On Bayesian One-sample Prediction of the Generalized Pareto Distribution based on Generalized Order Statistics

M.A.W. Mahmoud
Maths. Dept. Faculty of Science
Al-Azhar University
Nasr City 11884, Cairo, Egypt.

A.A.K. Saleh
Maths. Dept. Faculty of Women
Ain Shams University
Cairo, Egypt

ABSTRACT

Bayesian predictive functions for future observations from a generalized Pareto distribution based on generalized order statistics are obtained. Two cases are considered unknown one parameter and unknown two parameters. We also consider two cases fixed sample size and random sample size. The Bayesian predictive functions are specialized to ordinary order statistics, progressive type II censoring and upper record values. Examples are calculated for the lower and the upper bounds for the future observation based on ordinary order statistics, progressive type II censoring and upper record samples..

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Keywords

Bayesian prediction, generalized order statistics, generalized Pareto distribution, ordinary order statistics, predictive function, random sample size.

1. INTRODUCTION

The generalized Pareto distribution (GPD) was first introduced by [1]. The probability density function (pdf) and cumulative distribution function of the generalized Pareto are given by

$$f\left(x;\sigma,\theta\right)=\frac{1}{\sigma\theta}\left(1-\frac{x}{\sigma}\right)^{\frac{1}{\theta}-1} \quad ;\theta>0,0<x<\sigma, (1)$$

and
$$F\left(x;\sigma,\theta\right)=1-\left(1-\frac{x}{\sigma}\right)^{\frac{1}{\theta}}, \quad (2)$$

where θ is the shape parameter and σ is the scale parameter. See [2].

Generalized order statistic's concept was presented by [3] as a unified approach to different models of ordered random variables such models are order statistics (OOS), record values (RVs), k-th record values and sequential order statistics, among others. Authors [4], [5] among others utilized the generalized order statistics (GOS) in their works.

Let $X_1 < X_2 < ... < X_r$ be the first r Generalized order failure times in a random sample of n components from an absolutely continuous distribution function F, where $X_i \equiv X(i,n,m_i,k)$, $i=1,2,...,n$, m_i and k are real numbers and $k \geq 1$. The joint pdf of the first r GOS is given by

$$f_{X_1,X_2,...,X_r}(x_1,x_2,...,x_r) = C_{r-1}\left[\prod_{i=1}^{r-1}(1-F(x_i))^{m_i} f(x_i)\right]$$
$$\times\left[(1-F(x_r))^{\gamma_r-1} f(x_r)\right], \quad (3)$$

where $f(x_i)$ and $F(x_i)$ are the pdf and the cdf of X_i, and

$$C_{j-1}=\prod_{i=1}^{j}\gamma_i, \quad \gamma_i=k+n-i+\sum_{j=i}^{n-1}m_j \text{ and } r=1,2,...,n-1. \quad (4)$$

See [3], considering two cases for the real numbers m_i, the conditional density function $h(x_s \mid x_r)$ of the GOS x_s given that r GOSs had already failed is given by,

Case I: When $m_1 = m_2 = ... = m_{n-1} = m = -1$

$$h_1(x_s \mid x_r)=\frac{k^{s-r}}{(s-r-1)!C_{r-1}}\left[\ln(1-F(x_r))-\ln(1-F(x_s))\right]^{s-r-1}$$
$$\times\left[1-F(x_s)\right]^{k-1}\left[1-F(x_r)\right]^{-k} f(x_s), \quad (5)$$

Case II: When $m_i \neq m_j \quad for \; i \neq j,$

$$h_2(x_s \mid x_r)=\frac{C_{s-1}}{C_{r-1}}$$
$$\times\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{1-F(x_s)}{1-F(x_r)}\right)^{\gamma_i}\left(\frac{f(x_s)}{1-F(x_s)}\right), 1\leq r\leq n-1, \quad (6)$$

where
$$a_i^{(r)}(s)=\prod_{\substack{j=r+1\\j\neq i}}^{s}\frac{1}{\gamma_j-\gamma_i}, \qquad r+1\leq i\leq s. \quad (7)$$

See [3], [6]. The Bayes prediction of an unobserved random variable is frequently more significant for applications than interval estimation of parameters. Many authors have been discussed the prediction problems in general based on GOS, progressively type-II censored data, record values (RVs) and ordinary order statistics (OOS) see for example [7], [8]. The goal of this paper is to discuss the one-sample Bayesian prediction problem from generalized Pareto distribution based on GOS in two cases when the sample size is fixed (*FSS*) and when it is a random variable (*RSS*). In case of fixed sample size, The Bayes predictive density function of the future GOS $x_s, s=r+1, r+2, ..., n$ can be written as:

$$f^*(x_s \mid \underline{x})=\iint_{\sigma\;\theta}h(x_s \mid \theta,\sigma) f(\theta,\sigma \mid \underline{x})d\theta d\sigma, \quad (8)$$

where the conditional pdf of the future observation is $h(x_s \mid \theta,\sigma)$ and $f(\theta,\sigma \mid \underline{x})$ is the posterior pdf.

