The $\mathrm{L}(2,1)$-Labeling on γ-Product of Graphs and Improved Bound on the $\mathrm{L}(2,1)$-Number of γ-Product of Graphs

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ABSTRACT
The concept of $\mathrm{L}(2,1)$-labeling in graph come into existence with the solution of frequency assignment problem. In fact, in this problem a frequency in the form of nonnegative integers is to assign to each radio or TV transmitters located at various places such that communication does not interfere. This frequency assignment problem can be modeled with vertex labeling of graphs. An $\mathrm{L}(2,1)$-labeling (or distance two labeling) of a graph G is a function f from the vertex set V(G) to the set of all nonnegative integers such that $|\mathrm{f(u)} - \mathrm{f(v)}| \geq 2$ if $\mathrm{d(u, v)} = 1$ and $|\mathrm{f(u)} - \mathrm{f(v)}| \geq 1$ if $\mathrm{d(u, v)} = 2$, where $\mathrm{d(u, v)}$ denotes the distance between u and v in G. The $\mathrm{L}(2,1)$-labeling number $\lambda(\mathrm{G})$ of G is the smallest number k such that G has an $\mathrm{L}(2,1)$-labeling with $\max\{\mathrm{f(v)}: \mathrm{v} \in \mathrm{V(G)}\} = \mathrm{k}$. In this paper, upper bound for the L(2,1)-labeling number for the γ-product of two graphs has been obtained in terms of the maximum degrees of the graphs involved and improved this bound by using a dramatically new approach on the analysis of the adjacency matrices of the graphs. By the new approach, we have achieved more accurate result with significant improvement of this bound.

Keywords
Channel assignment, $L(2,1)$-labeling, $L(2,1)$-labeling number, Graph γ-product, Adjacency matrix of graphs

1. INTRODUCTION
The frequency assignment problem asks for assigning frequencies to transmitters in a broadcasting network with the aim of avoiding undesired interference**.** Hale [20] was first person who formulated this problem as a graph vertex coloring problem. By Roberts [7], In order to avoid interference, any two "close" transmitters must receive different channels and any two "very close" transmitters must receive channels that are at least two channels apart. To translate the problem into the language of graph theory, the transmitters are represented by the vertices of a graph; two vertices are "very close" if they are adjacent and "close" if they are of distance two in the graph. Based on this problem, Griggs and Yeh [10] considered an $L(2,1)$-labeling on a simple graph. An $L(2,1)$-labeling (or distance two labeling) of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(u) - f(v)| \geq 2$ if $d(u, v) = 1$ and $|f(u) - f(v)| \geq 1$ if $d(u, v) = 2$, where $d(u, v)$ denotes the distance between u and v in G. A $k - L(2,1)$-labeling is an $L(2,1)$-labeling such that no label is greater than k. The $L(2,1)$-labeling number of G, denoted by $\lambda(G)$or λ, is the smallest number k such that G has a $k - L(2,1)$ labeling.The $L(2,1)$-labeling has been extensively studied in recent past by many researchers [see 1, 4, 8, 9, 11, 12, 19]. The common trend in most of the research paper is either to determine the value of $L(2,1)$-labeling number or to suggest bounds for particular classes of graphs.

Griggs and Yeh [10] provided an upper bound of $\lambda(G)$ is $\Delta^2 + 2\Delta$ for a general graph with the maximum degree Δ. Later, Chang and Kuo[8], improved the bound to $\Delta^2 + \Delta$, while Kral and Skrekovski [2] reduced the bound to $\Delta^2 + \Delta - 1$. Furthermore, recently Gonccalves [1] proved the bound $\Delta^2 + \Delta - 2$ which is the present best record. If G is a graph of diameter 2 then $\lambda(G) \leq \Delta^2$. The upper bound is attainable for Moore graphs (diameter 2 graphs with order $\Delta^2 + 1$). (Such graphs exist only if $\Delta = 2,3,7$ and possibly 57). Thus Griggs and Yeh [10] conjectured that the best bound is Δ^2 for any graph G with the maximum degree $\Delta \geq 2$. (This is not true for $\Delta = 1$. For example, $\lambda(K_2) = 1$ but $\lambda(K_2) = 2$).

