Stability of k-Tribonacci Functional Equation in Non-Archimedean Space

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ABSTRACT
Throughout this paper, we investigate the Hyers-Ulam stability of k-Tribonacci functional equation if (k, x) = k f(k, x – 1) + f(k, x – 2) + f(k, x – 3) in the class of functions $f : N \times R \to X$ where X is real non-archimeadean Banach space.

Keywords
Hyers-Ulam Stability, Real Non-archimedean Banach Space,k-Tribonacci functional equation.

1. INTRODUCTION
They study of stability problems for functional equations is related to question of Ulam [20] concerning the stability of group homorophism and affirmatively answered for Banach Spaces by Hyers [2]. Aoki [21] presented the generalization of Hyers results by considering additive mappings and later on, Rassias [23] did for linear mapping by considering an unbounded Cauchy difference. The paper of Rassias has signification

influenced the development of what we now call the Hyers-Ulam-Rassias stability of functional equation. Various extension, generalization and applications obtained by many researchers [3,4,5,6,9,14,15,16,19,22,24].

In 2009, S.M. Jung [18] investigated the Hyer-Ulam stability of Fibonacci functional equation $f(x) = f(x-1) + f(x-2)$.

In 2011, Alvaro H. Salas [1] investigate about the K-Fibonacci number and their associated number. After that, M. Bidkhan and M. Hosslini [7] proved the stability of K-fibonacci functional equation. Later on, M. Bidkhan, M. Hosseini, C. Park and M. Eshaghi Gordzi [8] successed to prove the Hyers-Ulam stability of k,s-Fibonacci functional equation. First of all, in 2012, M. Gordji, M. Naderi and Th. M. Rassias [35] et. al. proved the stability of Tribonacci functional equation in non -archimedean space and in 2014, M.E.Gordji, Ali Divandi, M. Rostannian, C. Park and D.Y. Sin[10] also proved the stability of Tribonacci functional equation in 2-normed space.

In this paper, we denote by $F_{k,n}$ the nth k-Tribonacci number where

$$F_{k,n} = k\, F_{k,\, n-1} + F_{k,\, n-2} + F_{k,\, n-3} \text{ for } n \geq 3$$

with initial conditions $F_{k,\,0} = 0$, $F_{k,\,1} = 1$, $F_{k,\,2} = 1$. From this, we may derive a functional equation

$$f(k, x) = k\, f(k, x-1) + f(k, x-2) + f(k, x-3) \quad (1.1)$$

which if called the k-Tribonacci function equation if a function $f : N \times R \to X$ satisfies the above equation for all $x \in R$, $K \in N$, characteristic equation of the kth–Tribonacci sequence is $x^3 - kx^2 - x - 1 = 0$, and p, q, r denote the roots of characteristic equation where p is greater than one and q, r, $\in C$ and $|q| = |r|$.

We know that $p + q + r = K$, $pq + qr + pr = -1$,

$pqr = 1$. For each $x \in R$, [x] stands for he largest integer that does not exceed x.

Definition [13] : We recall that a field K, equipped with a function (non-Archimedean absolute value, valuation) $|\cdot|$ from K into $[0, \infty)$, is called a non-Archimedean field if the function $|\cdot| : K \to [0, \infty)$ satisfies the following conditions :

1. $|r| = 0$ if and only if $r = 0$;
2. $|r\, s| = |r|\, |s|$;
3. the strong triangle inequality, namely, $|r + s| \leq \max\{|r|, |s|\}$ for all r, s, $\in K$ clearly, $|1| = 1 = |-1|$ and $|n| \leq 1$ for all non-zero integer n.

Definition [12] : Let y be a vector space over the non-Archimedean field K with a non-trivial non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : y \to [0, \infty)$ is called a non-Archimedean norm (valuation) if it satisfies the following conditions:

1. $\|x\| = 0$ if and only if $x = 0$;
2. $\|rx\| = \|r\|\, \|x\|$ for all $x \in y$ and all $r \in K$;
3. the strong triangle inequality, namely, $\|x + y\| \leq \max(\|x\|, \|y\|)$ for all x, $y \in Y$.

