

Mathematical Modeling of Fuel System with Respect to Reliability

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ABSTRACT

The authors have considered here helps to transfer the fuel from the fuel tank to the engine and regulate its flow into the cylinder depending on speed and local requirement. Here, a multi-component fuel system in diesel engine, comprised two subsystems ψ_1 and ψ_2 in series has been considered. In this system the fuel is injected into the cylinder using an injection pump and injectors. The block diagram depicts the fuel system in an N -cylinder diesel engine. The diesel fuel from the tank is pump to the fuel injection device. This device consists of small plunger pumps operating to push the fuel through the injectors (nozzles) into the cylinders one can operated one pump from the fuel injection pumps at the appropriate time i.e., towards the end of the compression stroke the fuel is pushed through the small hole in the injector and is atomized as it enters into the cylinder.

Keywords

Reliability theory, stochastic processes, Laplace transforms, Cost profit function and availability function and reliability function.

1. INTRODUCTION

Suppose a multi-component fuel system in diesel engine, comprised two subsystems ψ_1 and ψ_2 in series has been considered. In this system the fuel is injected into the cylinder using an injection pump and injectors. The block diagram depicts the fuel system in an N -cylinder diesel engine. The diesel fuel from the tank is pump to the fuel injection device. This device consists of small plunger pumps operating to push the fuel through the injectors (nozzles) into the cylinders one can operated one pump from the fuel injection pumps at the appropriate time i.e., towards the end of the compression stroke the fuel is pushed through the small hole in the injector and is atomized as it enters into the cylinder.

Now N -cylinders is j -out-of- N : D system) more than $N-j + 1$ cylinders s must fail for ψ_1 to fail), while sub-system ψ_2 may fail due to failure of anyone of its units. Also the whole system can fail due to critical human error and environmental pressure from the normal efficiency and degraded state.

The lifetime of the active units depend on each other in having simultaneous failure of all the operating units and repair times are distributed quite generally. A failed unit is repaired at a single service channel. [3], [4], [5] Earlier researchers have studied the model for $j = 1, 2$; under the assumption that the repair of a unit is possible only when the system stops operating and the life times of the active units are mutually independent, and have studied the model with exponential failure distribution under the different repair policies. [4] The concept of human error and environmental pressure has been examined. The availability and reliability

function for the system are obtained simultaneously by using supplementary variable technique.

2. ASSUMPTIONS

- (i). Initially, all the units are operating.
- (ii). Failure rates are exponential.
- (iii). The system goes to complete breakdown if more than $N-j$ units in ψ_1 are simultaneously failed or if any failure in ψ_2 occurs, critical human error or under environmental disasters.
- (iv). ψ_1 failed unit is repaired at a single service channel.
- (v). ψ_2 repaired unit is put into the operation immediately.
- (vi). ψ_1 repaired unit is assumed to behave like new after repair.
- (vii). Repair times are distributed quite generally. The system has three modes; good, degraded and failed state.
- (viii). The system has three modes; good, degraded and failed state.

3. NOTATIONS

\int	: \int_0^{∞} Otherwise stated
M, N	: Number of units in subsystem A and B
$\lambda / \lambda' / \lambda'' / e_1$ $/ e_2 / h_1 / h_2$: Constant failure rates from state 0 to ψ_2 0 to 3/0 to 1/1 to 2/0 to 5/ 1 to 5/0 to 6/1 to 6.
$\phi_i(x), S_i(x)$: Transition repair rate of the subsystem ψ_2 /pdf of repair rate
$\phi(r)$: Transition repair rate of subsystem ψ_1 pdf of repair rate
$\phi_e(z)$: Transition repair rate for critical human error
$\phi_e(k)$: Transition repair rate for environmental Pressure
$\bar{f}(s)$: Laplace transform of $f(t)$