Graph products play an important role in connecting various useful networks and they also serve as natural tools for different concepts in many areas of research. In this paper, we have considered the graph formed by the γ-product of graphs [6] and obtained a general upper bound for $L(2,1)$-labeling number in terms of the maximum degrees of the graphs. In the case of γ-product of graphs, $L(2,1)$-labeling number of graph holds Griggs and Yeh's conjecture [10] with minor exception.

2. A LABELING ALGORITHM
A subset X of $V(G)$ is called an i-stable set (or i-independent set) if the distance between any two vertices in X is greater than i. An 1-stable (independent) set is a usual independent set. A maximal 2-stable subset X of a set Y is a 2-stable subset of Y such that X is not a proper subset of any 2-stable subset of Y.

Chang and Kuo [8] proposed the following algorithm to obtain an $L(2,1)$-labeling and the maximum value of that labeling on a given graph.

Algorithm

Input: A graph $G = (V, E)$

Output: The value k is the maximum label.

Idea: In each step i, find a maximal 2-stable set from the unlabeled vertices that are distance at least two away from those vertices labeled in the previous step. Then label all the vertices in that 2-stable set with the index i in the current stage. The label i starts from 0 and then increase by 1 in each step. The maximum label k is the final value of i.

Initialization: Set $X_{-1} = \phi; V = V(G); i = 0$.

Iteration:

1. Determine Y_i and X_i.

■ $Y_i = \{u \in V: u$ is unlabelled and $d(u,v) \geq 2 \ \forall \ v \in X_{i-1}\}$.

■ X_i is a maximal 2–stable subset of Y_i.

■ If $Y_i = \phi$ then set $X_i = \phi$.

2. Label the vertices of X_i (if there is any) with i.
3. $V \leftarrow V - X_i$.
4. If $V \neq \phi$, then $i \leftarrow i + 1$, go to step 1.
5. Record the current i as k(which is the maximum label). Stop.

Thus k is an upper bound on $\lambda(G)$. Let u be a vertex with largest label k obtained by above algorithm.Set

$I_1 = \{i : 0 \leq i \leq k-1$ and $d(u,v) = 1$ for some $v \in X_i\}$.

$I_2 = \{i : 0 \leq i \leq k-1$ and $d(u,v) \leq 2$ for some $v \in X_i\}$.

$I_3 = \{i : 0 \leq i \leq k-1$ and $d(u,v) \geq 3$ for all $v \in X_i\}$.

Then Chang and Kuo showed that $\lambda(G) \leq k \leq |I_2| + |I_3| \leq |I_2| + |I_1|$.

In order to find k, it suffices to estimate $B = |I_1| + |I_2|$ in term of $\Delta(G)$. We will investigate the value B with respect to a particular graph (γ-product of two graphs). The notations which have been introduced in this section will also be used in the following sections.

3. THE γ-PRODUCT OF GRAPHS

The γ-product $G \boxdot H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which the vertex (u, v) is adjacent to the vertex (u', v') if and only if either u is adjacent to u' in G or v is adjacent to v' in H. For example, we consider the Fig.1.

By the definition of the γ-product of two graphs G and H, if $\Delta(G) = 0$ or $\Delta(H) = 0$ then $G \boxdot H$ consists of disjoint copies of H or G. Thus $\lambda(G \boxdot H) = \lambda(H)$or $\lambda(G \boxdot H) = \lambda(G)$. Therefore we assume that $\Delta(G) \geq 1$ and $\Delta(H) \geq 1$.