In this case, the pair $(Y, \|\cdot\|)$ is called a non-Archimedean space. By a Banach non-Archimedean space we mean one in which every cauchy sequence is convergent. It follows from the strong triangle inequality that

$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\|;\ m \leq j \leq n-1\}$ for all x_n, $x_m \in Y$ and all m, $n \in N$ with $n > m$.

Therefore, a sequence $\{x_n\}$ is a cauchy sequence in non-Archimedean space $(Y, \|\cdot\|)$ if and only if the sequence $\{x_{n+1} - x_n\}$ converges to zero in the space $(Y, \|\cdot\|)$.

2. MAIN RESULT
As we see in [10], the general solution of the Tribonacci functional equation is strongly related to the Tribonacci numbers T_n .In the following theorem, we prove the Hyers-Ulam stability of the k-Tribonacci functional equation(1.1)

Theorem [2.1] : Let $(X, \|\cdot\|)$ be a non-Archimedean Banach space. If a function $f : R \to X$ satisfies the inequality

$$\|f(k, x) - k\, f(k, x-1) - f(k, x-2) - f(k, x-2)\| \leq \varepsilon \quad (2.1)$$

for all $x \in R$, $k \in N$ and for some $\varepsilon > 0$, then there exist a k-Tribonacci function

$H : N \times R \to X$ such that

$$\|f(k,x)-H(k,x)\| \le \frac{2(1+|q|)+|q|^2}{\|q^2(r-p)+r^2(p-q)+p^2(q-r)} \times \frac{\varepsilon}{1-|q|^2} . \quad (2.2)$$

Proof : Since, $p + q + r = k$, $pq + qr + pr = -1$ and $pqr = 1$. So from (1), we obtain

$$\|f(k, x) - (p + q + r) f(k, x-1) + (pq + qr + pr) f(k, x-2) - pqr\, f(k, x-3)\| \le \varepsilon. \quad (2.3)$$

for all $x \in R$, $k \in N$. Now it follows from (1.1) that

$$f(k, x) - p(f(k, x-1) - r f(k, x-2)) - r f(k, x-1) = q[f(k, x-1) - (r + p) f(k, x-2) + pr f(k, x-3)] \quad (2.4)$$

for all $k \in N$, $x \ge 0$. By mathematical induction, we verify that for all $x \ge 0$ and all m belonging to the set {0, 1, 2, ...}, we obtain,

$$f(k, x) - p(f(k, x-1) - r f(k, x-2)) - r f(K, x-1) = q^m [f(k, x-m) - r f(k, x-m-1) + pr f(k, x-m-2) - p f(k, x-m-1)] \quad (2.5)$$

$$f(k, x) - r[f(k, x-1) - q f(k, x-2)] - q f(k, x-1) = p^m [f(k, x-m) - r f(k, x-m-1) + qr f(k, x-m-2) - q f(k, x-m-1)] \quad (2.6)$$

$$f(k, x) - q [f(k, x-1) - p f(k, x-2)] - p f(k, x-1) = r^m [f(k, x-m) - p f(k, x-m-1) + qp f(k, x-m-2) - q f(k, x-m-1)] \quad (2.7)$$

for all $x \ge 0$ and all $m \in$ {0, 1, 2, ...}.

If we replace x by $x - \alpha$ in inequality (2.4) then we get

$$\|f(k, x-\alpha) - p[f(k, x-\alpha-1) - r f(k, x-\alpha-2)] - r f(k, x-\alpha-1) - qp f(K, x-\alpha-1) - (r + p) f(k, x-\alpha-2) + pr f(k, x-\alpha-3)\| \le \varepsilon$$

for all $x \in R$, $k \in N$.

Now multiplying both sides by q^α,

$$\|q^\alpha[f(k, x-\alpha) - p\{f(k, x-\alpha-1) - r f(k, x-\alpha-2)\} - r f(k, x-\alpha-1)] - q^{\alpha+1}[f(k, x-\alpha-1) - (r + p) f(k, x-\alpha-2) + pr f(k,x-\alpha-3)]\| \le \left|q^\alpha\right| \varepsilon \quad (2.8)$$

for all $x \in R$, $k \in N$. Furthermore, we have

$$\| f(k,x) - p (f(k, x-1) - r f(k, x-2)) - r f(k, x-1) - q^m[f(k, x-m) - (r + p) f(k, x-m-1) + pr f(k, x-m-2)] \|$$

