Best Proximity Point for Generalized Geraghty-Contractions with MT-Condition

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ABSTRACT
A best proximity point for a non-selfmapping is that point whose distance from its image is as small as possible. In mathematical language, if X is any space, A and B are two subsets of X and T: A → B is a mapping. We can say that x is best proximity point if d(x, Tx) = d(A, B) and this best proximity point reduces to fixed point if mapping T is a selfmapping.

The main objective in this paper is to prove the best proximity point theorem for the notion of Geraghty-contractions by using MT-function β which satisfies Mizoguchi-Takahashi's condition (equation (i)) in the context of metric space and we also provide an example to support our main result.

Keywords
Best proximity point, P-property, MT-condition.

1. INTRODUCTION
The significance of fixed point theory stems from the fact that it furnishes a unified treatment and is a vital tool for solving equations of the form Tx=x where T is a self mapping defined on some suitable space. If T is non-self mapping then it is probable that Tx = x has no solution, in that case best approximation theorems explore the existence of an approximate solution whereas best proximity point theorems provide the existence of an optimal approximate solution. When d(x, Tx) > 0 for all x∈ X in that case it is natural to ask the existence and uniqueness of the smallest value of d(x, Tx). This is the main motivation of a best proximity point. The subject has attracted attention of number of authors [1-19].

$$\lim_{s\to t^+} \sup\beta(s) < 1 \text{ for all } t \in [0,\infty) \qquad \text{(i)}$$

2. PRILIMINARIES
To establish our results of this section, we consider the following definitions and notations:

Definition 2.1: Let A and B be two nonempty sets. A point x* is called best proximity point if

$$d(x^*, Tx^*)= d(A, B).$$

where d(A, B) = inf{d(x, y) : x ∈ X, y ∈ B}.

Let A and B be two non void subsets of a metric space (X, d) ; we denote A_0 and B_0 by the following sets:

$$A_0 = \{x \in A: d(x, y) = d(A,B) \text{ for some } y \in B\},$$

$$B_0 = \{ y \in B: d(x, y) = d(A,B) \text{ for some } x \in A\}.$$

If A∩B ≠ ∅, then A_0 and B_0 are nonempty .

Definition 2.2[19]: A function β: [0,∞) → [0, 1) is said to be an MT function if it satisfies Mizoguchi-Takahashi's condition (i.e. $\lim_{s\to t^+} \sup\beta(s) < 1$ for all $t \in [0,\infty)$).

Clearly, a non-increasing or a non-decreasing function β: [0,∞) → [0, 1) is MT-function. So the set of MT- function is a rich class. But taking into account that there exist some functions which are not MT -functions.

For example:

Let β: [0,∞) → [0, 1) be defined by

$$\beta(t)= \begin{cases} \frac{\sin t}{t}, & if\ t \in [0,\infty) \\ 0, & otherwise. \end{cases}$$

Since $\lim_{s\to 0^+} \sup\beta(s) = 1$, β is not an MT-function.

Definition 2.3: Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property if and only if for any u, v ∈ A_0 and x, y ∈ B_0,

$$\left.\begin{aligned} d(u,x) = d(A,B) \\ d(v,y) = d(A,B) \end{aligned}\right\} \Rightarrow d(u, v) = d(x, y).$$

Example 2.4: Let A be a nonempty subsets of a metric space (X, d). It is clear that the pair (A, A) has the P-property. Let (A, B) be any pair of nonempty, closed, convex subsets of a real Hilbert space H. Then (A, B) has the P-property.

3. MAIN RESULT
3.1 Theorem
Let (X, d) be a complete metric space. Suppose that (A, B) is a pair of nonempty closed subsets of X and A_0 is nonempty. Suppose also that the pair (A, B) has the P-property. If a non-self mapping T: A → B satisfying

i. $d(Tx, Ty) \leq \beta(M_T(x, y))[M_T(x, y) - d(A, B)]$ for any x, y ∈ A, where

$M_T(x, y) = \max \{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/2 \}$ and β is an MT- function.

ii. $T(A_0) \subseteq B_0$.

Then there exists a unique best proximity point, that is, there exists x^* in A such that $d(x^*, Tx^*) = d(A, B)$.

Proof: Let us fix an element x_0 in A_0. Since $Tx_0 \in T(A_0) \subseteq B_0$, we can find $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$. Further, as $Tx_1 \in T(A_0) \subseteq B_0$, there is an element x_2 in A_0 such that $d(x_2, Tx_1) = d(A, B)$. Repeatedly, we obtain a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = d(A, B) \qquad (3.1)$$

for any $n \in N$.