4. UPPER BOUND FOR THE $L(2,1)$–LABELING NUMBER IN G $\boxdot$ H

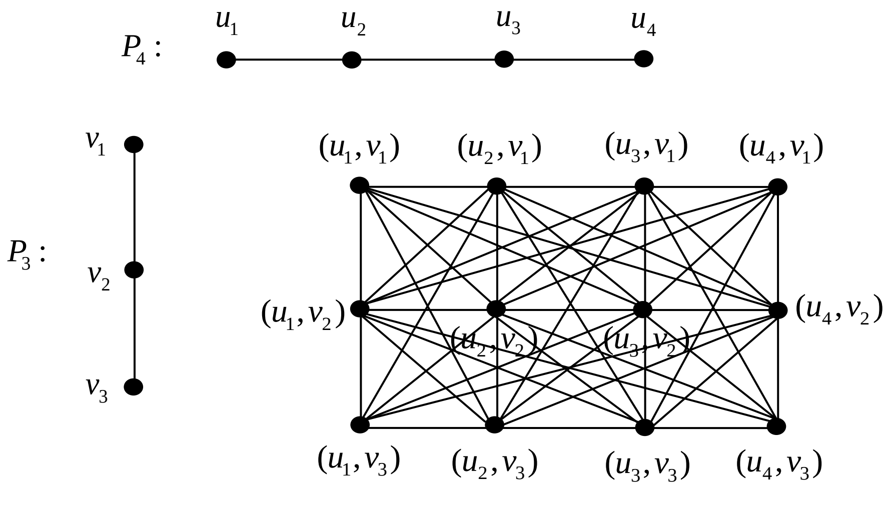
In this section, general upper bound for the $L(2,1)$–labeling number (λ-number) of γ–product $G \boxdot H$ in term of maximum degree of the graphs has been established. In this regard, we state and prove the following theorem.

Theorem 4.1. Let Δ, Δ_1, Δ_2 be the maximum degree of $G \boxdot H$, G, H and n, n_1, n_2 be the number of vertices of $G \boxdot H$, G, H respectively. Then

$$\lambda(G \boxdot H) \leq \Delta^2 - n_1\Delta_2(\Delta_1 - 1) - n_2\Delta_1(\Delta_2 - 1) - \Delta_1\Delta_2(\Delta_1 + \Delta_2 + 1) + 1.$$

Proof: Let $u_\gamma = (u, v)$ be any vertex in the graph $G \boxdot H$. Denote $d = deg_{G \boxdot H}(u_\gamma)$, $d_1 = deg_G(u)$,$d_2 = deg_H(v)$, $\Delta_1 = maxdeg(G)$, $\Delta_2 = maxdeg(H)$, $|V(G)| = n_1$ and $|V(H)| = n_2$. Then by the definition of γ-product we have the following results $d = n_1 d_2 + n_2 d_1 - d_1 d_2$ and $\Delta = n_1\Delta_2 + n_2\Delta_1 - \Delta_1\Delta_2$.

Let us consider the Fig.2. For any vertex v' in H with distance 2 from v, there must be a path $v'v''v$ of length two between v' and v in H; but the degree of u in G is d_1 i.e. u has d_1 adjacent vertices in G, by the definition of γ-product $G \boxdot H$, there must be $d_1 + 1$ internally-disjoint paths(two paths are said to be internally-disjoint if they do not intersect each other) of length two between (u, v') and (u, v). Hence for any vertex in H with distance 2 from v, there must be corresponding $d_1 + 1$ vertices with distance 2 from $u_\gamma = (u, v)$ which are coincided in $G \boxdot H$; on the contrary whenever there is no such vertex in H with distance 2 from v in H, the corresponding $d_1 + 1$ vertices with distance 2 from the vertex $u_\gamma = (u, v)$ which are



