

Unreliable Batch Arrival Retrial Queue with Negative Customers, Multi-Types of Heterogeneous Service, Setup Time and Reserved Time

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ABSTRACT

Batch arrival retrial queue with positive and negative customers is considered. Server provides M types of service. Positive customers arrive in batches according to Poisson process. If the server is idle upon the arrival of a batch, one of the customers in the batch receives any one the types immediately and others join the orbit. The server is subject to two different modes of failure. Mode 1 failure occurs due to the arrival of negative customer and Mode 2 due to random breakdown of the server. In both cases, repair starts after some random amount of time. The server failed under mode 2 continues the interrupted service or waits for the same customer after the repair completion. Generating function technique is employed to obtain joint distributions of the server state and orbit length. Expected system size, expected orbit size, availability of the server and failure frequency of the server are derived. Stochastic decomposition law is also verified.

Keywords

Retrial queue - negative customers - multi-types of heterogeneous service - setup time - reserved time.

1. INTRODUCTION

Retrial queueing system are characterized by the fact that an arriving customer who finds the server occupied is obliged to join a group of blocked customers and reapply after random intervals of time to obtain the service. Two extensive survey articles in retrial queue are due to Yang and Templeton [10] and Falin [3].

Queues with server subject to breakdown and repairs are often encountered in many practical applications. Aissani [1] and Kulkarni and Choi [6] considered retrial queueing systems with server breakdowns and repairs. Prakash Rani et al. [8] analysed retrial queueing models with server breakdown and delayed repair. Choudhury and Ke [2] considered a batch arrival retrial queue with general retrial time, Bernoulli vacation, unreliable server and delayed repair.

Queue with negative arrivals called G-queue was first introduced by Gelenbe [4] with a view to modeling neural networks. Wang and Zhang [9] considered a discrete time retrial queue with negative arrivals. Peng et al. [7] suggested an M/G/1 retrial G-queue with pre-emptive resume priority and collisions under linear retrial policy. In this paper, the article by Kirupa and Udaya Chandrika[5] is analyzed by including multi-types of heterogeneous service and reserved time.

2. MODEL DESCRIPTION

Single server queueing system with two types of arrival - positive and negative is considered. Positive customers arrive according to Poisson process with rate λ^+ in groups of random size having distribution function $P(Y=k)=C_k, k \geq 1$, the generating function $C(z)$ and first two moments m_1 and m_2 . The server provides M types of service. Customers opt type i service with probability $p_i (1 \leq i \leq M)$. If the arriving batch of positive customers finds the server free, then one of the arrivals in the batch receives service immediately and others join the orbit. Otherwise the arriving batch joins the retrial queue. The retrial time is generally distributed with distribution function $A(x)$, density function $a(x)$, Laplace Stieltje's transform $A^*(\bullet)$ and conditional completion rate $\eta(x) = a(x)/[1-A(x)]$.

The service time of type $i(i=1,2,\dots,M)$ follows a general distribution with distribution function $B_i(x)$, density function $b_i(x)$, Laplace Stieltje's transform $B_i^*(\bullet)$, n^{th} factorial moments $\mu_{i,n}$ and conditional completion rate $\mu_i(x) = b_i(x)/[1-B_i(x)]$. There are two different modes of server breakdown, say mode 1 and mode 2. Mode 1 is due to negative arrival which removes the positive customer being in service from the system and makes the server down. Mode 2 is random failure of the server. The lifetime of the server providing type i service is exponentially distributed with rate $\alpha_i (i=1,2,\dots,M)$. Once the system fails, the repair starts after a random amount of time. This time is referred as setup time. The setup times of the server failed under mode $l (l=1,2)$ are generally distributed with distribution function $S_1^{(l)}(x)$,

density function $s_1^{(l)}(x)$, Laplace Stieltje's transform $S_1^{(l)*}(\bullet)$, n^{th} factorial moments $\phi_{1,n}^{(l)}$ and conditional completion rate $\phi_1^{(l)}(x) = s_1^{(l)}(x)/[1-S_1^{(l)}(x)]$.

The repair times of mode l breakdown are generally distributed with distribution function $R_1^{(l)}(x)$, density function $r_1^{(l)}(x)$, Laplace Stieltje's transform $R_1^{(l)*}(\bullet)$, n^{th} factorial moments $\beta_{1,n}^{(l)}$ and conditional completion rate $\beta_1^{(l)}(x) = r_1^{(l)}(x)/[1-r_1^{(l)}(x)]$, for $i=1,2,\dots,M$ and $l=1,2$.

