The Minimum Monopoly Distance Energy of a Graph

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ABSTRACT

In a graph $G = (V, E)$, a set $M \subseteq V$ is called a monopoly set of G if every vertex $v \in V - M$ has at least $\frac{d(v)}{2}$ neighbors in M. The monopoly size $mo(G)$ of G is the minimum cardinality of a monopoly set among all monopoly sets in G. In this paper, the minimum monopoly distance energy $E_{Md}(G)$ of a connected graph G is introduced and minimum monopoly distance energies of some standard graphs are computed. Some properties of the characteristic polynomial of the minimum monopoly distance matrix of G are obtained. Finally. Upper and lower bounds for $E_{Md}(G)$ are established.

Keywords

Minimum monopoly set, minimum monopoly distance matrix, minimum monopoly distance eigenvalues, minimum monopoly distance energy.

MSC(2010): 05C50, 05C99, 11C08.

1. INTRODUCTION

In this paper, a graph $G = (V, E)$ mean a connected simple graph, that is nonempty, finite, having no loops no multiple and no directed edges also there is a path between any pair of its vertices. Let G be such a graph and let n and m be the number of its vertices and edges, respectively. The degree of a vertex v in a graph G, denoted by $d(v)$, is the number of vertices adjacent to v. For any vertex v of a graph G, the open neighborhood of v is the set $N(v) = \{u \in V : uv \in E\}$. For a subset $S \subseteq V$ the degree of a vertex $v \in V$ with respect to a subset S is $d_S(v) = |N(v) \cap S|$. The distance $d(u, v)$ between two vertices u and v of a graph G is the minimum length of the paths connecting them (i.e., the number of edges between them). For more terminologies and notations in graph theory, the reader is referred to [12].

A subset $M \subseteq G$ is called a monopoly set of G if for every vertex $v \in V - M$ has at least $\frac{d(v)}{2}$ neighbors in M. The monopoly size of G is the smallest cardinality of a monopoly set in G, denoted by $mo(G)$. A monopoly set of a graph G is minimum if for any other monopoly set M' of G, $|M| \leq |M'|$. In particular, monopolies are a dynamic monopoly (dynamos) that, when colored black at a certain time step, will cause the entire graph to be colored black in the next time step under an irreversible majority conversion process. Dynamos were first introduced by Peleg [18]. For more details in dynamos in graphs (see [4, 5, 8, 17]). In [14], the author defined a monopoly set of a graph G, proved that the $mo(G)$ for general graph is at least $\frac{n}{2}$, discussed the relationship between matchings and monopolies and he showed that any graph G admits a monopoly with at most $\alpha'(G)$ vertices.

The concept of energy of a graph was introduced by I. Gutman [9] in the year 1978. Let G be a graph with n vertices and m edges and let $A = (a_{ij})$ be the adjacency matrix of the graph. The eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of A, assumed in non increasing order, are the eigenvalues of the graph G. Let $\lambda_1, \lambda_2, ..., \lambda_t$ for $t \leq n$ be the distinct eigenvalues of G with multiplicity $m_1, m_2, ..., m_t$, respectively, the multiset of eigenvalues of $A(G)$ is called the spectrum of G and denoted by

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & ... & \lambda_t \\ m_1 & m_2 & ... & m_t \end{pmatrix}.$$

The energy $E(G)$ of G is defined to be the sum of the absolute values of the eigenvalues of G, i.e. $E(G) = \sum_{i=1}^{n} |\lambda_i|$. For more details on the mathematical aspects of the theory of graph energy see [2, 10, 16].

The distance matrix of G is the square matrix $A_d(G)$ whose (i, j) - entry is the distance between the vertices v_i and v_j . Let $\rho_1, \rho_2, ..., \rho_n$ be the eigenvalues of the distance matrix $A_d(G)$ of a graph G. The distance energy $E_d(G)$ of a graph G is defined by

$$E_d(G) = \sum_{i=1}^{n} |\rho_i|.$$

For more studies on distance energy see [6, 7, 11, 13, 20].

Recently C. Adiga et al. [1] defined the minimum covering energy, $E_C(G)$ of a graph which depends on its particular minimum cover set C. Kanna and et al. in [19] introduced minimum covering distance energy of a graph. Motivated by these papers, minimum monopoly distance energy, denoted $E_{Md}(G)$, of a connected graph G is introduced and minimum monopoly distance energies of some standard graphs are computed. Some properties of characteristic polynomial of a minimum monopoly distance matrix of a graph G are obtained. Finally, upper and lower bounds for $E_{Md}(G)$ are established. It is possible that the minimum monopoly distance energy that is considering in this paper may be have some applications in chemistry as well as in other areas.