$P_4 * P_3$

Fig.1 γ – product of graphs

coincided in $G \boxdot H$ will never exit. In the former case since such d_1+1 vertices with distance 2 from $u_\gamma = (u,v)$ coincide in $G \boxdot H$ and hence they can only be counted once and therefore we have to deduct $d_1+1-1=d_1$ from the value $d(\Delta-1)$ which is best possible number of vertices at distance 2 from a vertex $u_\gamma = (u,v)$ in $G \boxdot H$. Let the number of vertices in H with distance 2 from v be t, then $t \in [0, d_2(\Delta_2-1)]$. If we take $t = d_2(\Delta_2-1)$ which is best possible number of vertices at distance 2 from a vertex v in H, then to get the number of vertices at distance 2 from $u_\gamma = (u,v)$ in $G \boxdot H$, we will have to subtract at least $d_2(\Delta_2-1)d_1$ from the value $d(\Delta-1)$. For G, we can proceed in the similar way to get the number of vertices at distance 2 from $u_\gamma = (u,v)$ in $G \boxdot H$ and in this case subtract $d_1(\Delta_1-1)d_2$ from the value $d(\Delta-1)$. Hence the number of vertices at distance 2 from $u_\gamma = (u,v)$ in $G \boxdot H$ will decrease $d_1(\Delta_1-1)d_2 + d_2(\Delta_2-1)d_1$ from the value $d(\Delta-1)$ altogether. By the above analysis, the number $d(\Delta-1) - d_1(\Delta_1-1)d_2 - d_2(\Delta_2-1)d_1$ is now the best possible number of vertices at distance 2 from $u_\gamma = (u,v)$ in $G \boxdot H$.

Moreover by the definition of γ-product $G \boxdot H$, we can again analyse as follows:

Let ε be the number of edges of the subgraph F induced by the neighbours of u_γ.The edges of the subgraph F induced by the neighbours of u_γ can be divided into the following two cases.

Case I: Consider the Fig.3 for this case. For each neighbour vertex (u,v') (where v' is adjacent to v in H) of $u_\gamma = (u,v)$ and any vertex (u',v_t) (where u' is adjacent to u in G and v_t is any vertex of H), (u',v_t) must be the common neighbour of (u,v') and (u,v), then there must be an edge between (u',v_t) and (u,v') and an edge between (u',v_t) and (u,v) respectively. But there are at least $n_2 d_1$ neighbour vertices like (u',v_t) of $u_\gamma = (u,v)$ and there are totally d_2 neighbour vertices like (u,v') of $u_\gamma = (u,v)$. Hence the number of edges of the subgraph F induced by the neighbours of u_γ is at least $n_2 d_1 d_2$ i.e. $\varepsilon \geq n_2 d_1 d_2$. By a symmetric analysis, the neighbours of $u_\gamma = (u,v)$ should again add at least $(n_1 d_1 d_2 - 1)$ (excluding the coincided edge between (u',v) and (u,v')).