- $P_{M,N}(t)$: The probability that at time t , the system is operating in the state of normal efficiency.
- $P_{M-j,N}(y,t)\Delta$: The probability that at time t , the system is in degraded state due to the failure units of subsystem ψ_1 and elapsed repair time lies in the interval $(y, y + \Delta)$
- $P_{M-j+1,N}(r,t)\Delta$: The probability that at time t , the system is in the failed state due to the failure of $M-j+1$ units of subsystem ψ_1 . The elapsed repair time lies in the $(r, r + \Delta)$.
- $P_{M,i}(x,t)$: The probability that at time t , the system is in failed state due to the failure of i^{th} -unit of subsystem ψ_2 . The elapsed repair time lies in the interval $(x, x + \Delta)$.
- $P_{M-j,i}(x,t)\Delta$: The probability that at time t the system is in failed state. The elapsed repair time lies in the interval $(x, x + \Delta)$.
- $P_H(z,t)$: The probability that at time t the system is in failed state due to critical human error. The elapsed repair time lies in interval $(z, z + \Delta)$.
- $P_e(k,t)$: The probability that at time t the system is in failed state due to environmental disorders. The elapsed repair time lies in the interval $(k, k + \Delta)$.

$$+ \sum_{i=1}^M \int P_{M,i}(x,t) \phi_i(x) dx + \int P_{M-j+1,N}(r,t) \phi(r) dr + \int P_e(k,t) \phi_e(k) dk + \int P_H(z,t) \phi_H(z) dz \quad \dots(1)$$

$$[D_y + D_t + \lambda + \lambda' + e_2 + h_2 + \phi_p(y)] P_{M-j,N}(y,t) = 0 \quad \dots (2)$$

$$[D_r + D_t + \phi(r)] P_{M-j+1,N}(r,t) = 0 \quad \dots (3)$$

$$[D_x + D_t + \phi_i(x)] P_{M,i}(x,t) = 0 \quad \dots (4)$$

$$[D_x + D_t + \phi_i(x)] P_{M-j,i}(x,t) = 0 \quad \dots (5)$$

$$[D_k + D_t + \phi_e(k)] P_e(k,t) = 0 \quad \dots (6)$$

$$[D_z + D_t + \phi_H(z)] P_H(z,t) = 0 \quad \dots (7)$$

These equations can be solved under following boundary and initial conditions:

4.1 Boundary Conditions

$$P_{M-j,N}(0,t) = \lambda' P_{M,N}(t) \sum_{i=1}^M \int P_{M-j,i}(x,t) \phi_i(x) dx + \int P_e(k,t) \phi_e(k) dk + \int P_H(z,t) \phi_H(z) dz \quad \dots (8)$$

$$P_{M-j+1,N}(0,t) = \lambda'' P_{M-j,N}(t) \quad \dots (9)$$

$$P_{M,i}(0,t) = \lambda P_{M,N}(t) \quad \dots (10)$$

$$P_{M-j,i}(0,t) = \lambda P_{M-i,N}(t) \quad \dots (11)$$

$$P_e(0,t) = e_1 P_{M,N}(t) + e_2 P_{M-j,N}(t) \quad \dots (12)$$

$$P_H(0,t) = h_1 P_{M,N}(t) + h_2 P_{M-j,N}(t) \quad \dots (13)$$

4.2 Preliminary Condition

$$P_k(0) = 1, \text{ as } k = M, N \text{ otherwise zero} \quad \dots (14)$$

4. CONCEPT OF

MATHEMATICAL MODEL

The analysis crucially depends on the method of supplementary variables, and the supplementary variable \times denotes the elapsed time that a unit has been undergoing repair. Viewing the nature of the problem, the following set of difference-differential equations is obtained as follows:

$$[D_t + \lambda + \lambda' + e_1 + h_1] P_{M,N}(t) = \sum_{p=1}^j \int P_{M-j,N}(y,t) \phi_p(y) dy$$