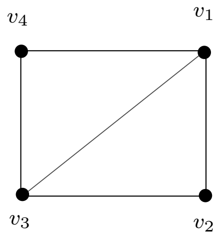

Fig. 1. A graph G

2. THE MINIMUM MONOPOLY DISTANCE ENERGY OF A GRAPH

Let G be a graph of order n with vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set E. Any monopoly set M of a graph G with minimum cardinality is called a minimum monopoly set. Let M be a minimum monopoly set of a graph G. The minimum monopoly distance matrix of G is the $n \times n$- matrix, denoted $A_{Md}(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } i = j \text{ and } v_i \in M; \\ d(v_i, v_j), & \text{othewise.} \end{cases}$$

The characteristic polynomial of $A_{Md}(G)$ is defined as

$$f_n(G, \rho) = det\left(\rho I - A_{Md}(G)\right).$$

The minimum monopoly distance eigenvalues of G are the eigenvalues of $A_{Md}(G)$. Since $A_{Md}(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\rho_1 \geq \rho_2 \geq ... \geq \rho_n$. The minimum monopoly distance energy of G is defined as:

$$E_{Md}(G) = \sum_{i=1}^{n} |\rho_i|$$

To illustrious this concept, the minimum monopoly distance energy of a graph G is computed as the following example.

EXAMPLE 1. *Let G be a graph in Fig. 1 with vertices $\{v_1, v_2, v_3, v_4\}$ and let we chose the minimum monopoly set $M_1 = \{v_1, v_3\}$ of G. Then*

$$A_{M_1d}(G) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of $A_{M_1d}(G)$ is

$$f_n(G, \rho) = \rho^4 - 2\rho^3 - 8\lambda^2.$$

The minimum monopoly distance eigenvalues of G are $\rho_1 = 4,\ \rho_2 = \rho_3 = 0,\ \rho_4 = -2$. Therefore the minimum monopoly distance energy of G is

$$E_{M_1d}(G) = 6.$$

Now, if we chose another minimum monopoly set of G, namely $M_2 = \{v_2, v_4\}$, then

$$A_{M_2d}(G) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

The characteristic polynomial of $A_{M_2d}(G)$ is

$$f_n(G, \rho) = \rho^4 - 2\rho^3 - 8\rho^2 - 6\rho - 1.$$

The minimum monopoly distance eigenvalues of G are $\rho_1 = 2 + \sqrt{5},\ \rho_2 = 2 - \sqrt{5},\ \rho_3 = \rho_4 = -1$. Therefore the minimum monopoly distance energy of G is

$$E_{M_1d}(G) = 2 + 2\sqrt{5}.$$

The examples above illustrate that the minimum distance monopoly energy of a graph G depends on the choice of the minimum monopoly set. i.e. the minimum monopoly distance energy is not a graph invariant.

3. SOME PROPERTIES OF MINIMUM MONOPOLY DISTANCE ENERGY OF GRAPHS

In this section, some properties of characteristic polynomials of minimum monopoly distance matrix of a graph G are introduced.

THEOREM 1. *Let G be a graph of order n, size m and monopoly size $mo(G)$ and let*

$$f_n(G, \rho) = c_0\rho^n + c_1\rho^{n-1} + c_2\rho^{n-2} + ... + c_n$$

be the characteristic polynomial of a minimum monopoly distance matrix of a graph G. Then

(*1*) $c_0 = 1$.
(*2*) $c_1 = -mo(G)$.
(*3*) $c_2 = \begin{pmatrix} mo(G) \\ 2 \end{pmatrix} - \sum_{1 \leq i < j \leq n} d^2(v_i, v_j)$.

PROOF. (1) From the definition of $f_n(G, \rho)$.

(2) Since the sum of diagonal elements of $A_{Md}(G)$ is equal to $|M| = mo(G)$, where M is a minimum monopoly set in G, and the sum of determinants of all 1×1 principal submatrices of $A_{Md}(G)$ is the trace of $A_{Md}(G)$, which evidently is equal to $mo(G)$. Then $(-1)^1 c_1 = mo(G)$.