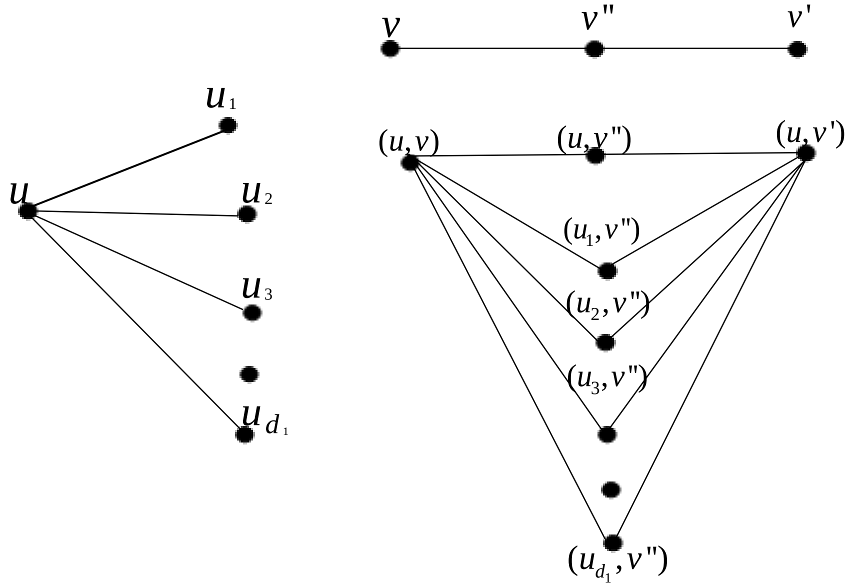


Fig 2

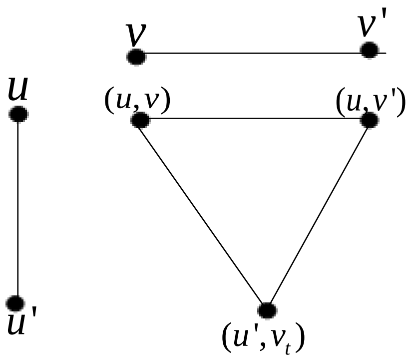


Fig 3

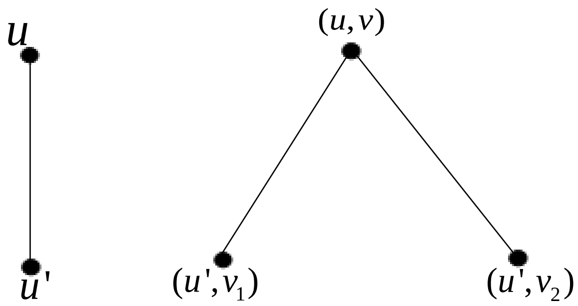


Fig 4

Case II: Consider the Fig.4 for this case. If u' is adjacent to u in G, then (u,v) must be adjacent to (u',v_1) and (u',v_2) where v_1 and v_2 are any two vertices of H, hence the vertices of the subgraph F induced by the neighbours of $u_\gamma = (u,v)$ should be all (u',v) where $v \epsilon V(H)$. Because $\Delta(H) = \Delta_2$ and there are totally d_1 neighbour vertices u' of u, then the number of edges of the subgraph F induced by the neighbours of $u_\gamma = (u,v)$ should be greater than $d_1\Delta_2$. Hence, at least $d_1\Delta_2$ should be added to the number of edges of the subgraph F induced by the neighbours of $u_\gamma = (u,v)$. By a symmetric analysis, the number of edges of the subgraph F induced by the neighbours of $u_\gamma = (u,v)$ must be increased by the number $d_2\Delta_1$ at least.

By the analysis of the above two cases, we have $\varepsilon \geq n_1 d_1 d_2 - 1 + n_2 d_1 d_2 + d_1\Delta_2 + d_2\Delta_1$.

Whenever there is an edge in F, the number of vertices with distance 2 from u_γ will decrease by 2, hence the number of vertices with distance 2 from $u_\gamma = (u,v)$ in $G \boxdot H$ will still need at least a decrease $n_1 d_1 d_2 - 1 + n_2 d_1 d_2 + d_1\Delta_2 + d_2\Delta_1$ from the value $d(\Delta - 1) - d_1(\Delta_1 - 1)d_2 - d_2(\Delta_2 - 1)d_1$. (The number $d(\Delta - 1) - d_1(\Delta_1 - 1)d_2 - d_2(\Delta_2 - 1)d_1$ is now the best possible for the number of vertices with distance 2 from $u_\gamma = (u,v)$ in $G \boxdot H$).

Hence for any vertex u_γ, the number of vertices with distance 1 from u_γ is no greater than Δ. The number of vertices with distance 2 from u_γ is no greater than

$d(\Delta - 1) - d_1(\Delta_1 - 1)d_2 - d_2(\Delta_2 - 1)d_1 - n_1 d_1 d_2 + 1 - n_2 d_1 d_2 - d_1\Delta_2 - d_2\Delta_1$.

Hence $|I_1| \leq d$. $|I_2| \leq d + d(\Delta - 1) - d_1(\Delta_1 - 1)d_2 - d_2(\Delta_2 - 1)d_1 - n_1 d_1 d_2 + 1 - n_2 d_1 d_2 - d_1\Delta_2 - d_2\Delta_1$.