In case of random sample size, the predictive density function

of x_s is given by

$$q(x_s \mid \underline{x}, n) = \frac{1}{\Pr(n \geq s)} \sum_{n=s}^{\infty} v(n) f^*(x_s \mid \underline{x}), \qquad (9)$$

where $v(n)$ is the probability mass function of n and $f^*(x_s|\underline{x})$ is defined in (8), see for example [9].

2. BAYESIAN PREDICTION BOUNDS BASED ON GOS IN CASE OF FSS

The likelihood function of the first r GOS failure times $x_1, x_2, ..., x_r$ in a random sample of n identical distributed random variables having the GPD defined in (1) can be obtained by substituting (1) and (2) into (3) as follows

$$L(\sigma, \theta; x) = C_{r-1} (\sigma\theta)^{-r} \eta(\sigma) e^{\frac{T}{\theta}}, \qquad (10)$$

where

$$\eta(\sigma) = \prod_{i=1}^{i=r} (1 - \frac{x_i}{\sigma})^{-1}, \ T = \sum_{i=1}^{i=r-1} (m_i + 1)\ln(1 - \frac{x_i}{\sigma}) + \gamma_r \ln(1 - \frac{x_r}{\sigma})$$

(11)

Considering that the scale parameters σ is known and the unknown shape parameter θ is having the inverted gamma prior distribution defined by

$$g_1(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\frac{\beta}{\theta}}, \qquad if \ \ \theta > 0, \ (\alpha, \beta), \quad (12)$$

the posterior pdf of θ can be written as

$$f_1(\theta \mid \underline{x}) = \frac{(\beta - T)^{\alpha+r}}{\Gamma(\alpha+r)} \theta^{-(r+\alpha+1)} e^{-\frac{\beta-T}{\theta}}, \qquad (13)$$

when the two parameters σ and θ are assumed to be unknown, the inverted gamma prior density defined in (12) is considered as a distribution for the unknown shape parameter θ and the uniform prior density is considered for the unknown scale parameters σ defined by

$$g_2(\sigma) = \frac{1}{\upsilon}, \qquad 0 \leq \sigma \leq \upsilon, \qquad (14)$$

the joint prior density for both the parameters θ and σ takes the form

$$g(\sigma, \theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\frac{\beta}{\theta}} \frac{1}{\upsilon}, \ 0 \leq \sigma \leq \upsilon, (\alpha, \beta, \theta, \upsilon) > 0, (15)$$

where α, β, and υ are known. The joint posterior pdf of θ and σ is given by

$$f_2(\sigma, \theta \mid \underline{x}) = \frac{\eta(\sigma)}{\Gamma(\alpha+r)\sigma^r I_0} \theta^{-(r+\alpha+1)} e^{-\frac{\beta-T}{\theta}}, \quad (16)$$

where $I_0 = \int_0^\upsilon \frac{\eta(\sigma)}{\sigma^r} (\beta-T)^{-(r+\alpha)} d\sigma.$ (17)

2.1 When σ is known

Now let x_s, $s = r+1, r+2, ..., n$ denotes the sth lifetime component from the remaining $(n$-$r)$ components having the GPD when the scale parameter σ is known, the conditional density function of the sth GOS given that r generalized order statistics had already failed can be obtained by substituting (1)

and (2) into (5) and (6) as follows:

Case **I:**

$$h_1(x_s \mid x_r) = \frac{k^{s-r}}{\sigma\Gamma(s-r)} \left(\frac{1}{\theta}\right)^{s-r} (1 - \frac{x_s}{\sigma})^{-1} A^{s-r-1} e^{-\frac{Ak}{\theta}}, (18)$$

Case **II:**

$$h_2(x_s \mid x_r) = \frac{C_{s-1}}{\sigma C_{r-1}} \left(1 - \frac{x_s}{\sigma}\right)^{-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \frac{1}{\theta} e^{A\frac{\gamma_i}{\theta}}, (19)$$

where $$A = -\ln\left[\frac{\sigma - x_s}{\sigma - x_r}\right]. \qquad (20)$$

The Bayes predictive density function of the future GOS x_s is obtained by substituting (18), (19) and (13) into (8)

Case I:

$$f_{11}^*(x_s \mid \underline{x}) = \frac{k^{s-r}}{\sigma} \frac{(\beta - kT_r)^{\alpha+r} \Gamma(\alpha+s)}{\Gamma(s-r)\Gamma(\alpha+r)}$$

$$\times \frac{(1 - \frac{x_s}{\sigma})^{-1} A^{s-r-1}}{\left(\beta - k\ln(1 - \frac{x_s}{\sigma})\right)^{\alpha+s}}. \qquad (21)$$

Case II:

$$f_{12}^*(x_s \mid \underline{x}) = \frac{(\alpha + r)C_{s-1}}{C_{r-1}} (\sigma - x_s)^{-1}$$

$$\sum_{i=r+1}^{s} \frac{a_i^{(r)}(s)(\beta - T)^{\alpha+r}}{(\beta - T + A\gamma_i)^{\alpha+r+1}}. \qquad (22)$$

By integrating with respect to x_s, the Bayesian prediction survival function $\Pr(x_s > t \mid \underline{x})$ for some t, can be written as follows:

Case **I:**

$$\Pr_1(x_s \geq t \mid \underline{x}) = R_{r,s} \left[\frac{1}{(\alpha+s-1)} \frac{(kT_r - kZ)^{s-r-1}}{(\beta - kZ)^{\alpha+s-1}} \right.$$

$$+ \frac{s-r-1}{(\alpha+s-1)(\alpha+s-2)} \frac{(kT_r - kZ)^{s-r-2}}{(\beta - kZ)^{\alpha+s-2}}$$

$$+ \frac{(s-r-1)(s-r-2)}{(\alpha+s-1)(\alpha+s-2)(\alpha+s-3)} \frac{(kT_r - kZ)^{s-r-3}}{(\beta - kZ)^{\alpha+s-3}} + ...$$

$$\left. + \frac{(s-r-1)!}{(\alpha+s-1)(\alpha+s-2)...(\alpha+r)} \frac{1}{(\beta - kZ)^{\alpha+r}} \right], (23)$$

Case II:

$$\Pr_2(x_s \geq t \mid \underline{x}) = \frac{C_{s-1}}{C_{r-1}} (\beta - T)^{\alpha+r}$$

$$\times \sum_{i=r+1}^{s} \frac{a_i^{(r)}(s)}{\gamma_i} [\beta - T - \gamma_i (Z - T_r)]^{-(\alpha+r)}, (24)$$

where $T_r = \ln(1 - \frac{x_r}{\sigma}), Z = \ln(1 - \frac{t}{\sigma}),$ (25)

and
$$R_{r,s} = \frac{(\beta - T_r)^{\alpha+r}\Gamma(\alpha+s)}{\Gamma(s-r)\Gamma(\alpha+r)} \ . \qquad (26)$$

Let L_s and U_s denote the lower and the upper limits of the $100\tau\%$ Bayesian prediction interval for the GOS x_s such that $pr[L_s \le x_s \le U_s] = \tau$ where,

$$\Pr(x_s \ge L_s \mid \underline{x}) = \frac{1+\tau}{2}, \ \Pr(x_s \ge U_s \mid \underline{x}) = \frac{1-\tau}{2}. \quad (27)$$

2.2 When θ and σ are Unknown parameters
In this subsection, The Bayes predictive density function of the GOS x_s is obtained by substituting (18), (19) and (16) into (8):

Case I:

$$f_{21}^{*}(x_s \mid \mathbf{x}) = \frac{k^{s-r}}{I_0\Gamma(s-r)}\frac{\Gamma(\alpha+s)}{\Gamma(\alpha+r)}$$
$$\int_0^{\upsilon}\frac{\eta(\sigma)(1-\frac{x_s}{\sigma})^{-1}A^{s-r-1}}{\left(\beta - k\ln(1-\frac{x_s}{\sigma})\right)^{\alpha+s}\sigma^{r+1}}d\sigma. \quad (28)$$

Case II:

$$f_{22}^{*}(x_s \mid \underline{x}) = \frac{(\alpha+r)C_{s-1}}{I_0 C_{r-1}}$$
$$\int_0^{\upsilon}\frac{\eta(\sigma)}{\sigma^r(\sigma-x_s)}\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{(\beta-T+A\gamma_i)^{(\alpha+r+1)}}d\sigma, \quad (29)$$

and the Bayesian prediction survival function of the future GOS x_s as follows:

CaseI:

$$\Pr_{21}(x_s \ge t \mid \underline{x}) = \frac{\Gamma(\alpha+s)}{I_0\ \Gamma(\alpha+r)\Gamma(s-r)}$$
$$\times \int_0^{\upsilon}\frac{\eta(\sigma)}{\sigma^r}\left[\frac{1}{(\alpha+s-1)}\frac{(kT_r-kZ)^{s-r-1}}{(\beta-kZ)^{\alpha+s-1}}\right.$$
$$+\frac{s-r-1}{(\alpha+s-1)(\alpha+s-2)}\frac{(kT_r-kZ)^{s-r-2}}{(\beta-kZ)^{\alpha+s-2}}$$
$$+\frac{(s-r-1)(s-r-2)}{(\alpha+s-1)(\alpha+s-2)(\alpha+s-3)}\frac{(kT_r-kZ)^{s-r-3}}{(\beta-kZ)^{\alpha+s-3}}+...$$
$$+\left.\frac{(s-r-1)!}{(\alpha+s-1)(\alpha+s-2)...(\alpha+r)}\frac{1}{(\beta-kZ)^{\alpha+r}}\right]d\sigma,(30)$$

CaseII:

$$\Pr_{22}(x_s \ge t \mid \underline{x}) = \frac{C_{s-1}}{I_0\ C_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)$$
$$\times\int_0^{\upsilon}\frac{\eta(\sigma)}{\gamma_i\sigma^r}[\beta-T-\gamma_i(Z-T_r)]^{-(\alpha+r)}. \qquad (31)$$

3. BAYESIAN PREDICTION BOUNDS BASED ON GOS IN CASE OF RSS
In this section, assuming that the sample size n is a random

variable distributed as a Poisson distribution with pmf given by

$$p(n;\lambda) = \frac{e^{-\lambda}\lambda^n}{n!}, \qquad n = 0,1,2,...\ and\ \lambda > 0. \quad (32)$$