(3) $(-1)^2 c_2$ is equal to the sum of determinants of all 2×2 principal submatrices of $A_{Md}(G)$, that is

$$\begin{aligned} c_2 &= \sum_{1 \leq i < j \leq n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \sum_{1 \leq i < j \leq n} (a_{ii}a_{jj} - a_{ij}a_{ji}) \\ &= \sum_{1 \leq i < j \leq n} a_{ii}a_{jj} - \sum_{1 \leq i < j \leq n} a_{ij}^2 \\ &= \begin{pmatrix} mo(G) \\ 2 \end{pmatrix} - \sum_{1 \leq i < j \leq n} d^2(v_i, v_j). \end{aligned}$$

THEOREM 2. *Let G be a graph of order n and let $\rho_1, \rho_2, ..., \rho_n$ be the eigenvalues of $A_{Md}(G)$. Then*

(*i*) $\sum_i^n \rho_i = mo(G)$.
(*ii*) $\sum_i^n \rho_i^2 = mo(G) + 2m + 2D$, *where*

$$D = \sum_{i<j,\ d(v_i, v_j) \neq 1} d^2(v_i, v_j).$$

PROOF. (i) Since the sum of the eigenvalues of $A_{Md}(G)$ is the trace of $A_{Md}(G)$, then

$$\sum_{i=1}^{n} \rho_i = \sum_{i=1}^{n} a_{ii} = |M| = mo(G).$$

(ii) Similarly the sum of squares eigenvalues of $A_{Md}(G)$ is the trace of $(A_{Md}(G))^2$. Then

$$\begin{aligned}\sum_{i=1}^{n} \rho_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\ &= \sum_{i=1}^{n} a_{ii}^2 + \sum_{i\neq j}^{n} a_{ij}a_{ji} \\ &= \sum_{i=1}^{n} a_{ii}^2 + 2\sum_{i<j}^{n} a_{ij}^2 \\ &= |M| + 2\sum_{1\leq i<j\leq n} d^2(v_i, v_j) \\ &= mo(G) + 2\sum_{1\leq i<j\leq n} d^2(v_i, v_j) \\ &= mo(G) + 2m + 2D,\end{aligned}$$

where $D = \sum_{i<j,\ d(v_i,v_j)\neq 1} d^2(v_i, v_j)$. □

LEMMA 3. *Let G be a graph with a minimum monopoly set M. If the minimum monopoly distance energy $E_{Md}(G)$ of G is a rational number, then*

$$E_{Md}(G) \equiv |M| \pmod 2.$$

PROOF. Let $\rho_1, \rho_2, ..., \rho_n$ be minimum monopoly distance eigenvalues of a graph G of which $\rho_1, \rho_2, ..., \rho_r$ are positive and the rest are non-positive, then

$$\begin{aligned}\sum_{i=1}^{n} |\rho_i| &= (\rho_1 + \rho_2 + ... + \rho_r) - (\rho_{r+1} + \rho_{r+2} + ... + \rho_n). \\ &= 2(\rho_1 + \rho_2 + ... + \rho_r) - (\rho_1 + \rho_2 + ... + \rho_n). \\ &= 2q - |M|.\ Where\ q = \rho_1 + \rho_2 + ... + \rho_r.\end{aligned}$$

Since $\rho_1, \rho_2, ..., \rho_r$ are algebraic integers, so is q. Therefore, $(\rho_1 + \rho_2 + ... + \rho_r)$ must be integer if $E_{Md}(G)$ is rational. Hence the Theorem. □

4. MINIMUM MONOPOLY DISTANCE ENERGY OF SOME STANDARD GRAPHS

In this section, the exact values of the minimum monopoly distance energy of some standard graphs are investigated.

THEOREM 4. *For the complete graph K_n for $n \geq 2$,*

$$E_{Md}(K_n) = \begin{cases} \frac{n-1}{2} + \sqrt{n^2-1}, & \text{if } n \text{ is odd;} \\ \frac{n-2}{2} + \sqrt{n^2-1}, & \text{if } n \text{ is even.}\end{cases}$$

PROOF. Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. Then the minimum monopoly size of the complete graph is

$$mo(K_n) = \left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n-1}{2}, & \text{if } n \text{ is odd;} \\ \frac{n}{2}, & \text{if } n \text{ is even.}\end{cases}$$

Hence the minimum monopoly set is $\{v_1, v_2, \cdots, v_{\frac{n-1}{2}}\}$ if n is odd and $\{v_1, v_2, \cdots, v_{\frac{n}{2}}\}$ if n is even. Therefore, we consider the following cases:

Case 1: n is odd,

$$A_{Md}(K_n) = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 0 \end{pmatrix}_{n\times n}$$

The respective characteristic polynomial is

$$\begin{aligned} f_n(K_n, \rho) &= \begin{vmatrix} \rho-1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ -1 & \rho-1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \rho-1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & \rho & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & -1 & \rho & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & \rho \end{vmatrix}_{n\times n} \\ &= \rho^{\frac{n-3}{2}}(\rho+1)^{\frac{n-1}{2}}\left(\rho^2 - (n-1)\rho - \frac{n-1}{2}\right).\end{aligned}$$

The spectrum of K_n is

$$MM\ Spec(K_n) = \begin{pmatrix} 0 & -1 & \frac{(n-1)+\sqrt{n^2-1}}{2} & \frac{(n-1)-\sqrt{n^2-1}}{2} \\ \frac{n-3}{2} & \frac{n-1}{2} & 1 & 1 \end{pmatrix}$$

Hence, the minimum monopoly distance energy of a complete graph of odd order is

$$E_{Md}(K_n) = \frac{n-1}{2} + \sqrt{n^2-1}.$$

Case 2: n is even,

$$A_{Md}(K_n) = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 0 \end{pmatrix}_{n\times n}$$

The respective characteristic polynomial is

$$f_n(K_n,\rho) = \begin{vmatrix} \rho-1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\ -1 & \rho-1 & \cdots & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \rho-1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & \rho & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & -1 & \cdots & \rho \end{vmatrix}_{n\times n}$$

$$= \rho^{\frac{n-2}{2}}(\rho+1)^{\frac{n-2}{2}}\left(\rho^2-(n-1)\rho-\frac{n}{2}\right).$$

The spectrum of K_n is

$$MM\ Spec(K_n) = \begin{pmatrix} 0 & -1 & \frac{(n-1)+\sqrt{n^2+1}}{2} & \frac{(n-1)-\sqrt{n^2+1}}{2} \\ \frac{n-2}{2} & \frac{n-2}{2} & 1 & 1 \end{pmatrix}$$

Hence, the minimum monopoly distance energy of a complete graph of even order is

$$E_{Md}(K_n) = \frac{n-2}{2} + \sqrt{n^2+1}.$$

From cases (1) and (2) the result is hold. □

THEOREM 5. *For the complete bipartite graph $K_{r,s}$ for $r \le s$*

$$E_{Md}(K_{r,s}) = (2s+r-3)+\sqrt{(4rs)^2-4rs+4r+4s^2-4s+1}.$$

PROOF. For the complete bipartite graph $K_{r,s}$, $(r \le s)$ with vertex set $V = \{v_1, v_2, \cdots, v_r, u_1, u_2, \cdots, u_s\}$. The minimum monopoly set is $M = \{v_1, v_2, \cdots, v_r\}$. Then

$$A_{Md}(K_{r,s}) = \begin{pmatrix} 1 & 2 & \cdots & 2 & 1 & 1 & \cdots & 1 \\ 2 & 1 & \cdots & 2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 0 & 2 & \cdots & 2 \\ 1 & 1 & \cdots & 1 & 2 & 0 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 2 & 2 & \cdots & 0 \end{pmatrix}_{(r+s)\times(r+s)}$$

The characteristic polynomial of $A_{Md}(K_{r,s})$, where $n = r+s$ is

$$f_n(K_{r,s},\rho) = \begin{vmatrix} \rho-1 & -2 & \cdots & -2 & -1 & -1 & \cdots & -1 \\ -2 & \rho-1 & \cdots & -2 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & -2 & \cdots & \rho-1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & \rho & -2 & \cdots & -2 \\ -1 & -1 & \cdots & -1 & -2 & \rho & \cdots & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 & -2 & -2 & \cdots & \rho \end{vmatrix}$$

$$= (\rho+1)^{r-1}(\rho+2)^{s-1}\left[\rho^2-(2r+2s-3)\rho-(3rs-4r-2s+2)\right].$$

Hence, the minimum monopoly distance eigenvalues of $K_{r,s}$ are $\rho_1 = -2[(s-1)\ times]$, $\rho_2 = -1[(r-1)\ times]$, $\rho_3 = \frac{(2r+2s-3)\pm\sqrt{(4rs)^2-4rs+4r+4s^2-4s+1}}{2}$ $[one\ time\ each]$. Therefore

$$E_{Md}(K_{r,s}) = (2s+r-3)+\sqrt{(4rs)^2-4rs+4r+4s^2-4s+1}.$$

□

THEOREM 6. *For $n \ge 2$, the minimum monopoly distance energy of a star graph $K_{1,n-1}$ is equal to $4n-7$.*