During type i service, if the server fails under mode 2 then the interrupted customer either remains in service position with

probability τ_i or leaves the system and keeps returning at times exponentially distributed with rate θ_i . As soon as the server is fixed, it continues the service of the interrupted customer or waits for the same customer to complete the remaining service. This waiting time of the server is referred as reserved time. The server is not allowed to accept new customer until the interrupted customer leaves the system.

3. JOINT DISTRIBUTION OF THE SERVER STATE

For $i=1,2,\dots,M$, we define the server state $J(t)$ as follows

$$J(t) = \begin{cases} 0, & \text{server is idle} \\ i, & \text{server is busy in type } i \text{ service} \\ M+i, & \text{server failed under mode 1 during } i^{\text{th}} \text{ type service is in setup time} \\ 2M+i, & \text{server failed due to mode 1 during } i^{\text{th}} \text{ type service is under repair} \\ 3M+i, & \text{server failed under mode 2 during } i^{\text{th}} \text{ type service is in setup time} \\ 4M+i, & \text{server failed due to mode 2 during } i^{\text{th}} \text{ type service is under repair} \\ 5M+i, & \text{server is under reserved time} \end{cases} \quad \square$$

Let $X(t)$ denote the number of customers in the orbit. The state of the interrupted customer $J^*(t)$ is defined as

$$J^*(t) = \begin{cases} 0, & \text{if the interrupted customer remains in service position} \\ 1, & \text{if the interrupted customer leaves the service area} \end{cases}$$

Define supplementary variable as

$\xi_0(t)$ = elapsed retrial time; $\xi_1(t)$ = elapsed service time ; $\xi_2(t)$ = elapsed setup time ; $\xi_3(t)$ = elapsed repair time;

$\xi_4(t)$ = elapsed reserved time

The state of the system at time t can be described by the Markov processes $\{N(t), t \geq 0\} = \{J(t), J^*(t), X(t), \xi_0(t), \xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t), t \geq 0\}$.

Define the probability densities

$$I_0(t) = P\{J(t)=0, X(t) = 0\}$$

$$I_n(x,t)dx = P\{J(t)=0, X(t) = n, x \leq \xi_0(t) < x+dx\}, n \geq 1$$

For $n \geq 0, i=1,2,\dots,M$ and $j = 0,1$

$$P_{i,n}(x,t)dx = P\{J(t)=i, X(t) = n, x \leq \xi_1(t) < x+dx\}$$

$$S_{i,n}^{(1)}(x,t)dx = P\{J(t)=M+i, X(t) = n, x \leq \xi_2(t) < x+dx\}$$

$$R_{i,n}^{(1)}(x,t)dx = P\{J(t)=2M+i, X(t) = n, x \leq \xi_3(t) < x+dx\}$$

$$S_{i,j,n}^{(2)}(x,y,t)dxdy = P\{J(t)=3M+i, J^*(t)=j, X(t) = n, x \leq \xi_1(t) < x+dx, y \leq \xi_2(t) < y+dy\}$$

$$R_{i,j,n}^{(2)}(x,y,t)dxdy = P\{J(t)=4M+i, J^*(t)=j, X(t) = n, x \leq \xi_1(t) < x+dx, y \leq \xi_3(t) < y+dy\}$$

$$Q_{i,n}(x,y,t)dxdy = P\{J(t)=5M+i, X(t) = n, x \leq \xi_1(t) < x+dx, y \leq \xi_4(t) < y+dy\}$$

The governing equations of the model under steady state are

$$\lambda^+ I_0 = \sum_{i=1}^M \left[\int_0^\infty P_{i,0}(x) \mu_i(x) dx + \int_0^\infty R_{i,0}^{(1)}(x) \beta_i^{(1)}(x) dx \right] \quad (1)$$

$$\frac{d}{dx} I_n(x) = -(\lambda^+ + \eta(x)) I_n(x), n \geq 1 \quad (2)$$

$$\frac{d}{dx} P_{i,n}(x) = -(\lambda^+ + \lambda_i^- + \alpha_i + \mu_i(x)) P_{i,n}(x) + \lambda^+ \sum_{k=1}^n C_k P_{i,n-k}(x) + \int_0^\infty R_{i,0,n}^{(2)}(x,y) \beta_i^{(2)}(y) dy + \theta_i \int_0^\infty Q_{i,n}(x,y) dy, n \geq 0, i=1,2,\dots,M \quad (3)$$