5. CYLINDRICAL FUEL SYSTEM AND TRANSITION STATE DIAGRAM

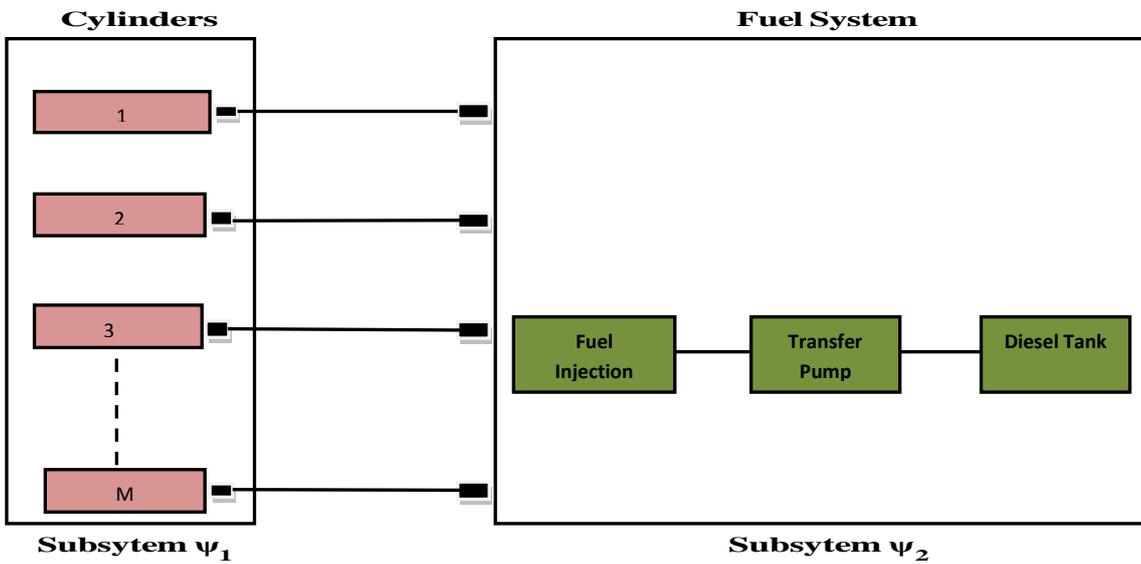


Figure 1: Block Diagram of Cylindrical Fuel System

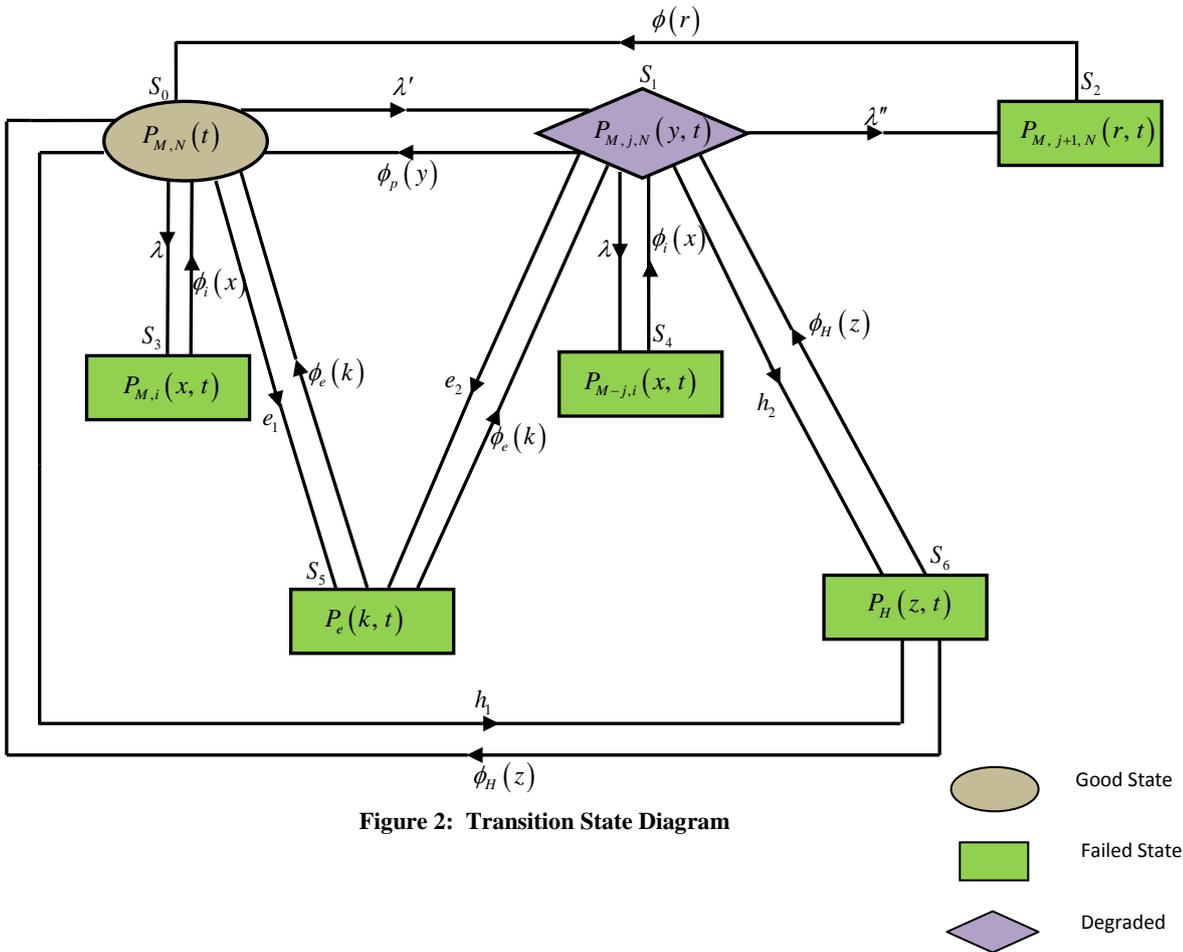


Figure 2: Transition State Diagram

- Good State
- Failed State
- Degraded

6. SOLUTIONS OF MATHEMATICAL MODEL

Taking Laplace transforms of the equations (1) through (13) and using initial conditions we may get:

$$[s + \lambda + \lambda' + e_1 + h_1] \bar{P}_{M,N}(s) = 1 + \sum_{p=1}^j \int \bar{P}_{M-j,N}(y,s) \phi_p(y) dy + \sum_{i=1}^M \int \bar{P}_{M,i}(x,s) \phi_i(x) dx + \int \bar{P}_{M-j+1,N}(r,s) \phi(r) dr + \int \bar{P}_e(k,s) \phi_e(k) dk + \int \bar{P}_H(z,s) \phi_H(z) dz \quad \dots (15)$$

$$[D_y + s + \lambda + \lambda' + e_2 + h_2 + \phi_p(y)] \bar{P}_{M-j,N}(y,s) = 0 \quad \dots (16)$$

$$[D_r + s + \phi(r)] \bar{P}_{M-j+1,N}(r,s) = 0 \quad \dots (17)$$