Due to fact that the pair (A, B) has the P-property, we obtain that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \quad (3.2)$$

for any $n \in N$.

From equation (3.1), we obtain

$$d(x_{n-1}, Tx_{n-1}) \leq d(x_{n-1}, x_n) + d(x_n, Tx_{n-1}) = d(x_{n-1}, x_n) + d(A, B).$$

On the other hand, by using equation (3.1) and (3.2), we obtain that

$$d(x_n, Tx_n) \leq d(x_n, Tx_{n-1}) + d(Tx_{n-1}, Tx_n) = d(A, B) + d(x_n, x_{n+1}) = d(x_n, x_{n+1}) + d(A, B).$$

We have

$$M_T(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), [d(x_{n-1}, Tx_n)+ d(x_n, Tx_{n-1})]/2\}$$

$$\leq \max\{d(x_{n-1},x_n)d(x_{n-1},x_n)+d(A,B),d(x_n,x_{n+1})+d(A,B), [d(x_{n-1},x_n)+d(x_n,x_{n+1}) +d(x_{n+1}, Tx_n) + d(A,B)]/2\}.$$

$$\leq \max\{d(x_{n-1}, x_n) + d(A,B), d(x_n,x_{n+1})+d(A,B), [d(x_{n-1},x_n) +d(x_n,x_{n+1}) +d(A, B) + d(A, B)]/2\}$$

$$=\max\{d(x_{n-1}, x_n) + d(A,B), d(x_n,x_{n+1})+d(A,B), [d(x_{n-1},x_n) +d(x_n,x_{n+1})]/2 + d(A, B)\}.$$

$$=\max\{d(x_{n-1},x_n),d(x_n,x_{n+1}),[d(x_{n-1},x_n) +d(x_n,x_{n+1})]/2\} +d(A, B).$$

$$=\max\{d(x_{n-1}, x_n), d(x_n,x_{n+1})\} + d(A, B).$$

$$M_T(x_{n-1}, x_n) \leq \max\{d(x_{n-1}, x_n), d(x_n,x_{n+1})\} + d(A, B). \quad (3.3)$$

Let us suppose that if there exists $n_0 \in$ such that $d(x_{n_0}, x_{n_0+1}) = 0$, then by using equation (3.2),

$$0 = d(x_{n_0}, x_{n_0+1}) = d(Tx_{n_0-1}, Tx_{n_0}) \quad (3.4)$$

which provides that

$$d(A, B) = d(x_{n_0}, Tx_{n_{0-1}}) = d(x_{n_0}, Tx_{n_0}). \quad (3.5)$$

Then proof is completed.

For the other case, i.e., when $d(x_n, x_{n+1}) > 0$ for any $n \in$.

Considering the fact that T satisfies

$d(Tx, Ty) \leq \beta(M_T(x, y))[M_T(x, y) - d(A, B)]$ for any $x, y \in A$.

Then by using (3.2), $d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$

$$\leq \beta(M_T(x_{n-1}, x_n))[M_T(x_{n-1}, x_n) - d(A, B)]$$

$$< M_T(x_{n-1}, x_n) - d(A, B) \quad (3.6)$$

Then by using inequalities (3.3) and (3.6), we get

$$d(x_n, x_{n+1}) < M_T(x_{n-1}, x_n) - d(A, B) \leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.$$

We know there is two possible values of $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$. Suppose if

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1}).$$

Then $d(x_n, x_{n+1}) < d(x_n, x_{n+1})$ which is contradiction.

By above discussion, we conclude that

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n)$$

and hence

$$M(x_{n-1}, x_n) \leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} + d(A, B) = d(x_{n-1}, x_n) + d(A, B) \quad (3.7)$$

We get

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \beta(M_T(x_{n-1}, x_n))d(x_{n-1}, x_n) < d(x_{n-1},x_n)$$

This implies

$$d(x_n, x_{n+1}) < d(x_{n-1},x_n) \text{ for all } n \in \quad . \quad (3.8)$$

As a result, we find that $\{d(x_n,x_{n+1})\}$ is a non increasing sequence and bounded below. Thus, there exists $L \geq 0$ such that $\lim_{n\to\infty} (d(x_n,x_{n+1})) = L$. Now we shall show that $L = 0$. Suppose on the contrary that $L > 0$, then by the equation (3.8), we have

$$\frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \leq \beta(M_T(x_{n-1}, x_n))$$

which implies that when $n \to \infty$ then $\beta(M_T(x_{n-1}, x_n)) \geq 1$ which contradicts that β is an MT- function and hence