$$\frac{d}{dx} S_{i,n}^{(1)}(x) = -(\lambda^+ + \phi_i^{(1)}(x)) S_{i,n}^{(1)}(x) + \lambda^+ \sum_{k=1}^n C_k S_{i,n-k}^{(1)}(x), n \geq 0, i=1,2,\dots,M \quad (4)$$

$$\frac{d}{dx} R_{i,n}^{(1)}(x) = -(\lambda^+ + \beta_i^{(1)}(x)) R_{i,n}^{(1)}(x) + \lambda^+ \sum_{k=1}^n C_k R_{i,n-k}^{(1)}(x), n \geq 0, i=1,2,\dots,M \quad (5)$$

$$\frac{d}{dy} S_{i,j,n}^{(2)}(x,y) = -(\lambda^+ + \phi_i^{(2)}(y)) S_{i,j,n}^{(2)}(x,y) + \lambda^+ \sum_{k=1}^n C_k S_{i,j,n-k}^{(2)}(x,y), n \geq 0, i=1,2,\dots,M, j=0,1 \quad (6)$$

$$\frac{d}{dy} R_{i,j,n}^{(2)}(x,y) = -(\lambda^+ + \beta_i^{(2)}(y)) R_{i,j,n}^{(2)}(x,y) + \lambda^+ \sum_{k=1}^n C_k R_{i,j,n-k}^{(2)}(x,y), n \geq 0, i=1,2,\dots,M, j=0,1 \quad (7)$$

$$\frac{d}{dy} Q_{i,n}(x,y) = -(\lambda^+ + \theta_i) Q_{i,n}(x,y) + \lambda^+ \sum_{k=1}^n C_k Q_{i,n-k}(x,y), n \geq 0, i=1,2,\dots,M \quad (8)$$

with boundary conditions

$$I_n(0) = \sum_{i=1}^M \left[\int_0^\infty P_{i,n}(x) \mu_i(x) dx + \int_0^\infty R_{i,n}^{(1)}(x) \beta_i^{(1)}(x) dx \right], n \geq 1 \quad (9)$$

$$P_{i,0}(0) = p_i \left[\lambda^+ c_1 I_0 + \int_0^\infty I_1(x) \eta(x) dx \right], i=1,2,\dots,M \quad (10)$$

$$P_{i,n}(0) = p_i \left[\lambda^+ c_n I_0 + \int_0^\infty I_{n+1}(x) \eta(x) dx + \lambda^+ \sum_{k=1}^n C_k \int_0^\infty I_{n-k+1}(x) dx \right], n \geq 1, i=1,2,\dots,M \quad (11)$$

$$S_{i,n}^{(1)}(0) = \lambda^- \int_0^\infty P_{i,n}(x) dx, n \geq 0, i=1,2,\dots,M \quad (12)$$

$$R_{i,n}^{(1)}(0) = \int_0^\infty S_{i,n}^{(1)}(x) \phi_i^{(1)}(x) dx, n \geq 0, i=1,2,\dots,M \quad (13)$$

$$S_{i,0,n}^{(2)}(x,0) = \tau_i \alpha_i P_{i,n}(x), n \geq 0, i=1,2,\dots,M \quad (14)$$

$$S_{i,1,n}^{(2)}(x,0) = (1 - \tau_i) \alpha_i P_{i,n}(x), n \geq 0, i=1,2,\dots,M \quad (15)$$

$$R_{i,j,n}^{(2)}(x,0) = \int_0^\infty S_{i,j,n}^{(2)}(x,y) \phi_i^{(2)}(y) dy, n \geq 0, i=1,2,\dots,M, j=0,1 \quad (16)$$

$$Q_{i,n}(x,0) = \int_0^\infty R_{i,1,n}^{(2)}(x,y) \beta_i^{(2)}(y) dy, n \geq 0, i=1,2,\dots,M \quad (17)$$

The normalizing condition is

$$I_0 + I(0) + \sum_{i=1}^M \left[P_i(0) + S_i^{(1)}(0) + R_i^{(1)}(0) + S_{i,0}^{(2)}(0) + S_{i,1}^{(2)}(0) + R_{i,0}^{(2)}(0) + R_{i,1}^{(2)}(0) + Q_i(0) \right] = 1 \quad (18)$$