$$[D_x + s + \phi_i(x)] \bar{P}_{M,i}(x,s) = 0 \quad \dots (18)$$

$$[D_x + s + \phi_i(x)] \bar{P}_{M,j,i}(x,s) = 0 \quad \dots (19)$$

$$[D_k + s + \phi_e(k)] \bar{P}_e(k,s) = 0 \quad \dots (20)$$

$$[D_z + s + \phi_H(z)] \bar{P}_H(z,s) = 0 \quad \dots (21)$$

$$\bar{P}_{M-j,N}(0,s) = \lambda' \bar{P}_{M,N}(s) + \sum_{i=1}^M \int \bar{P}_{M-j,i,s}(x,s) \phi_i(x) dx + \int \bar{P}_e(k,s) \phi_e(k) dk + \int \bar{P}_H(z,s) \phi_H(z) dz \quad \dots (22)$$

$$\bar{P}_{M-j+1,N}(0,s) = \lambda'' \bar{P}_{M-j,N}(s) \quad \dots (23)$$

$$\bar{P}_{M,i}(0,s) = \lambda \bar{P}_{M,N}(s) \quad \dots (24)$$

$$\bar{P}_{M-j,i}(0,s) = \lambda \bar{P}_{M-j,N}(s) \quad \dots (25)$$

$$\bar{P}_e(0,s) = e_1 \bar{P}_{M,N}(s) + e_2 \bar{P}_{M-j,N}(s) \quad \dots (26)$$

$$\bar{P}_H(0,s) = h_1 \bar{P}_{M,N}(s) + h_2 \bar{P}_{M-j,N}(s) \quad \dots (27)$$