$$\le \max_{0\le\alpha\le m-1} \|q^\alpha[f(k, x-\alpha) - p\{f(k, x-\alpha-1) - r f(k, x-\alpha-2)\} - r f(k, x-\alpha-1)] - q^{\alpha+1}[f(k, x-\alpha-1) - (r + p) f(k, x-\alpha-2) + pr f(k, x-\alpha-3)]\|$$

$$\le \max_{0\le\alpha\le m-1} \|q^\alpha\|\varepsilon = \varepsilon$$

for all $x \in R$, $m \in N$, $k \in N$. (2.9)

Let $x \in R$ be fixed, than (2.8) implies that $\{q^m [f(k, x-m) - p (f(k, x-m-1) - r f(k, x-m-2)) - r f(k, x-m-1))]$ is a cauchy sequence ($|q| < 1$). So by the completeness of X, we may define a function $H_1 : R \to X$ such that

$$H_1(k,x) = \lim_{m\to\infty} q^m[f(k, x-m) - (p + r) f(k, x-m-1) + pr f(k, x-m-2)]$$

for all $x \in R$, $k \in N$. Applying the definition of H_1, we introduce the k-Tribonacci function

$$k H_1(k, x-1) + H_1(k, x-2) + H_1(k, x-3) =$$

$$kq^{-1} \lim_{m\to\infty} q^{m+1}[f(k, x-(m+2)) - (p + r) f(k, x-(m+1)-1) + pr f(k, x-(m+1)-2)]$$

$$+q^{-2} \lim_{m\to\infty} q^{m+2}[f(k, x-(m+2)) - (p + r) f(k, x-(m+2)-1) + pr f(k, x-(m+2)-2)]$$

$$+q^{-3} \lim_{m\to\infty} q^{m+3}[f(k, x-(m+3)) - (p + r) f(k, x-(m+3)-1) + pr f(k, x-(m+3)-2)]$$

$$= kq^{-1}H_1(k, x) + q^{-2}H_1(k, x) + q^{-3}H_1(k, x)$$

$= H_1(k, x)$ for all $x \in R$, $k \in N$. Hence H_1 is a k-Tribonacci function.

If $m \to \infty$, then from (2.9), we obtain

$$|f(k, x) - (p + r) f(k, x-1) + pr f(k, x-2) - H_1(k,x)\| \le \frac{1}{1-|q|}\varepsilon \quad (2.10)$$

for all $x \in R$, $k \in N$. Furthermore, it follows from (2.1) that

$$\|f(k, x) - q (f(k, x-1) - p f(k, x-2)) - p f(k, x-1) - r[f(k, x-1) - p f(k, x-2) + pq f(k, x-3) - q f(k, x-2)]\| \le \varepsilon$$

for all $x \in R$, $k \in N$. Now, we replace x by $x - \alpha$ in above inequality, we have

$$\|f(k, x-\alpha) - q (f(k, x-\alpha-1) - p f(k, x-\alpha-2)) - p f(k, x-\alpha-1) - r[f(k, x-\alpha-1) - p f(k, x-\alpha-2) + pq f(k, x-\alpha-3) - q f(k, x-\alpha-2)]\| \le \varepsilon$$

and now multiplying by r^α on both sides.

$$\|r^\alpha[f(k, x-\alpha) - q(f(k, x-\alpha-1) - p f(k, x-\alpha-2)) - p f(k, x-\alpha-1)] - r^{\alpha+1}[f(K, x-\alpha-1) - p f(K, x-\alpha-2) + pq f(k, x-\alpha-3) - q f(k, x-\alpha-2)]\|$$

$$\le |r^\alpha|\varepsilon \quad (2.11)$$

for all $x \in R$, $\alpha \in Z$. Now, we have

$$\|f(k, x) - q(f(k, x-1) - p f(k, x-2)) - p f(k, x-1) - r^m [f(k, x-m) - (q + p) f(k, x-m-1) + p q (f(k, x-m-2)]\| \le \max_{1\le\alpha\le m}$$