PROOF. Let $K_{1,n-1}$ be a star graph with vertex set $V = \{v_0, v_1, v_2, \cdots, v_{n-1}\}$, where v_0 is the center vertex, and the minimum monopoly set is $M = \{v_0\}$. Then

$$A_{Md}(K_{1,n-1}) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 2 & \cdots & 2 \\ 1 & 2 & 0 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 2 & \cdots & 0 \end{pmatrix}_{n\times n}$$

The characteristic polynomial of $A_{Md}(K_{1,n-1})$ is

$$f_n(K_{1,n-1},\rho) = \begin{vmatrix} \rho-1 & -1 & -1 & \cdots & -1 \\ -1 & \rho & -2 & \cdots & -2 \\ -1 & -2 & \rho & \cdots & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -2 & -2 & \cdots & \rho \end{vmatrix}$$

$$= (\rho+2)^{n-2}(\rho^2-(2n-3)\rho+(n-3)).$$

Then $MM\ Spec(K_{1,n-1})$ is

$$\begin{pmatrix} -2 & \frac{(2n-3)+\sqrt{4n^2-16n+21}}{2} & \frac{(2n-3)+\sqrt{4n^2-16n+21}}{2} \\ n-2 & 1 & 1 \end{pmatrix}$$

Therefore $E_{Md}(K_{1,n-1}) = 4n-7$. □

DEFINITION 7. *The double star graph $S_{n,m}$ is the graph constructed from union $K_{1,n-1}$ and $K_{1,m-1}$ by join whose centers v_0 with u_0. A vertex set is $V(S_{n,m}) = V(K_{1,n-1}) \cup V(K_{1,m-1})$ and edge set is $E(S_{n,m}) = \{v_0u_0, v_0v_i, u_0u_j : 1 \le i \le n-1,\ 1 \le j \le m-1\}$. Therefore, double star graph is bipartite graph.*

THEOREM 8. *For the double star graph $S_{m,m}$ for $m \ge 3$*

$$E_{Md}(S_{m,m}) = (9m-13)+\sqrt{m^2+6m-3}.$$

PROOF. For the double star graph $S_{n,n}$ with vertex set $V = \{v_0, v_1, ..., v_{m-1}, u_0, u_1, ..., u_{m-1}\}$ the minimum monopoly set is $M = \{v_0, u_0\}$. Then

$$A_{Md}(S_{m,m}) = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 2 & \cdots & 2 \\ 1 & 2 & \cdots & 2 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \cdots & 2 & 2 & 3 & \cdots & 3 \\ 1 & 2 & \cdots & 2 & 1 & 1 & \cdots & 1 \\ 2 & 3 & \cdots & 3 & 1 & 2 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 3 & \cdots & 3 & 1 & 2 & \cdots & 2 \end{pmatrix}_{m\times m}$$

The characteristic polynomial of $A_{Md}(S_{m,m})$ is

$$f_n(S_{m,m},\rho) = \begin{vmatrix} \rho-1 & -1 & \cdots & -1 & -1 & -2 & \cdots & -2 \\ -1 & \rho-2 & \cdots & -2 & -2 & -3 & \cdots & -3 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -2 & \cdots & \rho-2 & -2 & -3 & \cdots & -3 \\ -1 & -2 & \cdots & -2 & \rho-1 & -1 & \cdots & -1 \\ -2 & -3 & \cdots & -3 & -1 & -2 & \cdots & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & -3 & \cdots & -3 & -1 & -2 & \cdots & \rho-2 \end{vmatrix}$$

$= (\rho+2)^{2m-4}[\rho^2+(m+1)\rho-(m-1)][\rho^2-(5m-5)\rho+(m-5)]$.
Hence, the minimum monopoly distance eigenvalues of $S_{m,m}$ are $\rho_1 = -2[(2m-4)\ times]$, $\rho_2 = \frac{-(m+1)\pm\sqrt{m^2+6m-3}}{2}$ $[one\ time\ each]$, and $\rho_3 = \frac{(5m-5)\pm\sqrt{25m^2-54m+24}}{2}$ $[one\ time\ each]$.
Therefore, $E_{Md}(S_{m,m}) = (9m-13) + \sqrt{m^2+6m-3}$. □

5. BOUNDS FOR MINIMUM MONOPOLY DISTANCE ENERGY OF A GRAPH

In this section, some upper and lower bounds for minimum monopoly distance energy of graphs are established.