Replacing $v(n)$ in (9) by $p(n;\lambda)$ defined in (32), we get

$$q_p(x_s \mid \underline{x},n) = \frac{1}{e^{\lambda}-\sum_{n=0}^{s-1}\frac{\lambda^n}{n!}}\sum_{n=s}^{\infty}\frac{\lambda^n}{n!}f^{*}(x_s\mid\underline{x}). \quad (33)$$

3.1 When σ is known
Assuming that σ is known, substituting (21) and (22) into (33), the bayes predictive functions take the forms:

Case I:

$$q_{p11}(x_s \mid \underline{x},n) = \frac{\Gamma(\alpha+s)(\sigma-x_s)^{-1}}{\Gamma(\alpha+r)\Gamma(s-r)\left[e^{\lambda}-\sum_{n=0}^{s-1}\frac{\lambda^n}{n!}\right]}$$
$$\times\sum_{n=s}^{\infty}\frac{\lambda^n}{n!}\frac{k^{s-r}(\beta-kT_r)^{\alpha+r}A^{s-r-1}}{\left(\beta-k\ln(1-\frac{x_s}{\sigma})\right)^{\alpha+s}}, \qquad (34)$$

Case II:

$$q_{p12}(x_s \mid \underline{x},n) = \frac{(\alpha+r)}{e^{\lambda}-\sum_{n=0}^{s-1}\frac{\lambda^n}{n!}}(\sigma-x_s)^{-1}$$
$$\times\sum_{n=s}^{\infty}\sum_{i=r+1}^{s}\frac{\lambda^n}{n!}\frac{C_{s-1}}{C_{r-1}}a_i^{(r)}(s)\frac{(\beta-T)^{\alpha+r}}{(\beta-T+A\gamma_i)^{(\alpha+r+1)}}, \ (35)$$

and hence integrating (34) and (35) with respect to x_s (from some t to σ), the Bayesian prediction survival function is obtained:

CaseI:
$$\Pr_{p11}(x_s \ge t \mid \underline{x}) =$$
$$\frac{1}{e^{\lambda}-\sum_{n=0}^{s-1}\frac{\lambda^n}{n!}}\sum_{n=s}^{\infty}\left[\frac{\lambda^n}{n!}R_{r,s}\left[\frac{1}{(\alpha+s-1)}\frac{(kT_r-kZ)^{s-r-1}}{(\beta-kZ)^{\alpha+s-1}}\right.\right.$$
$$+\frac{s-r-1}{(\alpha+s-1)(\alpha+s-2)}\frac{(kT_r-kZ)^{s-r-2}}{(\beta-kZ)^{\alpha+s-2}}$$
$$+\frac{(s-r-1)(s-r-2)}{(\alpha+s-1)(\alpha+s-2)(\alpha+s-3)}\frac{(kT_r-kZ)^{s-r-3}}{(\beta-kZ)^{\alpha+s-3}}$$
$$+...+\left.\left.\frac{(s-r-1)!}{(\alpha+s-1)(\alpha+s-2)...(\alpha+r)}\frac{1}{(\beta-kZ)^{\alpha+r}}\right]\right],(36)$$

CaseII:

$$\Pr_{p12}(x_s \geq t \mid \underline{x}) = \frac{1}{e^{\lambda} - \sum\limits_{n=0}^{s-1} \frac{\lambda^n}{n!}}$$

$$\sum_{n=s}^{\infty} \sum_{i=r+1}^{s} \frac{\lambda^n C_{s-1}}{n! C_{r-1}} \frac{a_i^{(r)}(s)(\beta - T)^{\alpha+r}}{\gamma_i \left[\beta - T - \gamma_i (Z - T_r)\right]^{(\alpha+r)}}. \quad (37)$$

3.2 When θ and σ are Unknown parameters

When the two parameters having posterior pdf given by (16) and the sample size n has a Poisson distribution with pmf given by (32), The Bayes predictive density function of the GOS x_s is obtained by substituting (28) and (29) into (33) as follows:

CaseI:

$$q_{p21}(x_s \mid \underline{x}, n) = \frac{\Gamma(\alpha + s)}{I_0 \Gamma(\alpha + r)\Gamma(s - r)} \frac{k^{s-r}}{e^{\lambda} - \sum\limits_{n=0}^{s-1} \frac{\lambda^n}{n!}}$$

$$\sum_{n=s}^{\infty} \frac{\lambda^n}{n!} \int\limits_0^{\upsilon} \frac{\eta(\sigma)(\sigma - x_s)^{-1} A^{s-r-1}}{\sigma^r \left(\beta - k \ln(1 - \frac{x_s}{\sigma})\right)^{\alpha+s}} d\sigma, \quad (38)$$