$$\|r^\alpha[f(k, x-\alpha) - q\{f(k, x-\alpha-1) - p f(k, x-\alpha-2)\} - p f(k, x-\alpha-1)] - r^{\alpha+1}[f(k, x-\alpha-1) - (p + q) f(k, x-\alpha-2) + pq f(k, x-\alpha-3)]\|$$

$$\le \max_{0\le\alpha\le m-1} \{|r|^\alpha\}\varepsilon = \varepsilon \quad (2.12)$$

for all $x \in R$ and $m \in N$. We have $\{r^m[f(k, x-m) - (q + p) f(k, x-m-1) + pq f(k, x-m-2)]\}$ is a cauchy sequence ($|r| < 1$) for all $x \in R$. Hence, we can define a function $H_2 : R \to X$ by

$$H_2(k,x) = \lim_{m\to\infty} r^m[f(k, x-m) - (q + p) f(k, x-m-1) + pq f(k, x-m-2)]$$

for all $x \in R$. Using the above definition of H_2, we have

$$kH_2(k, x-1) + H_2(k, x-2) + H_2(k, x-3) =$$

$$kr^{-1} \lim_{m\to\infty} r^{m+1}[f(k, x-(m+1)) - (q + p) f(k, x-(m+1)-1) + pq f(k, x-(m+1)-2)] + r^{-2} \lim_{m\to\infty} r^{m+2}[f(k, x-(m+2)) - (q + p) f(k, x-(m+2)-1) + pq f(k, x-(m+2)-2)] + r^{-3} \lim_{m\to\infty} r^{m+3}[f(k, x-(m+3)) - (q + p) f(k, x-(m+3)-1) + pq f(k, x-(m+3)-2)]$$

$$= k r^{-1}H_2(k, x) + r^{-2} H_2(k, x) + r^{-3}H_2(k, x)$$

$= H_2(k, x)$ for all $x \in R$.

So, we can say that H_2 is also a k-Tribonacci function. If m tends to ∞, then from (2.12), we have

||f(k, x) – q(f(k, x – 1) – pf(k, x – 2)) - p f(k, x – 1) – H_2(k, x)||

$\leq \frac{1}{1-|r|}$ OR

||f(k, x) – (q + p) f(k, x – 1) + q p f(k, x – 2) – H_2(k, x)|| ≤

$\frac{1}{1-|r|}\varepsilon = \frac{1}{1-|q|}\varepsilon.$ (2.13)

for all x ∈ R. Finally, Analogus to (2.1), we obtain

||f(k, x) – r (f(k, x – 1) – q f(k, x – 2)) – q f(k, x – 1)

-p[f(k, x – 1) – r f(k, x – 2) + q r f(k, x – 3) – q f(k, x – 2)]|| ≤ ε

for all x ∈ R.

Now we replace x by x + α in above inequality, that we have

||f(k, x + α) – r(f(k, x + α - 1) – qf(k, x + α - 2)) – qf(k, x + α - 1) – p[f(k, x + α - 1) – (r + q) f(k, x - α - 2) + qrf(k, x + α - 3)]|| ≤ ε

and

$||p^{-\alpha}$[f(k, x + α) – r(f(k, x + α -1) – q f(k, x + α - 2)) – q f(k, x + α - 1)] – $p^{-\alpha+1}$[f(k, x + α-1) – (r + q) f(k, x – α - 2)) + qr f(k, x + α - 3)]|| ≤ $|\alpha^{-1}|^k$ ε (2.14)

for all x ∈ R and α ∈ Z. Applying (2.14), we obtain that

$||p^{-m}$[f(k, x + m) – r(f(k, x + m – 1) – q f(k, x + m – 2)) – q f(k, x + m – 1)]-[f(k, x) – (r + q) f(k, x – 1) + rq f(k, x – 2)]||

$\leq \max_{1\leq k\leq n}$ {$||p^{-\alpha}$[f(k, x + α) – r(f(k, x + α – 1) – q f(k, x + α - 2))-q f(k, x + α - 1)] – $p^{-\alpha+1}$[f(k, x + α - 1) – (r + q) f(k, x + α - 2)+qr f(k, x + α - 3)]