THEOREM 9. *Let G be a graph of order n and size m. Then*

$$\sqrt{mo(G)+2m+2s} \le E_{Md}(G) \le \sqrt{n\,[mo(G)+2m+2s]}.$$

Where $s = \sum_{i<j,\ d(v_i,v_j)\neq 1} d^2(v_i,v_j)$.

PROOF. Consider the Couchy-Schwartiz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

By choose $a_i = 1$ and $b_i = |\rho_i|$, we get

$$\begin{aligned}(E_{Md}(G))^2 &= \left(\sum_{i=1}^{n}|\rho_i|\right)^2\\ &\le \left(\sum_{i=1}^{n}1\right)\left(\sum_{i=1}^{n}\rho_i^2\right)\\ &\le n\,[mo(G)+2m+2s]\,.\end{aligned}$$

Where $s = \sum_{i<j,\ d(v_i,v_j)\neq 1} d^2(v_i,v_j)$. Therefore, the upper bound is hold.
Now, since

$$\left(\sum_{i=1}^{n}|\rho_i|\right)^2 \ge \sum_{i=1}^{n}\rho_i^2.$$

Then

$$(E_{Md}(G))^2 \ge \sum_{i=1}^{n}\rho_i^2 = mo(G)+2m+2s.$$

Therefore.

$$E_{Md}(G) \ge \sqrt{mo(G)+2m+2s}.$$

where $s = \sum_{i<j,\ d(v_i,v_j)\neq 1} d^2(v_i,v_j)$. □

THEOREM 10. *Let G be a graph of order n, size m and monopoly size mo(G). If* $D = det(A_{Md}(G))$*, then*

$$E_{Md}(G) \ge \sqrt{mo(G)+2m+2s+n(n-1)D^{2/n}}.$$

Where $s = \sum_{i<j,\ d(v_i,v_j)\neq 1} d^2(v_i,v_j)$.

PROOF. Since

$$\begin{aligned}(E_M(G))^2 &= \left(\sum_{i=1}^{n}|\rho_i|\right)^2\\ &= \left(\sum_{i=1}^{n}|\rho_i|\right)\left(\sum_{i=1}^{n}|\rho_i|\right)\\ &= \sum_{i=1}^{n}|\rho_i|^2 + 2\sum_{i\neq j}|\rho_i||\rho_j|.\end{aligned}$$

Employing the inequality between the arithmetic and geometric means, we get

$$\frac{1}{n(n-1)}\sum_{i\neq j}|\rho_i||\rho_j| \ge \left(\prod_{i\neq j}|\rho_i||\rho_j|\right)^{1/[n(n-1)]}.$$

Thus

$$\begin{aligned}(E_M(G))^2 &\ge \sum_{i=1}^{n}|\rho_i|^2 + n(n-1)\left(\prod_{i\neq j}|\rho_i||\rho_j|\right)^{1/[n(n-1)]}\\ &\ge \sum_{i=1}^{n}|\rho_i|^2 + n(n-1)\left(\prod_{i=j}|\rho_i|^{2(n-1)}\right)^{1/[n(n-1)]}\\ &= \sum_{i=1}^{n}|\rho_i|^2 + n(n-1)\left|\prod_{i\neq j}\rho_i\right|^{2/n}\\ &= mo(G)+2m+2s+n(n-1)D^{2/n}.\end{aligned}$$

Where $s = \sum_{i<j,\ d(v_i,v_j)\neq 1} d^2(v_i,v_j)$. □

6. CONCLUSION

In this paper, the minimum monopoly distance energy of a connected graph is studied. The exact value of the minimum monopoly distance energies of some standard graphs and also Some properties of characteristic polynomial of a minimum monopoly distance matrix of a graph are obtained. Upper and lower bounds for minimum monopoly distance energy of a graph are established. For the Path and the cycle graphs, computing the exact value of their minimum monopoly distance energy is completed and hence it is still open problem.
The minimum monopoly distance energy of several other families of graphs and binary operations of two graphs are open problems. The relationships between the minimum monopoly distance energy and the minimum distance (resp. minimum dominating distance) energy of a graph are also open problems.

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