Integrating (18) and using (24) one obtain:

$$\bar{P}_{M,j}(x,s) = \lambda \bar{P}_{M,N}(s) \exp[-sx - \int_0^x \phi_i(x) dx] \quad \dots (28)$$

$$\bar{P}_{M,i}(s) = \lambda \bar{P}_{M,N}(s) D_{\phi_i}(s) \quad \dots (28)$$

Integrating equation (18) by making use of (24) we get:

$$\bar{P}_{M-j,i}(x,s) = \lambda \bar{P}_{M-j,N}(s) \exp[-sx - \int_0^x \phi_i(x) dx] \quad \dots (29)$$

$$\bar{P}_{M-j,i}(s) = \lambda \bar{P}_{M-j,N}(s) D_{\phi_i}(s) \quad \dots (29)$$

Integrating (17) using (23) one may get

$$\bar{P}_{M-j+1,i}(r,s) = \lambda'' \bar{P}_{M-j,N}(s) \exp[-sx - \int_0^r \phi(r) dr] \quad \dots (30)$$

$$\bar{P}_{M-j+1,i}(s) = \lambda'' \bar{P}_{M-j,N}(s) D_{\phi}(s) \quad \dots (30)$$

$$\bar{P}_e(k,s) = [e_1 \bar{P}_{M,N}(s) + e_2 \bar{P}_{M-j,N}(s)] \exp[-sk - \int_0^k \phi_e(k) dk] \quad \dots (31)$$

$$\bar{P}_e(s) = [e_1 \bar{P}_{M,N}(s) + e_2 \bar{P}_{M-j,N}(s)] D_{\phi_e}(s) \quad \dots (31)$$

$$\bar{P}_H(z,s) = [h_1 \bar{P}_{M,N}(s) + h_2 \bar{P}_{M-j,N}(s)] \exp[-sz - \int_0^z \phi_H(z) dz] \quad \dots (32)$$

$$\bar{P}_H(s) = [h_1 \bar{P}_{M,N}(s) + h_2 \bar{P}_{M-j,N}(s)] D_{\phi_H}(s) \quad \dots (32)$$

Integrating (16) by using relevant relations one may obtain:

$$\bar{P}_{M-j,N}(y,s) = G(s) \bar{P}_{M,N}(s) \quad \dots (33)$$

$$G(s) = [\lambda' + e_1 \bar{S}_{\phi_e}(s) + h_1 \bar{S}_{\phi_H}(s)] \times D_{\phi_e}(s + \lambda' + \lambda'' + e_2 + h_2 - \lambda \bar{S}_{\phi_e}(s) - e_2 \bar{S}_{\phi_e}(s) - h_2 \bar{S}_{\phi_H}(s)) \quad \dots (33)$$

$$\text{Where, } D_k(s) = \frac{1 - \bar{S}_k(s)}{s}, \forall k$$

Lastly, equation (15) gives by using relevant relations

$$\bar{P}_{M,N}(s) = \frac{1}{A(s)} \quad \dots (34)$$

$$\text{Where, } A(s) = s + \lambda + \lambda' + e_1 [1 - \bar{S}_{\phi_e}(s)] + h_1 [1 - \bar{S}_{\phi_H}(s)] - (\lambda' + e_1 \bar{S}_{\phi_e}(s) + h_1 \bar{S}_{\phi_H}(s)) \bar{S}_p(s) [s + \lambda' + \lambda'' + e_2 + h_2 - \lambda \bar{S}_{\phi_e}(s) - e_2 \bar{S}_{\phi_e}(s) - h_2 \bar{S}_{\phi_H}(s)] \times [s + \lambda' + \lambda'' + e_2 + h_2 - \lambda \bar{S}_{\phi_e}(s) - e_2 \bar{S}_{\phi_e}(s) - h_2 \bar{S}_{\phi_H}(s)] [e_2 \bar{S}_{\phi_e}(s) + h_2 \bar{S}_{\phi_H}(s) + \lambda'' \bar{S}_{\phi}(s)] G(s) \quad \dots (34)$$