$\leq \max_{1\leq k\leq n}$ {$|p^{-1}|^{\alpha}$ ε} = $\alpha^{-1}\varepsilon$. (2.15)

for all x ∈ R, m ∈ N. By using (2.15) we see that

{p^{-m}[f(k, x + m) – (r + q) f(k, x + m – 1) + qr f(k, x + m – 2)]} is a cauchy sequence by definition of completeness for a fixed x ∈ R. Hence, we may define a function $H_3 : R \to X$ by

H_3(k,x) = $\lim_{m\to\infty} p^{-m}$[f(k, x+m) – (r+q) f(k, x+m+1)+qr f(k, x + m – 2)]

for all x ∈ R. In view of above definition of H_3, we obtain

kH_3(k, x – 1) + H_3(k, x – 2) + H_3(k, x – 3)

= $kp^{-1} \lim_{m\to\infty} p^{-(m-1)}$[f(k, x+m – 1) – (r+q) f(k, x+(m-1) -1)+qr f(k, x+(m -1) – 2)]

+ $p^{-2} \lim_{m\to\infty} p^{-(m-2)}$[f(k, x+m – 2) – (r+q) f(k, x+(m-2) - 1)+qr f(k, x+(m -2) – 2)]

+ $p^{-3} \lim_{m\to\infty} p^{-(m-3)}$[f(k, x+m – 3) – (r+q) f(k, x+(m-3) -1)+qr f(k, x+(m -3) – 2)]

=$kp^{-1}H_3$(k, x) + $p^{-2}H_3$(k, x) + $p^{-3}H_3$(k, x)

= H_3(k, x) for all x ∈ R, k ∈ N.

Hence, we can say that H_3 is also a k-Tribonacci function. If we suppose, m tends to infinity in (2.15) then we have

||H_3(k, x) – f(k, x) + (r+q) f(k, x – 1) – qr f(k, x – 2)||

$$\leq \frac{\alpha^{-1}}{1-|\alpha^{-1}|}\varepsilon \quad (2.16)$$

for all x ∈ R. From (2.9), (2.12) and (2.15), we observe that

$$\left\| f(k,x) - \left[\frac{q^2(r-p)H_1(k,x) + r^2(p-q)H_2(k,x) - p^2(q-r)H_3(k,x)}{q^2(r-p)+r^2(p-q)+p^2(q-r)} \right] \right\|$$

$$= \frac{1}{|q^2(r-p)+r^2(p-q)+p^2(q-r)|}$$

||(q^2(r-p)+r^2(p - q)+p^2(q – r) f(k, x) – q^2(rp)H_1(k, x) – r^2(p-q) H_2(k, x) + p^2(q-r) H_3(k, x)||

For convince, we assume that

$$\frac{1}{|q^2(r-p)+r^2(p-q)+p^2(q-r)|} = \frac{1}{|A|} \quad (2.17)$$

$$\leq \frac{1}{|A|}[\| q^2(r-p)f(k,x) - q^2(r^2-p^2)f(k,x-1) + q^2(r-p)prf(k,x-2) - q^2(r-p)H_1(k,x)\|$$

$$+\| r^2(p-q)f(k,x) - r^2(p^2-q^2)f(k,x-1) + r^2(p-q)qpf(k,x-2) - r^2(p-q)H_2(k,x)\|$$

$$+\| p^2(q-r)f(k,x) - p^2(q^2-r^2)f(k,x-1) + p^2(q-r)qrf(k,x-2) - p^2(q-r)H_3(k,x)\|$$

$$] \quad = \frac{1}{|A|}\left[\frac{1}{1-|q|} + \frac{1}{1-|q|} + \frac{|q^2|}{1-|q^2|}\right]\varepsilon$$

$$= \frac{1}{|A|}\left[\frac{2}{1-|q|} + \frac{|q^2|}{1-|q^2|}\right]\varepsilon$$

$$= \frac{1}{|A|}\left[\frac{2(1+|q|)+|q|^2}{1-|q^2|}\right]\varepsilon$$

Putting the value of |A| from (2.17) we get the required result.

Hence,

$$H(k, x) = \frac{q^2(r-p)H_1(k,x) + r^2(p-q)H_2(k,x) - p^2(q-r)H_3(k,x)}{q^2(r-p)+r^2(p-q)+p^2(q-r)}$$

for all x ∈ R. It is not difficult to show that H is a k-Tribonacci function satisfying (2.2).

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