Case II:

$$q_{p22}(x_s \mid \underline{x}, n) = \frac{(\alpha + r)}{e^{\lambda} - \sum\limits_{n=0}^{s-1} \frac{\lambda^n}{n!}} \sum_{n=s}^{\infty} \sum_{i=r+1}^{s} \frac{\lambda^n}{n!} \frac{a_i^{(r)}(s)C_{s-1}}{I_0 C_{r-1}}$$

$$\int\limits_0^{\upsilon} \frac{\eta(\sigma)(\sigma - x_s)^{-1}}{\sigma^r (\beta - T + A\gamma_i)^{(\alpha+r+1)}} d\sigma, \quad (39)$$

and the Bayesian prediction survival function of the future GOS x_s, $s=r+1, r+2, …, n$ in this case take the forms:

CaseI:
$$\Pr_{p21}(x_s \geq t \mid \underline{x}) =$$

$$\frac{\Gamma(\alpha + s)}{I_0 \Gamma(\alpha + r)\Gamma(s - r)} \frac{1}{e^{\lambda} - \sum\limits_{n=0}^{s-1} \frac{\lambda^n}{n!}}$$

$$\times \sum_{n=s}^{\infty} \frac{\lambda^n}{n!} \int\limits_0^{\upsilon} \left[\frac{\eta(\sigma)}{\sigma^r} \left[\frac{1}{(\alpha + s - 1)} \frac{(kT_r - kZ)^{s-r-1}}{(\beta - kZ)^{\alpha+s-1}} \right. \right.$$

$$+ \frac{s - r - 1}{(\alpha + s - 1)(\alpha + s - 2)} \frac{(kT_r - kZ)^{s-r-2}}{(\beta - kZ)^{\alpha+s-2}}$$

$$+ \frac{(s - r - 1)(s - r - 2)}{(\alpha + s - 1)(\alpha + s - 2)(\alpha + s - 3)} \frac{(kT_r - kZ)^{s-r-3}}{(\beta - kZ)^{\alpha+s-3}} + …$$

$$\left. \left. + \frac{(s - r - 1)!}{(\alpha + s - 1)(\alpha + s - 2)…(\alpha + r)} \frac{1}{(\beta - kZ)^{\alpha+r}} \right] \right] d\sigma, (40)$$

Case II:

$$\Pr_{p22}(x_s \geq t \mid \underline{x}) = \frac{1}{e^{\lambda} - \sum\limits_{n=0}^{s-1} \frac{\lambda^n}{n!}} \sum_{n=s}^{\infty} \sum_{i=r+1}^{s} \frac{a_i^{(r)}(s)C_{s-1}}{I_0 \, C_{r-1}} \frac{\lambda^n}{n!}$$

$$\times \int\limits_0^{\upsilon} \frac{\eta(\sigma)}{\gamma_i \sigma^r} \left[\beta - T - \gamma_i (Z - T_r)\right]^{-(\alpha+r)} d\sigma, \quad (41)$$

4. BAYESIAN PREDICTION BASED ON UPPER RVS IN CASE OF FSS

It is well known that the record value is a GOS when $k = 1$, and $m = -1$.

4.1 When σ is known

The Bayes predictive density function of the future upper RVs x_s, $s = r+1$, $r+2$, …, n, is given by putting $(k = 1, m = -1)$ in (21).

$$f_{RV1}^*(x_s \mid \underline{x}) = \frac{\Gamma(\alpha + s)}{\sigma \Gamma(s - r)\Gamma(\alpha + r)} (\beta - T_r)^{\alpha+r}$$

$$\frac{(1 - \frac{x_s}{\sigma})^{-1} A^{s-r-1}}{\left(\beta - \ln(1 - \frac{x_s}{\sigma})\right)^{\alpha+s}}, \quad (42)$$

the Bayesian prediction survival function of the upper RVs x_s can be obtained from (23) as follows:

$$\Pr_{RV1}(x_s \geq t \mid \underline{x}) =$$

$$\frac{\Gamma(\alpha + s)(\beta - T_r)^{\alpha+r}}{\Gamma(s - r)\Gamma(\alpha + r)} \left[\frac{1}{(\alpha + s - 1)} \frac{(T_r - Z)^{s-r-1}}{(\beta - Z)^{\alpha+s-1}} \right.$$

$$+ \frac{s - r - 1}{(\alpha + s - 1)(\alpha + s - 2)} \frac{(T_r - Z)^{s-r-2}}{(\beta - Z)^{\alpha+s-2}}$$

$$+ \frac{(s - r - 1)(s - r - 2)}{(\alpha + s - 1)(\alpha + s - 2)(\alpha + s - 3)} \frac{(T_r - Z)^{s-r-3}}{(\beta - Z)^{\alpha+s-3}} + …$$

$$\left. + \frac{(s - r - 1)!}{(\alpha + s - 1)(\alpha + s - 2)…(\alpha + r)} \frac{1}{(\beta - Z)^{\alpha+r}} \right]. (43)$$

The lower and the upper limits of the $100\tau\%$ Bayesian prediction interval for the future upper RV x_s are obtained by substituting (43) into (27).