Thus finally we have

$$\bar{P}_{M,N}(s) = \frac{1}{A(s)} \quad \dots (35)$$

$$\bar{P}_{M-j,N}(s) = \frac{G(s)}{A(s)} \quad \dots (36)$$

$$\bar{P}_{M-j+1,N}(s) = \frac{\lambda'' G(s)}{A(s)} D_{\phi}(s) \quad \dots (37)$$

$$\bar{P}_{M,j}(s) = \frac{\lambda}{A(s)} D_{\phi}(s) \quad \dots (38)$$

$$\bar{P}_{M-j,i}(s) = \frac{\lambda G(s)}{A(s)} D_{\phi}(s) \quad \dots (39)$$

$$\bar{P}_e(s) = \frac{1}{A(s)} [e_1 + e_2 G(s)] D_{\phi_e}(s) \quad \dots (40)$$

$$\bar{P}_H(s) = \frac{1}{A(s)} [h_1 + h_2 G(s)] D_{\phi_H}(s) \quad \dots (41)$$

7. ERGODIC BEHAVIOR OF THE SYSTEM

Using Abel's Lemma in L.T. viz.

$$\lim_{s \rightarrow 0} s \bar{F}(s) = \lim_{t \rightarrow \infty} F(t) = F(Say) \quad \text{provided the limit on R.H.S. exists, the time independent probabilities are obtained as follows:}$$

$$P_{M,N} = \frac{1}{A'(0)} \quad \dots (42)$$

$$P_{M-j,N} = \frac{G(0)}{A'(0)} \quad \dots (43)$$

$$P_{M-j+1,N} = \frac{\lambda'' G(0)}{A'(0)} M_{\phi}(s) \quad \dots (44)$$

$$P_{M,i} = \frac{\lambda}{A'(0)} M_{\phi} \quad \dots (45)$$

$$P_{M-j,t} = \frac{\lambda G(0)}{A'(0)} M_\phi \quad \dots (46)$$

$$P_e = \frac{1}{A'(s)} [e_1 + e_2 G(0)] M_\phi \quad \dots (47)$$

$$P_e = \frac{1}{A'(0)} [h_1 + h_2 G(0)] M_\phi \quad \dots (48)$$

Where,

$$M_k = -\bar{S}_k'(0) = \text{Mean time to repair } k^{\text{th}} \text{ unit}$$

8. SOME PARTICULAR CASES

When the repair follow exponential time distribution

$$\text{Setting } \bar{S}_{\phi_i}(s) = \frac{\phi_i}{s + \phi_i}, \bar{S}_{\phi_e}(s) = \frac{\phi_e}{s + \phi_e} \text{ etc. in equation (35)}$$

through (41) one obtains:

$$\bar{P}_{M,N}(s) = \frac{1}{E(s)} \quad \dots (49)$$

$$\bar{P}_{M-j,N}(s) = \frac{g(s)}{E(s)} \quad \dots (50)$$

$$\bar{P}_{M-j+1,N}(s) = \frac{\lambda'' g(s)}{E(s)} \frac{1}{s + \phi} \quad \dots (51)$$

$$\bar{P}_{M,i}(s) = \frac{\lambda}{E(s)} \frac{1}{s + \phi_i} \quad \dots (52)$$

$$\bar{P}_{M-j,i}(s) = \frac{\lambda g(s)}{E(s)} \frac{1}{s + \phi_i} \quad \dots (53)$$

$$\bar{P}_e(s) = \frac{1}{E(s)} [e_1 + e_2 g(s)] \frac{1}{s + \phi_i} \quad \dots (54)$$

$$\bar{P}_H(s) = \frac{1}{E(s)} \frac{1}{s + \phi_H} [h_1 + h_2 g(s)] \quad \dots (55)$$

$$\text{Where, } E(s) = [A(s)]_{\bar{S}_k(s) = \frac{k}{s+k}}, g(s) = [G(s)]_{\bar{S}_k(s) = \frac{k}{s+k}}$$

