \alpha\}$. Hence, V_m is a star.
(2) Let $m = p_1 p_2$. Then $V = \{p_1, p_2, p_1 p_2\}$. The prime vertices p_1, p_2 are not adjacent one to the other and the vertex $p_1 p_2$ is adjacent to both p_1 and p_2. Hence, V_m is a star.

For the other side, suppose that G is a star. The proof of the first side tell us V_m is a star if and only if $m = p^\alpha$ or $m = p_1 p_2$ for some positive integer α and some primes p_1, p_2 and p. Since, $G = S_n$ for some number of vertices n, then there exists an Arithmetic graph V_m where $m = p^n$ such that $V_m \cong G = S_n$. So, G is m-Arithmetical graph. □

4. CONCLUSION

In this paper, we have studied some of the basic properties of V_n-Arithmetic graph which will greatly help facilitate the study of many other properties and other parameters for this type of graphs, also they will help to conclude the sufficient and necessary conditions for any graph to be m-Arithmetical, which the authors will study it soon.

5. REFERENCES

[1] F. Harary, Graph theory, Addison-Wesley, Reading Mass (1969).

[2] M.Manjuri and B.Maheswari, Strong Dominating Sets of Some Arithmetic Graphs. International Journal of Computer Applications 83(3):36-40, December 2013.

[3] M.Manjuri and B.Maheswari, Some Domination Parameters of Arithmetic Vn Graph. International Journal of Computer Applications 92(11):29-32, April 2014.

[4] M.Manjuri and B Maheswari, Strong Dominating Sets of Direct Product Graph of Cayley Graphs with Arithmetic Graphs. International Journal of Applied Information Systems 7(4):15-21, June 2014.

[5] Nathanson, Melvyn B. Connected Components of Arithmetic Graphs. Monatshefte fr Mathematik. 1980, Vol. 89, Issue 3, pp. 219-222.

[6] S.Uma Maheswari and B.Maheswari, Some Domination parameters of Arithmetic Graph Vn, IOSR, Journal of Mathematics, Volume 2,Issue 6,(Sep- Oct 2012), 14 -18.

[7] S.Uma Maheswari, B.Maheswari and M.Manjuri, Some Domination Parameters of Direct Product Graphs of Cayley Graphs with Arithmetic Graphs. International Journal of Computer Applications 58(9):32-38, November 2012.

[8] S.Uma Maheswari, B.Maheswari and M.Manjuri, Matching Dominating Sets of Direct Product Graphs of Cayley Graphs with Arithmetic Graphs. International Journal of Computer Applications 60(11):11-16, December 2012.

[9] N.Vasumath. Number theoretic graphs. Ph.D. Thesis, S.V.University, Tirupati, India 1994.