4.2 When θ and σ are Unknown parameters

When the two parameters are unknown and having joint prior pdf given by (16), putting $(k = 1, m = -1)$ in (28) and (30) the Bayes predictive density function and the Bayesian prediction survival function of the upper RVs x_s are obtained as follows:

$$f_{RV2}^*(x_s \mid \underline{x}) = \frac{1}{I_1} \frac{\Gamma(\alpha + s)}{\Gamma(s - r)\Gamma(\alpha + r)}$$

$$\times \int\limits_0^{\upsilon} \frac{\eta(\sigma)(1 - \frac{x_s}{\sigma})^{-1} A^{s-r-1}}{\left(\beta - \ln(1 - \frac{x_s}{\sigma})\right)^{\alpha+s} \sigma^{r+1}} d\sigma, \quad (44)$$

$$\Pr_{RV2}(X_s \ge t \mid \underline{x}) = \frac{\Gamma(\alpha+s)}{I_1\,\Gamma(\alpha+r)\Gamma(s-r)}$$

$$\times \int_0^{\upsilon} \frac{\eta(\sigma)}{\sigma^r}\left[\frac{1}{(\alpha+s-1)}\frac{(T_r-Z)^{s-r-1}}{(\beta-Z)^{\alpha+s-1}}\right.$$

$$+\frac{s-r-1}{(\alpha+s-1)(\alpha+s-2)}\frac{(T_r-Z)^{s-r-2}}{(\beta-Z)^{\alpha+s-2}}$$

$$+\frac{(s-r-1)(s-r-2)}{(\alpha+s-1)(\alpha+s-2)(\alpha+s-3)}\frac{(T_r-Z)^{s-r-3}}{(\beta-Z)^{\alpha+s-3}}+\ldots$$

$$\left.+\frac{(s-r-1)!}{(\alpha+s-1)(\alpha+s-2)\ldots(\alpha+r)}\frac{1}{(\beta-Z)^{\alpha+r}}\right]d\sigma,\ (45)$$

where
$$I_1 = \int_0^{\upsilon}\frac{\eta(\sigma)}{\sigma^r}(\beta-T_r)^{-(r+\alpha)}d\sigma. \qquad (46)$$

5. BAYESIAN PREDICTION BASED ON UPPER RVS IN CASE OF RSS

It is clear that the equations obtained in Section 4 are all independent of the sample size *n*.

5.1 When σ is known

Assuming that the sample size is a random variable with the probability mass function $v(n)$, substituting by (42) into (9), the Bayes predictive density function is obtained as follows:

$$q_{RV1}(x_s\mid\underline{x},n) = \frac{1}{\Pr(n\ge s)}\sum_{n=s}^{\infty}v(n)f_{RV1}^*(x_s\mid\underline{x})$$

$$= \frac{f_{RV1}^*(x_s\mid\underline{x})}{\Pr(n\ge s)}\sum_{n=s}^{\infty}v(n)$$

$$= \frac{f_{RV1}^*(x_s\mid\underline{x})}{\Pr(n\ge s)}\Pr(n\ge s) = f_{RV1}^*(x_s\mid\underline{x}). \quad (47)$$

Similarly, the survival function of the upper RV x_s has the same formulae of equation (43) weather the sample size is fixed or random variable.

5.2 When θ and σ are Unknown parameters

Similarly, the Bayes predictive density function and the survival function of the upper RV x_s, can be obtained from equations (44) and (45) respectively.

6. BAYESIAN PREDICTION BASED ON PROGRESSIVELY TYPE-II CENSORED ORDER STATISTICS IN CASE OF FSS

Let $x_{1:\ell:n}, x_{2:\ell:n},\ldots,x_{\ell:\ell:n}$ denote a progressively Type-II censored sample, from GPD defined in (1) and (2), with censoring scheme $\underline{R}=(R_1,R_2,\ldots,R_\ell)$ then the conditional density function of the future progressively Type-II censored order statistic x_s can be obtained by putting $m_i = R_i$ for $1\le i\le \ell-1$, $\ell\in\{1,2,\ldots,n-1\}$ and $\gamma_\ell = k = R_\ell+1$ in (19), for more details see [10] and [11].

6.1 When σ is known

The Bayes predictive density function and the Bayesian prediction survival function in this case will take the forms:

$$f_{prog1}^*(x_s\mid\underline{x}) = \frac{C_{s-1}}{\sigma C_{r-1}}(\alpha+r)\left(1-\frac{x_s}{\sigma}\right)^{-1}$$

$$\sum_{i=r+1}^{s}\frac{a_i^{*(r)}(s)(\beta-T_R)^{\alpha+r}}{(\beta-T_R-A\delta_i)^{(\alpha+r+1)}}, \qquad (48)$$

and

$$\Pr_{prog1}(x_s\ge t\mid\underline{x}) = \prod_{k=r+1}^{s}\delta_k(\beta-T_R)^{\alpha+r}$$

$$\sum_{i=r+1}^{s}a_i^{*(r)}(s)\frac{1}{\delta_i}\left[\beta-T_R-\delta_i\ln\left[\frac{\sigma-t}{\sigma-x_r}\right]\right]^{-(\alpha+r)},\ (49)$$

where

$$T_R = \sum_{i=1}^{i=r-1}(R_i+1)\ln(1-\frac{x_i}{\sigma})+\delta_r\ln(1-\frac{x_r}{\sigma}) \qquad (50)$$

$$\delta_i = \sum_{j=i}^{\ell}(R_j+1)=n-\sum_{j=1}^{i-1}(R_j+1), \qquad (51)$$

and $\quad a_i^{*(r)}(s) = \prod_{\substack{j=r+1\\j\ne i}}^{s}\frac{1}{\delta_j-\delta_i},\ r+1\le i\le s\cdot$ $\qquad$ (52)