Then $B = |I_1| + |I_2| \leq d + d\Delta - d_1(\Delta_1 - 1)d_2 - d_2(\Delta_2 - 1)d_1 - n_1 d_1 d_2 + 1 - n_2 d_1 d_2 - d_1\Delta_2 - d_2\Delta_1 = (n_1 d_2 + n_2 d_1 - d_1 d_2)(n_1\Delta_2 + n_2\Delta_1 - \Delta_1\Delta_2 + 1) - d_1(\Delta_1 - 1)d_2 - d_2(\Delta_2 - 1)d_1 - n_1 d_1 d_2 + 1 - n_2 d_1 d_2 - d_1\Delta_2 - d_2\Delta_1$.

Define $f(s,t) = (n_1 t + n_2 s - st)(n_1\Delta_2 + n_2\Delta_1 - \Delta_1\Delta_2 + 1) - s(\Delta_1 - 1)t - t(\Delta_2 - 1)s - n_1 st + 1 - n_2 st - s\Delta_2 - t\Delta_1$.

Then $f(s,t)$ has the absolute maximum at (Δ_1, Δ_2) on $[0, \Delta_1] \times [0, \Delta_2]$.

$$
\begin{aligned}
f(\Delta_1, \Delta_2) = {} & (n_1\Delta_2 + n_2\Delta_1 \\
& - \Delta_1\Delta_2)(n_1\Delta_2 + n_2\Delta_1 - \Delta_1\Delta_2 + 1) \\
& - \Delta_1(\Delta_1 - 1)\Delta_2 - \Delta_2(\Delta_2 - 1)\Delta_1 \\
& - n_1\Delta_1\Delta_2 + 1 - n_2\Delta_1\Delta_2 - \Delta_1\Delta_2 \\
& - \Delta_2\Delta_1 \\
= {} & \Delta(\Delta + 1) - (n_1 + n_2 + \Delta_1 + \Delta_2)\Delta_1\Delta_2 + 1 \\
= {} & \Delta^2 + (n_1\Delta_2 + n_2\Delta_1 - \Delta_1\Delta_2) \\
& - (n_1 + n_2 + \Delta_1 + \Delta_2)\Delta_1\Delta_2 \\
& + 1 \\
= {} & \Delta^2 - n_1\Delta_2(\Delta_1 - 1) \\
& - n_2\Delta_1(\Delta_2 - 1) \\
& - \Delta_1\Delta_2(\Delta_1 + \Delta_2 + 1) + 1.
\end{aligned}
$$

The $\lambda(G \boxdot H) \leq k \leq B \leq \Delta^2 - n_1\Delta_2(\Delta_1 - 1) - n_2\Delta_1(\Delta_2 - 1) - \Delta_1\Delta_2(\Delta_1 + \Delta_2 + 1) + 1$.

5. IMPROVED BOUND FOR THE $\mathbf{L(2,1)}$–LABELING NUMBER IN G $\boxdot$ H

In this section, we shall improve the upper bound obtained in theorem 4.1 of the $L(2,1)$-labeling number on the γ-product $G \boxdot H$ of two graphs G and H on the analysis of the adjacency matrices of the graph involved.

Suppose A_1 and A_2 are the adjacency matrices of G and H respectively. Then the adjacency matrix of γ-product $G \boxdot H$ of the graphs G and H with (mod 2) can be written as $A = (A_1 \otimes A_2) + (J_1 \otimes A_2) + (A_1 \otimes J_2)$ where J_1 is the square matrix of order n_1 with all entries 1 and J_2 is the square matrix of order n_2 with all entries 1. These matrices involve the Kronecker product $\otimes$ of matrices, $(A_1 \otimes A_2)$ is the Kronecker product $\otimes$ of matrices A_1 and A_2.Similarly $(J_1 \otimes A_2)$ and $(A_1 \otimes J_2)$ are Kronecker product of matrices involved in it. Note that the rules of algebra of Kronecker product $\otimes$ of matrices can be found in [5].