Define the joint probability generating functions

$$I(x, z) = \sum_{n=1}^{\infty} I_n(x) z^n,$$

$$P_i(x, z) = \sum_{n=0}^{\infty} P_{i,n}(x) z^n,$$

$$S_i^{(1)}(x, z) = \sum_{n=0}^{\infty} S_{i,n}^{(1)}(x) z^n,$$

$$R_i^{(1)}(x, z) = \sum_{n=0}^{\infty} R_{i,n}^{(1)}(x) z^n,$$

$$S_{i,j}^{(2)}(x, y, z) = \sum_{n=0}^{\infty} S_{i,j,n}^{(2)}(x, y) z^n,$$

$$R_{i,j}^{(2)}(x, y, z) = \sum_{n=0}^{\infty} R_{i,j,n}^{(2)}(x, y) z^n,$$

$$Q_i(x, y, z) = \sum_{n=0}^{\infty} Q_{i,n}(x, y) z^n, \quad i=1 \text{ to } M, j=0,1.$$

Multiplying equations (2) to (17) by z^n and summing over for all possible values of n , we get

$$\left(\frac{d}{dx} + \lambda^+ + \eta(x) \right) I(x, z) = 0 \quad (19)$$

$$\left(\frac{d}{dx} + \lambda^+ + \lambda^- + \alpha_i + \mu_1(x) - \lambda^+ C(z) \right) P_i^{(i)}(x, z) = \int_0^{\infty} R_{i,0}^{(2)}(x, y, z) \beta_i^{(2)}(y) dy + \theta_i \int_0^{\infty} Q_i(x, y, z) dy, \quad i=1,2,\dots,M \quad (20)$$

$$\left(\frac{d}{dx} + \lambda^+ + \phi_i^{(1)}(x) - \lambda^+ C(z) \right) S_i^{(1)}(x, z) = 0, \quad i=1,2,\dots,M \quad (21)$$

$$\left(\frac{d}{dx} + \lambda^+ + \beta_i^{(1)}(x) - \lambda^+ C(z) \right) R_i^{(1)}(x, z) = 0, \quad i=1,2,\dots,M \quad (22)$$

$$\left(\frac{d}{dy} + \lambda^+ + \phi_i^{(2)}(y) - \lambda^+ C(z) \right) S_{i,j}^{(2)}(x, y, z) = 0, \quad i=1,2,\dots,M, j=0,1 \quad (23)$$

$$\left(\frac{d}{dy} + \lambda^+ + \beta_i^{(2)}(y) - \lambda^+ C(z) \right) R_{i,j}^{(2)}(x, y, z) = 0, \quad i=1,2,\dots,M, j=0,1 \quad (24)$$

$$\left(\frac{d}{dy} + \lambda^+ + \theta_i - \lambda^+ C(z) \right) Q_i(x, y, z) = 0, \quad i=1,2,\dots,M \quad (25)$$

$$I(0, z) = \sum_{i=1}^M \left[\int_0^{\infty} P_i(x, z) \mu_i(x) dx + \int_0^{\infty} R_i^{(1)}(x, z) \beta_i^{(1)}(x) dx \right] - \lambda^+ I_0 \quad (26)$$

$$P_i(0, z) = \frac{P_i}{z} \left[\lambda^+ C(z) I_0 + \int_0^{\infty} I(x, z) \eta(x) dx + \lambda^+ C(z) \int_0^{\infty} I(x, z) dx \right], \quad i=1,2,\dots,M \quad (27)$$

$$S_i^{(1)}(0, z) = \lambda^- \int_0^{\infty} P_i(x, z) dx, \quad i=1,2,\dots,M \quad (28)$$

$$R_i^{(1)}(0, z) = \int_0^{\infty} S_i^{(1)}(x, z) \phi_i^{(1)}(x) dx, \quad i=1,2,\dots,M \quad (29)$$

$$S_{i,0}^{(2)}(x, 0, z) = \tau_i \alpha_i P_i(x, z), \quad i=1,2,\dots,M \quad (30)$$

$$S_{i,1}^{(2)}(x, 0, z) = (1 - \tau_i) \alpha_i P_i(x, z), \quad i=1,2,\dots,M \quad (31)$$

$$R_{i,j}^{(2)}(x, 0, z) = \int_0^{\infty} S_{i,j}^{(2)}(x, y, z) \phi_i^{(2)}(y) dy, \quad i=1,2,\dots,M, j=0,1 \quad (32)$$

$$Q_i(x, 0, z) = \int_0^{\infty} R_{i,1}^{(2)}(x, y, z) \beta_i^{(2)}(y) dy, \quad i=1,2,\dots,M \quad (33)$$