6.2 When θ and σ are Unknown parameters

The Bayes predictive density function and the Bayesian prediction survival function of the future progressively type-II censored x_s, s = r+1, r+2, …, n will take the forms

$$f_{pro2}^*(x_s\mid\underline{x}) = \frac{C_{s-1}}{I_3 C_{r-1}}\int_0^{\upsilon}\frac{\eta(\sigma;\underline{x})}{\sigma^r(\sigma-x_s)}$$

$$\sum_{i=r+1}^{s}\frac{a_i^{*(r)}(s)(\alpha+r)}{(\beta-T_R-A\delta_i)^{(\alpha+r+1)}}d\sigma, \qquad (53)$$

and

$$\Pr_{pro2}(x_s\ge t\mid\underline{x}) = \frac{C_{s-1}}{I_3\,C_{r-1}}$$

$$\sum_{i=r+1}^{s}\int_0^{\upsilon}\frac{a_i^{*(r)}(s)\eta(\sigma;\underline{x})}{\sigma^r\delta_i[\beta-T_R-\delta_i(Z-T'')]^{(\alpha+r)}}d\sigma. \quad (54)$$

Where $\quad I_3 = \int_0^{\upsilon}\frac{\eta(\sigma)}{\sigma^r}(\beta-T_R)^{-(r+\alpha)}d\sigma. \qquad (55)$

7. BAYESIAN PREDICTION BASED ON PROGRESSIVELY TYPE-II CENSORED ORDER STATISTICS IN CASE OF RSS

7.1 When σ is known

The Bayes predictive density function and the Bayesian prediction survival function will take the forms

$$q_{pro1P}(x_s \mid \underline{x},n) = \frac{(\alpha + r)}{e^\lambda - \sum_{n=0}^{s-1} \frac{\lambda^n}{n!}} (\sigma - x_s)^{-1}$$

$$\times \sum_{n=s}^{\infty} \sum_{i=r+1}^{s} \frac{\lambda^n}{n!} \frac{C_{s-1}}{C_{r-1}} a_i^{*(r)}(s) \frac{(\beta - T_R)^{\alpha+r}}{(\beta - T_R - A\delta_i)^{(\alpha+r+1)}}, \quad (56)$$

and

$$\Pr_{pro1P}(x_s \geq t \mid \underline{x}) = \frac{1}{e^\lambda - \sum_{n=0}^{s-1} \frac{\lambda^n}{n!}}$$

$$\sum_{n=s}^{\infty} \sum_{i=r+1}^{s} \frac{\lambda^n C_{s-1}}{n! C_{r-1}} \frac{a_i^{*(r)}(s)(\beta - T_R)^{\alpha+r}}{\delta_i [\beta - T_R - \delta_i(Z - T'')]^{(\alpha+r)}}. \quad (57)$$

7.2 When θ and σ are Unknown parameters

The Bayes predictive density function and the Bayesian prediction survival function of the future progressively type-II censored OS x_s, $s = r+1, r+2, \ldots, n$ in this case take the forms

$$q_{pro2P}(x_s \mid \underline{x},n) = \frac{(\alpha + r)}{e^\lambda - \sum_{n=0}^{s-1} \frac{\lambda^n}{n!}}$$

$$\sum_{n=s}^{\infty} \sum_{i=r+1}^{s} \frac{\lambda^n}{n!} \frac{a_i^{*(r)}(s)C_{s-1}}{I_R C_{r-1}} \int_0^{\upsilon} \frac{\eta(\sigma; \underline{x})(\sigma - x_s)^{-1} \sigma^{-r}}{(\beta - T_R - A\delta_i)^{(\alpha+r+1)}} d\sigma,$$

$$\quad (58)$$

$$\Pr_{pro2P}(x_s \geq t \mid \underline{x}) = \frac{1}{e^\lambda - \sum_{n=0}^{s-1} \frac{\lambda^n}{n!}}$$

and $\quad \times \sum_{n=s}^{\infty} \sum_{i=r+1}^{s} \frac{a_i^{*(r)}(s)C_{s-1}}{I_3 C_{r-1}} \frac{\lambda^n}{n!}$

$$\int_0^{\upsilon} \frac{\eta(\sigma; \underline{x})}{\gamma_i \sigma^r} [\beta - T_R - \delta_i(Z - T'')]^{-(\alpha+r)} d\sigma. \quad (59)$$

8. 10. Numerical analysis

In this section, Bayesian prediction bounds for the sth observation in samples of progressively type-II censored and upper record sample with fixed and random sample sizes are obtained.