9. EVALUATION OF UP AND DOWN STATE PROBABILITIES

$$\begin{aligned} \bar{P}_{UP}(s) &= \bar{P}_{M,N}(s) + \bar{P}_{M-j,N}(s) \\ &= \frac{1}{s + \lambda + \lambda' + e_1 + h_1} \left[1 + \frac{\lambda''}{s + \lambda' + \lambda'' + e_2 + h_2} \right] \end{aligned}$$

Inverting this we have

$$\begin{aligned} P_{up}(t) &= (1 + A) \exp[-(\lambda + \lambda' + e_1 + h_1)t] \\ &\quad + B \exp[-(\lambda' + \lambda'' + e_2 + h_2)t] \quad \dots (A) \end{aligned}$$

$$\text{And } P_{down}(t) = 1 - P_{up}(t) \quad \dots (B)$$

10. COST ANALYSIS FUNCTION

The cost analysis function is defined as

$$G(t) = C_1 \int_0^t P_{up}(t) dt - C_2 t$$

Where, C_1 = revenue cost per unit up time
 C_2 = repair cost per unit time

After integration we get,

$$\begin{aligned} G(t) &= C_1(1 + A) \left[\frac{1 - \exp(-(\lambda + \lambda' + e_1 + h_1)t)}{\lambda + \lambda' + e_1 + h_1} \right] \\ &\quad + C_1 B \left[\frac{1 - \exp(-(\lambda' + \lambda'' + e_2 + h_2)t)}{\lambda' + \lambda'' + e_2 + h_2} \right] - C_2 t \quad \dots (C) \end{aligned}$$

$$\text{Where, } A = \frac{\lambda''}{\lambda'' + e_2 + h_2 - \lambda - e_1 - h_1} \text{ and}$$

$$B = \frac{\lambda''}{\lambda + e_1 + h_1 - \lambda'' - e_2 - h_2} \quad \dots (D)$$

11. NUMERICAL COMPUTATION

Let $\lambda = h_1 = 0.01, \lambda' = h_2 = 0.02, \lambda'' = 0.03, e_1 = 0.04, e_2 = 0.05$
and $C_1 = 2, C_2 = 1$

Putting these values in equations (D), (B), (C) and (A), we get

$$\begin{aligned} A &= \frac{\lambda''}{\lambda'' + e_2 + h_2 - \lambda - e_1 - h_1} \\ &= \frac{0.02}{0.03 + 0.05 + 0.02 - 0.01 - 0.04 - 0.01} = 0.5 \\ B &= \frac{\lambda''}{\lambda + e_1 + h_1 - \lambda'' - e_2 - h_2} \\ &= \frac{0.02}{0.01 + 0.04 + 0.01 - 0.03 - 0.05 - 0.02} = -0.5 \end{aligned}$$

$$P_{up}(t) = 1.5 \exp(-0.08)t - 0.5 \exp(-0.12)t$$

$$\text{and } G(t) = 3 \left[\frac{1 - \exp(-0.08)t}{0.08} \right] - \left[\frac{1 - \exp(-0.12)t}{0.12} \right] - t$$

12. INTERPRETATION

Table 1 computes the availability of the system decreases as the time increases and Figure 3 shows the graph for the same.

Table 2 & Figure 4 shows that cost function increases with time t .