Solving the partial differential equations (19),(21) – (25), we get

$$I(x, z) = I(0, z) e^{-\lambda^+ x} [1 - A(x)] \quad (34)$$

$$S_i^{(1)}(x, z) = S_i^{(1)}(0, z) e^{-\lambda^+ (1 - C(z))x} [1 - S_i^{(1)}(x)] \quad (35)$$

$$R_i^{(1)}(x, z) = R_i^{(1)}(0, z) e^{-\lambda^+ (1 - C(z))x} [1 - R_i^{(1)}(x)] \quad (36)$$

$$S_{i,j}^{(2)}(x, y, z) = S_{i,j}^{(2)}(x, 0, z) e^{-\lambda^+ (1 - C(z))y} [1 - S_{i,j}^{(2)}(y)], \quad i=1,2,\dots,M, j=0,1 \quad (37)$$

$$R_{i,j}^{(2)}(x, y, z) = R_{i,j}^{(2)}(x, 0, z) e^{-\lambda^+ (1 - C(z))y} [1 - R_{i,j}^{(2)}(y)], \quad i=1,2,\dots,M, j=0,1 \quad (38)$$

$$Q_i(x, y, z) = Q_i(x, 0, z) e^{-[\lambda^+ (1 - C(z)) + \theta_i]y}, \quad i=1,2,\dots,M \quad (39)$$

Using equations (30), (31) and (37) in equation (32) we obtain,

$$\begin{aligned} R_{i,0}^{(2)}(x, 0, z) &= \int_0^{\infty} S_{i,0}^{(2)}(x, 0, z) e^{-\lambda^+ (1 - C(z))y} [1 - S_{i,0}^{(2)}(y)] \phi_i^{(2)}(y) dy \\ &= S_{i,0}^{(2)}(x, 0, z) S_{i,0}^{(2)*}[h(z)] \\ &= \tau_i \alpha_i P_i(x, z) S_{i,0}^{(2)*}[h(z)], \quad i=1,2,\dots,M \end{aligned} \quad (40)$$

$$R_{i,1}^{(2)}(x, 0, z) = (1 - \tau_i) \alpha_i P_i(x, z) S_{i,1}^{(2)*}[h(z)], \quad i=1,2,\dots,M \quad (41)$$

where, $h(z) = \lambda^+ - \lambda^+ C(z)$

Using equations (38) and (41), equation (33) becomes

$$Q_i(x, 0, z) = (1 - \tau_i) \alpha_i P_i(x, z) S_{i,1}^{(2)*}[h(z)] R_{i,1}^{(2)*}[h(z)], \quad i=1,2,\dots,M. \quad (42)$$

Substituting the expressions of $R_{i,0}^{(2)}(x, y, z)$ and $Q_i(x, y, z)$ in equation (20) and solving, we get

$$P_i(x, z) = P_i(0, z) e^{-(g(z))x} [1 - B_i(x)], \quad i=1,2,\dots,M \quad (43)$$

where

$$g(z) = h(z) + \lambda^- + \alpha_i - \alpha_i S_{i,0}^{(2)*}(h(z)) R_{i,1}^{(2)*}(h(z)) \left(\frac{h(z) \tau_i + \theta_i}{h(z) + \theta_i} \right)$$

Using equations (28),(29),(35),(36) and (43) in equation (26), we get

$$I(0,z) = \sum_{i=1}^M P_i(0,z) \left[B_i^*(g(z)) + \lambda^- B_i^*(g(z)) S_i^{(1)*}(h(z)) R_i^{(1)*}(h(z)) \right] - \lambda^+ I_0 \quad (44)$$

Using equations (34) and (44) in equation (27) and on simplifying we get

$$P_i(0,z) = I_0 \lambda^+ p_i A^*(\lambda^+) g(z) (C(z) - 1) / D(z), i = 1, 2, \dots, M \quad (45)$$

Substituting the expression of $P_i(0,z)$ in equation (44), we obtain

$$I(0,z) = \lambda^+ I_0 \left[\begin{array}{l} C(z) \sum_{i=1}^M p_i [g(z) B_i^*(g(z)) + \\ \lambda^- (1 - B_i^*(g(z)) S_i^{(1)*}(h(z)) R_i^{(1)*}(h(z)))] - z g(z) \end{array} \right] / D(z) \quad (46)$$

where $D(z) =$

$$z g(z) - [A^*(\lambda^+) + C(z)(1 - A^*(\lambda^+))] \sum_{i=1}^M p_i [g(z) B_i^*(g(z)) + \lambda^- (1 - B_i^*(g(z)) S_i^{(1)*}(h(z)) R_i^{(1)*}(h(z)))]$$