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0. \quad (3.9)$$

Since $d(x_n,Tx_{n-1}) = (A, B)$ holds for all $n \in$ and the pair (A, B) satisfy P- property, then for all $m, n \in$, we can write

$$d(x_m, x_n) = d(Tx_{m-1}, Tx_{n-1}).$$

We also have

$$d(x_n, Tx_n) = d(x_n, x_{n+1}) + d(A, B) \text{ for all } n \in \quad .$$

It follows that

$$M_T(x_m, x_n) = \max\{d(x_m, x_n), d(x_m, Tx_m), d(x_n, Tx_n), [d(x_m, Tx_n) +d(x_n, Tx_m)]/2\}$$

$$\leq \max\{d(x_m, x_n), d(x_m, x_{m+1}) +d(A, B), d(x_n, x_{n+1}) + d(A, B), [d(x_m, x_n)+ d(x_n, Tx_n)+ d(x_m, x_n)+d(x_m, Tx_m)]/2\}$$

$$= \max\{ d(x_m, x_n), d(x_m, x_{m+1}), d(x_n, x_{n+1}), d((x_m, x_n) + [d(x_n, x_{n+1}) + d(x_m, x_{m+1})]/2 + d(A, B)\}.$$

By using equation (3.9) and taking limits $m, n\to\infty$,

$$\lim_{m, n\to\infty} M_T(x_m, x_n) \leq \lim_{m, n\to\infty} d(x_m, x_n) + d(A, B) \quad (3.10).$$

Now, we shall show that $\{x_n\}$ is a Cauchy sequence. Suppose on the contrary, we have

$$\varepsilon = \lim_{m,n\to\infty}\sup(d(x_n, x_m)) > 0 \quad (3.11)$$

Due to triangular inequality, we have

$$d(x_n,x_m) \leq d(x_n,x_{n+1})+d(x_{n+1},x_{m+1})+d(x_{m+1},x_m) \quad (3.12)$$

$$d(x_n, x_m) \leq d(x_n,x_{n+1})+d(Tx_n,Tx_m)+d(x_{m+1},x_m)$$

$\leq d(x_n, x_{n+1}) + \beta(M_T(x_n, x_m))[M_T(x_n, x_m) - d(A, B)] + d(x_{m+1}, x_m)$ (3.13)

Taking (3.9), (3.10) and (3.13) into account, we derive that

$\lim_{m,n\to\infty} d(x_n, x_m) \leq \beta(M_T(x_n, x_m))[M_T(x_n, x_m) - d(A, B)]$

$\leq \beta(M_T(x_n, x_m)) \lim_{m,n\to\infty} d(x_n, x_m)$.

This implies

$$\frac{\lim_{m,n\to\infty} d(x_n, x_m)}{\lim_{m,n\to\infty} d(x_m, x_n)} \leq \lim_{m,n\to\infty} \beta(M_T(x_n, x_m))$$

$$1 \leq \lim_{m,n\to\infty} \beta(M_T(x_n, x_m)).$$

which is contradiction.

Hence, we conclude that the sequence $\{x_n\}$ is Cauchy. Since A is a closed subset of the complete metric space (X, d) and $\{x_n\} \subset A$ and we can find $x^* \in A$ such that $x_n \to x^*$ as $n \to \infty$. We have to prove that $d(x^*, Tx^*) = d(A, B)$. Suppose on the contrary that $d(x^*, Tx^*) > d(A, B)$.

Consider

$$d(x^*, Tx^*) \leq d(x^*, Tx_n) + d(Tx_n, Tx^*)$$
$$\leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx_n) + d(Tx_n, Tx^*)$$
$$= d(x^*, x_{n+1}) + d(A, B) + d(Tx_n, Tx^*)$$

Taking $n \to \infty$, we conclude that

$$d(x^*, Tx^*) - d(A, B) \leq \lim_{n\to\infty} d(Tx_n, Tx^*).$$

On the other hand, we have

$d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) = d(x_n, x_{n+1}) + d(A, B)$.

Taking limit as $n \to \infty$ in above inequality, we have

$$\lim_{n\to\infty} d(x_n, Tx_n) \leq d(A, B).$$

So we deduce that $\lim_{n\to\infty} d(x_n, Tx_n) = d(A, B)$. As a result, we derive

$\lim_{n\to\infty} M_T(x_n, x^*) = \max\{\lim_{n\to\infty} d(x_n, x^*), \lim_{n\to\infty} d(x_n, Tx_n), d(x^*, Tx^*), \lim_{n\to\infty} [d(x_n, Tx^*) + d(x^*, Tx_n)]/2\}$.

$\leq \max\{\lim_{n\to\infty} d(x_n, x^*), \lim_{n\to\infty} d(x_n, Tx_n), d(x^*, Tx^*), \lim_{n\to\infty} [d(x_n, x^*) + d(x^*, Tx^*) + d(x^*, x_n) + d(x_n, Tx_n)]/2\}$.