Table 1. Availability function with respect to time

S.No.	t	Pup(t)
1	0	1
2	1	0.9412143
3	2	0.8849018
4	3	0.8311036
5	4	0.7798319
6	5	0.7310743
7	6	0.684799
8	7	0.6409583
9	8	0.5994922
10	9	0.5603306
11	10	0.5233963
12	11	0.4886067
13	12	0.4558754

14	13	0.425114
15	14	0.3962327
16	15	0.3691419

Table 2. Cost function with respect to time

S.No.	t	G(t)
1	0	0
2	1	0.9408073
3	2	1.7665068
4	3	2.4820913
5	4	3.092606
6	5	3.6030952
7	6	4.0185583
8	7	4.3439145
9	8	4.5839748
10	9	4.7434198
11	10	4.8267823
12	11	4.838435
13	12	4.7825814
14	13	4.66325
15	14	4.4842908
16	15	4.2493745

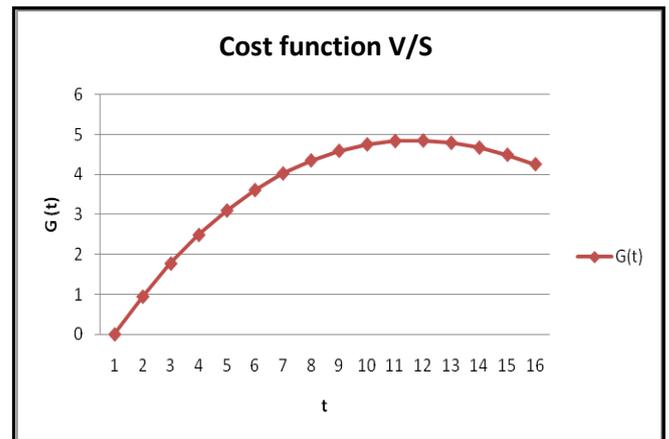


Figure 4 : Cost Profit as function of time

13. CONCLUSION

Table 1 and Figure 3 provide information how availability of the complex engineering repairable system changes with respect to the time when failure rate increases availability of the system decreases.

Table 2 and Figure 4 when revenue cost per unit time C_1 and C_2 are fixed, then one can conclude by observing this graph that as cost increases, when time increases.

Hence the present study clearly proves the importance of fuel system of [5-6] which seem to be possible in many engineering systems when it is analyzed with the help of the copula. The further research area is widely open, where one may think of the application of MTTF and sensitivity analysis.

14. REFERENCES

- [1] Gupta, P.P.; Agarwal, S.C. : ‘Cost function analysis of three state repairable system’, Microelectron. Reliability Vol. 24, pp 51-53 (1984).
- [2] Jindal, Deepika : ‘Operational readiness of a multistage complex repairable system’, Ph.D. Thesis submitted to C.C.S. Univ. Meerut, India (2000).
- [3] Sharma, V., Operational Readiness of a multistage Complex Repairable System.; Bulletin of Pure and Applied Sciences Vol. 31 EP 47-55 (2012)
- [4] Sharma, U.; Panigrahi, S.C.; Mishra, R.B. : ‘Reliability evaluation of a communication system considering a multi-state model’, Micro election. Reliability. Vol. 30, pp 701-704 (1980).
- [5] Gupta, P.P.; Kumar, A ; Mittal, S.K. : ‘Stochastic behavior of three state complex repairable system with three types of failures’, Microelectron. Reliability, Vol. 25, pp 853-858 (1985).
- [6] Tyagi, P.K.: ‘Cost benefit models of reliability models’, Ph.D. Thesis submitted to C.C.S.Univ. Meerut, India (2003).

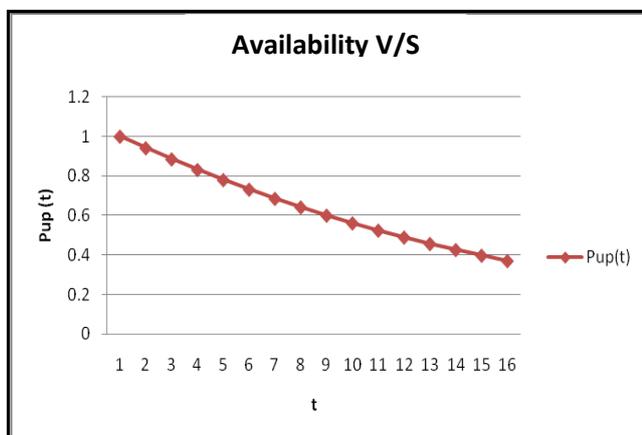


Figure 3: Availability as function of time