In order to find k, it suffices to estimate $B = |I_1| + |I_2|$ in term of $\Delta(G)$ (using labeling algorithm). Before eliminating the upper bound k, we introduce a notation first. Let M be a matrix with n rows. For $1 \leq i \leq n$, $r_i(M)$ denote the number of nonzero entries in the ith row of M excluding the diagonal entry.

Let A be the adjacency matrix of G with respect to a list of vertices $\{v_1, v_2, \dots \dots, v_n\}$. Then it is well known that the $(i,j)th$ entry of A^k is the number of different (v_i, v_j) walks in G of length k, for $k \geq 0$.

Thus $r_i(A) = \deg(v_i)$, $r_i(A^2)$ is the number of vertices joining by a walk of length 2 from v_i excluding v_i itself and $r_i(A^2 + A)$ is the number of vertices of distance 1 or 2 from v_i. So that

$$r_i(A^2) \leq \deg(v_i)(\Delta(G)-1) \quad (1)$$

$$r_i(A^2+A) \leq \deg(v_i)\Delta(G) \quad (2)$$

For convenience, the notations which have been introduced in this section will also be used in the following section.

6. MAIN RESULT

Theorem 5.1: Let Δ_1, Δ_2 be the maximum degree of G, H and n_1, n_2 be the number of vertices of G, H respectively. Then

$$\lambda(G \boxdot H) \leq \Delta_1^2\Delta_2^2 - \Delta_1^2\Delta_2 - \Delta_2^2\Delta_1 + (n_2-1)\Delta_1^2 + (n_1-1)\Delta_2^2 + \Delta_1\Delta_2 + n_1\Delta_2 + n_2\Delta_1.$$

Proof: From the above discussion in section 5, we get that the adjacency matrix of $G \boxdot H$ is $A = (A_1 \otimes A_2) + (J_1 \otimes A_2) + (A_1 \otimes J_2)$. Then

$$\begin{aligned} A^2 + A &= [(A_1\otimes A_2) + (J_1\otimes A_2) + (A_1\otimes J_2)]^2 \\ &\quad + (A_1\otimes A_2) + (J_1\otimes A_2) + (A_1\otimes J_2) \\ &= (A_1^2\otimes A_2^2) + (J_1^2\otimes A_2^2) + (A_1^2\otimes J_2^2) \\ &\quad + 2(J_1\otimes A_2)(A_1\otimes J_2) + 2(A_1J_1\otimes A_2^2) \\ &\quad + 2(A_1^2\otimes A_2J_2) + (A_1\otimes A_2) \\ &\quad + (J_1\otimes A_2) + (A_1\otimes J_2) \\ &= (A_1^2\otimes A_2^2) + n_1(J_1\otimes A_2^2) + \\ &\quad n_2(A_1^2\otimes J_2) + 2(J_1A_1\otimes A_2J_2) + \\ &\quad 2(A_1J_1\otimes A_2^2) + 2(A_1^2\otimes A_2J_2) + \\ &\quad (A_1\otimes A_2) + (J_1\otimes A_2) + (A_1\otimes J_2). \end{aligned}$$

Note that the rules of algebra of Kronecker product $\otimes$ of matrices can be found in [5].

Since all entries of the involved matrices are nonnegative, then the number of non-zero entries in the $(u_i, v_j)th$ entry of $n_2(A_1^2\otimes J_2) + 2(A_1^2\otimes A_2J_2) + (A_1\otimes J_2) + n_1(J_1\otimes A_2^2) + 2(A_1J_1\otimes A_2^2) + (J_1\otimes A_2) + 2(J_1A_1\otimes A_2J_2)$ is the same as that of $(A_1^2\otimes J_2) + (A_1\otimes J_2) + (J_1\otimes A_2^2) + (J_1\otimes A_2) = (A_1^2 + A_1)\otimes J_2 + J_1\otimes(A_2^2 + A_2)$.