$= \max\{d(A, B), d(x^*, Tx^*), [d(x^*, Tx^*) + d(A, B)]/2\}$.

$= \max\{d(A, B), d(x^*, Tx^*)\}$

$= d(x^*, Tx^*)$. [because $d(x^*, Tx^*) > d(A, B)$].

And hence

$\lim_{n\to\infty} M_T(x_n, x^*) - d(A,B) = d(x^*, Tx^*) - d(A,B)$ (3.14)

Also $d(x^*, Tx^*) - d(A, B) \leq \lim_{n\to\infty} d(Tx_n, Tx^*)$

$\leq \lim_{n\to\infty} \beta(M_T(x_n, x^*))[M_T(x_n, x^*) - d(A, B)]$ (3.15)

Since $d(x^*, Tx^*) - d(A, B) > 0$, we get $1 \leq \lim_{n\to\infty} \beta(M_T(x_n, x^*))$, we get a contradiction.

So $d(x^*, Tx^*) - d(A, B) \leq 0$ and hence $d(x^*, Tx^*) - d(A, B) = 0$.

This implies $d(x^*, Tx^*) = d(A, B)$.

x^* is a best proximity point of T. Hence, we conclude that T has a best proximity point.

Now we claim that best proximity point of T is unique. Suppose on the contrary, x^* and y^* are two distinct best proximity point of T. Thus, we have

$d(x^*, Tx^*) = d(A,B) = d(y^*, Ty^*)$ (3.16)

By using P-property, we have

$$d(x^*, y^*) = d(Tx^*, Ty^*)$$

and

$M_T(x^*, y^*) = \max\{ d(x^*, y^*), d(x^*, T x^*), d(y^*, Ty^*), [d(x^*, Ty^*) + d(y^*, Tx^*)]/2 \}$

$\leq \max\{d(x^*, y^*), d(A, B), d(A, B), d(x^*, y^*) + d(A, B)\}$

$$M_T(x^*, y^*) \leq d(x^*, y^*) + d(A, B)$$

$$M_T(x^*, y^*) - d(A, B) \leq d(x^*, y^*).$$

Now by using given condition $d(x^*, y^*) = d(Tx^*, Ty^*)$

$\leq \beta(M_T(x^*, y^*))[M_T(x^*, y^*) - d(A, B)]$

$= \beta(M_T(x^*, y^*))\, d(x^*, y^*)$

$\leq d(x^*, y^*)$

a contradiction. This completes the proof.

3.2 Example

We present the following example to support our main result.

Example: Let $X = R^2$ with the metric

$$d((x, y), (u,v)) = \max\{|x - u|, |y - v|\}$$

and consider the closed subsets

$$A = \{(x,0) : 0 \leq x \leq 1\},$$

$$B = \{(x,0) : -1 \leq x \leq 0\}$$

And let $T: A \to B$ be the following mapping defined by

$$T((x, 0)) = ((-x)/(1 + x), 0).$$

It is clear that $d(A, B) = 0$, then pair (A, B) has the P-property.

We have to notice that $A_0 = (0, 0)$ and $B_0 = (0, 0)$ and $T(A_0) \subseteq B_0$.

Also

$d(T(x, 0), T(u, 0)) = d(((-x)/(1 + x), 0), ((-u)/(1 + u), 0))$

$$= \frac{|u-x|}{(1+x)(1+u)} = \frac{|u-x|}{(1+x)(1+u)}$$

and as $(1+x)(1+u) \geq 1 + |u-x|$, we have d(T(x, 0), T(u, 0)) = $\frac{|u-x|}{(1+x)(1+u)} \leq \frac{|u-x|}{1+|x-u|}$

$=\beta(|x-u|)$ =β(d((x, 0), (u,0))) where β : [0,∞) → [0,1) is defined as β(t) = $\frac{t}{1+t}$.

Therefore, d(T(x, 0), T(u, 0)) ≤ β(d((x, 0), (u, 0)))

$\leq \beta(M_T((x, 0),(u,0)))$

$\leq \beta(M_T((x, 0),(u,0)))\ [M_T((x, 0),(u,0)) - d(A, B)]$.

Therefore all the assumption of theorem 3.1 are satisfied, so there exists a unique (x*, 0) ∈ A such that

d((x*, 0), T(x*, 0)) = 0 = d(A, B).

Here the point (x*, 0) ∈ A is (0, 0) ∈ A.

4. AUTHORS' CONTRIBUTIONS

Both authors contributed equally and significantly in writing this paper.

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