Using equation (46) in equation (34) and integrating with respect to x from 0 to ∞ , we get

$$I(z) = I_0 (1 - A^*(\lambda^+)) \left[\begin{array}{l} C(z) \sum_{i=1}^M p_i [g(z) B_i^*(g(z)) + \\ \lambda^- (1 - B_i^*(g(z)) S_i^{(1)*}(h(z)) R_i^{(1)*}(h(z)))] - z g(z) \end{array} \right] / D(z) \quad (47)$$

Using equation (45) in equation (43) and integrating with respect to x from 0 to ∞ , we get

$$P_i(z) = I_0 \lambda^+ p_i A^*(\lambda^+) (C(z) - 1) [1 - B_i^*(g(z))] / D(z), i = 1, 2, \dots, M \quad (48)$$

Using equations (28),(35) and (47) and integrating with respect to x from 0 to ∞ , yields

$$S_i^{(1)}(z) = -I_0 \lambda^- p_i A^*(\lambda^+) [1 - B_i^*(g(z))] [1 - S_i^{(1)*}(h(z))] / D(z), i = 1, 2, \dots, M \quad (49)$$

Using equations (29) and (49) in equation (36) and integrating with respect to x from 0 to ∞ , yields

$$R_i^{(1)}(z) = -I_0 \lambda^- p_i A^*(\lambda^+) [1 - B_i^*(g(z))] [1 - R_i^{(1)*}(h(z))] S_i^{(1)*}(h(z)) / D(z), \quad (50)$$

$i = 1, 2, \dots, M$

Using equations (30),(31) and (47) and substituting the result in equation (37)-(39), integrating with respect to x and y from 0 to ∞ we obtain

$$S_{i,0}^{(2)}(z) = -I_0 \tau_i p_i \alpha_i A^*(\lambda^+) [1 - B_i^*(g(z))] [1 - S_i^{(2)*}(h(z))] / D(z), \quad (51)$$

$i = 1, 2, \dots, M$

$$S_{i,1}^{(2)}(z) = -I_0 (1 - \tau_i) p_i \alpha_i A^*(\lambda^+) [1 - B_i^*(g(z))] [1 - S_i^{(2)*}(h(z))] / D(z), \quad (52)$$

$i = 1, 2, \dots, M$

$$R_{i,0}^{(2)}(z) = -I_0 \tau_i p_i \alpha_i A^*(\lambda^+) [1 - B_i^*(g(z))] S_i^{(2)*}(h(z)) [1 - R_i^{(2)*}(h(z))] / D(z), \quad (53)$$

$i = 1, 2, \dots, M$

$$R_{i,1}^{(2)}(z) = -I_0 (1 - \tau_i) p_i \alpha_i A^*(\lambda^+) [1 - B_i^*(g(z))] S_i^{(2)*}(h(z)) [1 - R_i^{(2)*}(h(z))] / D(z), \quad (54)$$

$i = 1, 2, \dots, M$

$$Q_i(z) = -I_0 (1 - \tau_i) p_i \alpha_i A^*(\lambda^+) h(z) [1 - B_i^*(g(z))] S_i^{(2)*}(h(z)) R_i^{(2)*}(h(z)) / [h(z) + \theta_i] D(z), \quad (55)$$

$i = 1, 2, \dots, M$

Using normalizing condition I_0 is obtained as

$$I_0 = \frac{\lambda^- (1 - m_1 (1 - A^*(\lambda^+))) - \lambda^+ m_1 \sum_{i=1}^M p_i (1 - B_i^*(\lambda^-)) (\lambda^- (\phi_{i,1}^{(1)} + \beta_{i,1}^{(1)}) + T_1)}{\lambda^- A^*(\lambda^+)} \quad (56)$$

The probability generating function $P_q(z)$ of the number of customers in the orbit is

$$P_q(z) = I_0 A^*(\lambda^+) g(z) (z - 1) / D(z) \quad (57)$$

The probability generating function $P_s(z)$ of the number of customers in the system is

$$P_s(z) = I_0 A^*(\lambda^+) (z - 1) \left(g(z) \sum_{i=1}^M p_i B_i^*(g(z)) + \lambda^- \sum_{i=1}^M p_i (1 - B_i^*(g(z))) \right) / D(z) \quad (58)$$

4. PERFORMANCE MEASURES

The steady state probability that the system is empty is I_0 .