Let k be the maximum label obtained by the algorithm (in section 2). Let $(u_i, v_j)\,\epsilon\, V(G)\times V(H)$ be the vertex with the label k. We look at the $(u_i, v_j)th$ row of the matrix $A^2 + A$. We have

$$\begin{aligned} r_{(u_i,v_j)}(A^2+A) \leq r_{(u_i,v_j)}&(A_1^2\otimes A_2^2) \\ &+ r_{(u_i,v_j)}\big((A_1^2+A_1)\otimes J_2\big) \\ &+ r_{(u_i,v_j)}\big(J_1\otimes(A_2^2+A_2)\big) \\ &+ r_{(u_i,v_j)}(A_1\otimes A_2) \end{aligned}$$

$= r_i(A_1^2)r_j(A_2^2) + r_i(A_1^2+A_1)r_j(J_2) + r_i(J_1)r_j(A_2^2+A_2) + r_i(A_1)r_j(A_2) = deg_G(u_i)(\Delta_1-1)deg_H(v_j)(\Delta_2-1) + deg_G(u_i)\Delta_1(n_2-1) + (n_1-1)deg_H(v_j)\Delta_2 + deg_G(u_i)deg_H(v_j)$.

Note that the last equality is obtained by applying equation (1) and (2).

Thus, the number of non-zero entries in the $(u_i, v_j)th$ entry of (A^2+A) excluding the diagonal entry is at most $deg_G(u_i)(\Delta_1-1)deg_H(v_j)(\Delta_2-1) + deg_G(u_i)\Delta_1(n_2-1) + (n_1-1)deg_H(v_j)\Delta_2 + deg_G(u_i)deg_H(v_j)$. Also we have known that $|I_1| \leq \Delta(G \boxdot H) = n_1\Delta_2 + n_2\Delta_1 - \Delta_1\Delta_2$.

Thus $\lambda(G \boxdot H) \leq |I_2| + |I_1| \leq \Delta_1(\Delta_1-1)\Delta_2(\Delta_2-1) + \Delta_1\Delta_1(n_2-1) + (n_1-1)\Delta_2\Delta_2 + \Delta_1\Delta_2 + n_1\Delta_2 + n_2\Delta_1 - \Delta_1\Delta_2$.

Hence $\lambda(G \boxdot H) \leq \Delta_1^2\Delta_2^2 - \Delta_1^2\Delta_2 - \Delta_2^2\Delta_1 + (n_2-1)\Delta_1^2 + (n_1-1)\Delta_2^2 + \Delta_1\Delta_2 + n_1\Delta_2 + n_2\Delta_1$.

This completes the proof.

7. CONCLUSION

In theorem 4.1, we have proved that $\lambda(G \boxdot H) \leq \Delta^2 - n_1\Delta_2(\Delta_1-1) - n_2\Delta_1(\Delta_2-1) - \Delta_1\Delta_2(\Delta_1+\Delta_2+1) + 1$, where the maximum degree of $G \boxdot H$ is $n_1\Delta_2 + n_2\Delta_1 - \Delta_1\Delta_2$. Since $(\Delta^2 - n_1\Delta_2(\Delta_1-1) - n_2\Delta_1(\Delta_2-1) - \Delta_1\Delta_2(\Delta_1+\Delta_2+1) + 1) - (\Delta_1^2\Delta_2^2 - \Delta_1^2\Delta_2 - \Delta_2^2\Delta_1 + (n_2-1)\Delta_1^2 + (n_1-1)\Delta_2^2 + \Delta_1\Delta_2 + n_1\Delta_2 + n_2\Delta_1) = (n_1\Delta_2 + n_2\Delta_1)^2 - (n_2-1)\Delta_1^2 - (n_1-1)\Delta_2^2 - \Delta_1\Delta_2(2n_1\Delta_2 + 2n_2\Delta_1 + n_1 + n_2 + 2) + 1$. We have thus reduced the bound by $(n_1\Delta_2 + n_2\Delta_1)^2 - (n_2-1)\Delta_1^2 - (n_1-1)\Delta_2^2 - \Delta_1\Delta_2(2n_1\Delta_2 + 2n_2\Delta_1 + n_1 + n_2 + 2) + 1$.

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