The steady state probability that the server is idle in non-empty system is

$$I = \frac{(1 - A^*(\lambda^+)) (\lambda^- (m_1 - 1) + \lambda^+ m_1 \sum_{i=1}^M p_i (1 - B_i^*(\lambda^-)) (\lambda^- (\phi_{i,1}^{(1)} + \beta_{i,1}^{(1)}) + T_1))}{\lambda^- A^*(\lambda^+)} \quad (59)$$

The steady state probability that the server is busy is

$$P = \sum_{i=1}^M P_i = \lambda^+ m_1 \sum_{i=1}^M p_i (1 - B_i^*(\lambda^-)) / \lambda^- \quad (60)$$

The steady state probability that the server is under mode 1 repair

$$H = \sum_{i=1}^M (S_i^{(1)} + S_i^{(1)} + R_i^{(1)} + R_i^{(1)}) = \lambda^+ m_1 \sum_{i=1}^M p_i (1 - B_i^*(\lambda^-)) (\phi_{i,1}^{(1)} + \beta_{i,1}^{(1)}) \quad (61)$$

The steady state probability that the server is under mode 2 repair

$$F = \sum_{i=1}^M (S_{i,0}^{(2)} + S_{i,1}^{(2)} + R_{i,0}^{(2)} + R_{i,1}^{(2)}) = \lambda^+ m_1 \sum_{i=1}^M \alpha_i p_i (1 - B_i^*(\lambda^-)) (\phi_{i,1}^{(2)} + \beta_{i,1}^{(2)}) / \lambda^- \quad (62)$$

The steady state probability that the server is under reserved time is

$$Q = \sum_{i=1}^M Q_i = \lambda^+ m_1 \sum_{i=1}^M p_i (1 - B_i^*(\lambda^-)) \alpha_i (1 - \tau_i) / \lambda^- \theta_i \quad (63)$$

The mean number of customers in the orbit L_q is given by

$$L_q = P'_q(1) = \frac{[D'(1)Nr'(1) - Nr'(1)D''(1)]}{2D'(1)^2} \quad (64)$$

Let $Nr(z)$ and $D(z)$ represent the numerator and denominator of $P_q(z)$.

where,

$$Nr'(1) = I_0 \lambda^- A^*(\lambda^+)$$

$$Nr''(1) = -2\lambda^+ m_1 I_0 A^*(\lambda^+) \sum_{i=1}^M p_i T_1$$

$$D'(1) = \lambda^-(1 - m_1(1 - A^*(\lambda^+)) - \lambda^+ m_1 \sum_{i=1}^M p_i(1 - B_i^*(\lambda^-))(\lambda^-(\phi_{i,1}^{(1)} + \beta_{i,1}^{(1)}) + T_1)$$

$$D''(1) = -\lambda^- m_2(1 - A^*(\lambda^+)) - 2\lambda^+ m_1 T_1 + T \left(1 - \sum_{i=1}^M p_i B_i^*(\lambda^-) \right) + 2\lambda^+ m_1^2 (1 - A^*(\lambda^+)) \sum_{i=1}^M p_i (B_i^*(\lambda^-) T_1 - \lambda^-(1 - B_i^*(\lambda^-))(\phi_{i,1}^{(1)} + \beta_{i,1}^{(1)})) + - \sum_{i=1}^M p_i \left[-2\lambda^+ m_1^2 (\mu_{i,1} T_1^2 - \lambda^- \left((\phi_{i,1}^{(1)} + \beta_{i,1}^{(1)}) \mu_{i,1} T_1 - (1 - B_i^*(\lambda^-)) \phi_{i,1}^{(1)} \beta_{i,1}^{(1)} \right)) \right] - \lambda^- \sum_{i=1}^M p_i (1 - B_i^*(\lambda^-)) T_2$$

$$T = -\alpha_i T_2 - 2\alpha_i \lambda^+ m_1^2 \left((\phi_{i,1}^{(2)} + \beta_{i,1}^{(2)}) \left(\frac{1 - \tau_i}{\theta_i} \right) + \phi_{i,1}^{(2)} \beta_{i,1}^{(2)} \right)$$

$$- \frac{\alpha_i \lambda^+ (1 - \tau_i)}{\theta_i^2} [m_2 \theta_i + 2\lambda^+ m_1^2] - \lambda^+ m_2$$

$$T_1 = \left[1 + \alpha_i (\phi_{i,1}^{(2)} + \beta_{i,1}^{(2)}) + \frac{1 - \tau_i}{\theta_i} \right]$$

$$T_2 = \lambda^+ m_1^2 (\phi_{i,1,2}^{(2)} + \beta_{i,1,2}^{(2)}) + \lambda^+ m_2 (\phi_{i,1}^{(2)} + \beta_{i,1}^{(2)})$$

The mean number of customers in the system L_s is given by

$$L_s = P'_s(1) = \frac{[D'(1)N_{r1}'(1) - N_{r1}'(1)D''(1)]}{2D'(1)^2} \quad (65)$$

Let $N_{r1}(z)$ and $D(z)$ represent the numerator and denominator of $P_q(z)$.

where,

$$N_{r1}'(1) = I_0 \lambda^- A^*(\lambda^+)$$

$$N_{r1}(1) = -2\lambda^+ m_1 I_0 \lambda^- A^*(\lambda^+) \sum_{i=1}^M p_i B_i^* \left[1 + \alpha_i (\phi_{i,1}^{(2)} + \beta_{i,1}^{(2)}) + \frac{1 - \tau_i}{\theta_i} \right]$$

5. RELIABILITY INDICES

Let $A(t)$ be the system availability at time t , that is the probability that the server is idle or working for a customer. Then under steady state condition,

The availability of the server is given by

$$A = 1 - (H + F + Q) \quad (66)$$

$$= 1 - [\lambda^+ m_1 \sum_{i=1}^M p_i (1 - B_i^*(\lambda^-)) (\theta_i (\lambda^-(\phi_{i,1}^{(1)} + \beta_{i,1}^{(1)})) + \alpha_i (\phi_{i,1}^{(2)} + \beta_{i,1}^{(2)})) + \alpha_i (1 - \tau_i) / \theta_i \lambda^-]$$

The steady state failure frequency of the server is

$$F = \lambda^+ m_1 \sum_{i=1}^M p_i (1 - B_i^*(\lambda^-)) (\lambda^- + \alpha_i) / \lambda^- \quad (67)$$

6. STOCHASTIC DECOMPOSITION

Theorem: The expected number of customers in the system L_s can be expressed as sum of two independent random variables, one of which is the expected number of customers (L) in the unreliable batch arrival classical queueing system with positive and negative arrivals, multi-optional service, two types of breakdown and repair and the other is the expected number of customers in the orbit during idle time (L_i).

Proof:

The probability generating function $\pi(z)$ of the system size of the classical unreliable batch arrival queue with multi-optional service, two types of breakdown and repair is given by

$$\pi(z) = \lim_{A^*(\lambda^+) \rightarrow 1} P'_s(z) \quad (68)$$

$$= I_1(z-1) \left(\sum_{i=1}^M p_i B_i^*(g(z)) g(z) + \lambda^- \sum_{i=1}^M p_i (1 - B_i^*(g(z))) \right) / D_1(z)$$

where

$$I_1 = \frac{\lambda^- - \lambda^+ m_1 \sum_{i=1}^M p_i (1 - B_i^*(\lambda^-)) (\lambda^-(\phi_{i,1}^{(1)} + \beta_{i,1}^{(1)}) + T_1)}{\lambda^-}$$

$$D_1(z) = zg(z) - \sum_{i=1}^M p_i [g(z) B_i^*(g(z)) + \lambda^- (1 - B_i^*(g(z))) S_i^{(1)*}(h(z)) R_i^{(1)*}(h(z))]$$

The probability generating function $\chi(z)$ of the number of customers in the orbit when the system is idle is given by

$$\chi(z) = \frac{I_0 + I(z)}{I_0 + I(1)}$$

$$= \frac{(zg(z) - \sum_{i=1}^M p_i [g(z) B_i^*(g(z)) + \lambda^- (1 - B_i^*(g(z))) S_i^{(1)*}(h(z)) R_i^{(1)*}(h(z))]) (\lambda^- (1 - m_1 (1 - A^*(\lambda^+))) - \lambda^+ m_1 \sum_{i=1}^M p_i (1 - B_i^*(\lambda^-)) (\lambda^-(\phi_{i,1}^{(1)} + \beta_{i,1}^{(1)}) + T_1))}{D(z) (\lambda^- - \lambda^+ m_1 \sum_{i=1}^M p_i (1 - B_i^*(\lambda^-)) (\lambda^-(\phi_{i,1}^{(1)} + \beta_{i,1}^{(1)}) + T_1))} \quad (69)$$

From equations (58), (68) and (69) we see that $P'_s(z) = \pi(z) \bullet \chi(z)$ and hence $L_s = L + L_i$.

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