8.1 Unknown one parameter

By taking the parameter σ is known ($\sigma = 0.1$ and 0.5), one can find the 90% and 95% BPI for the future observation $x_{s:n}$ using Monte Carlo simulation study according to the following steps:

1. For arbitrary given values of the prior parameters $(\alpha, \beta) = (1, 0.1)$, we generate θ using (12).

2. According to an algorithm introduced by [12], two progressively type –II censored samples with given censoring scheme (1, 0, 2, 1, 0, 3, 0, 2, 0, 1) from GPD with α, β and σ as in step (1) and (2) are generated {0.0003, 0.0006, 0.0007, 0.0025, 0.0036, 0.004, 0.0085, 0.0117, 0.0138, 0.0295} and

{0.00017, 0.00019, 0.0002, 0.0005, 0.0006, 0.0008, 0.0013, 0.0015, 0.0021, 0.0021}. Taking the number of observed failures is r = 7 and 9, the 90% and 95% BPI for the future $x_{s:n}$, s=8, 9 and 10 by evaluating (49) with (27) are constructed, these are given in Table 1.

3. Using generated θ obtained as in step (1), two samples of upper records from GPD given in (1) with $\sigma = 0.1$ and 0.5 and fixed sample size $n = 10$ are generated, {0.0204, 0.0239, 0.0329, 0.1644, 0.1882, 0.1965, 0.2698, 0.2870, 0.3026, 0.3421} and {0.0332, 0.0462, 0.0562, 0.0618, 0.0629, 0.0689, 0.0690, 0.0712, 0.0715, 0.0745}. Taking the number of observed upper records is r = 7 and 8, 90% and 95% BPI for the future observation $x_{s:n}$, s=8, 9 and 10 are constructed by evaluating (43) with (27), noting that the results doesn't depend on the sample size n. Table 3 represents these results.

4. Repeat step (4) with initial sample size $n = 7$ from Poisson distribution with $\lambda = 0.9$, two progressively type II censoring samples are generated. The constructed 90% and 95% BPI for the future $x_{s:n}$, s=8, 9 and 10 by evaluating (55) with (27) are in Table 2.

8.2 Unknown two parameters

Assuming that the parameters θ and σ are unknown, the 90% and 95% BPI for the future observation $x_{s:n}$ using the Monte Carlo simulation study are computed according to the following steps:

1. 90% and 95% BPI for the future upper RVs observation $x_{s:n}$, s=8, 9 and 10 are obtained from two different samples in Table 4.

2. Two progressively type-II censored samples with given censoring scheme (1, 0, 2, 1, 0, 3, 0, 2, 0, 1) from GPD are generated. Taking the number of observed failures is r = 7 and 8, the 90% and 95% BPI for the future $x_{s:n}$, s=8, 9 and 10 are constructed, these are given in Table 5.

9. CONCLUDING REMARKS

Based on the one-sample scheme, Bayesian prediction interval is presented in this paper based on a sample of GOS from the GPD. Results based on progressively type II censored and upper RVs are obtained. From the results we may observe

1. From the Tables one can observe that, the BPI increase when the value of the parameter σ increase, and that is because the observation values from the GPD are bounded by σ, this leads to when σ has a small value, the data will be in a small range and thus minimize the length of the BPI.

2. Tables show that the lengths of BPI are sensitive to the size of known observations, i.e, increasing the available information leads to shrinking the intervals. Noted that when confidence level increases the length of intervals also increases.

3. The width of the BPI increases by increasing s, i.e, the width of the BPI for a specific future observation is wider than the previous one when other parametric values are fixed.

4. It has been observed from the Tables 4 and 5 that, all the properties of the BPI in case of unknown two

parameters are similar to the case when one parameter is known.

5. Different values of both of the prior parameters and

sample size rather than those appeared in the Tables have been considered but did not change the previous conclusion.

Table 1: 90% and 95% BPI for future progressive type II censoring observations x_s in case of *FSS*

r	*s*	σ =0.5		σ =0.1	
		90% BPI (Width)	95% BPI (Width)	90% BPI (Width)	95% BPI (Width)
7	**8**	0.0086, 0.0189 (0.0103)	0.0086, 0.0218 (0.0132)	0.0013, 0.0032 (0.0019)	0.0023, 0.0053 (0.003)
	9	0.0099, 0.0344 (0.0245)	0.0094, 0.0408 (0.0313)	0.0016, 0.0061 (0.0045)	0.0015, 0.0073 (0.0058)
	10	0.0126, 0.0552 (0.0425)	0.0115, 0.0653 (0.0537)	0.0021, 0.0100 (0.0079)	0.0019, 0.0119 (0.0100)
8	**9**	0.0120, 0.0320 (0.0199)	0.0119, 0.0376 (0.0256)	0.0016, 0.0051 (0.0034)	0.0016, 0.0060 (0.0044)
	10	0.0142, 0.0529 (0.0386)	0.0134, 0.0621 (0.0487)	0.0020, 0.0087 (0.0067)	0.0018, 0.0103 (0.0084)

Table 2: 90% and 95% BPI for progressive type-II censoring observation x_s in case of Poisson *RSS*.

r	*s*	σ =0.5		σ =0.1	
		90% BPI (Width)	95% BPI (Width)	90% BPI (Width)	95% BPI (Width)
7	**8**	0.3241, 0.3378 (0.0136)	0.3240, 0.3416 (0.0175)	0.0723, 0.0745 (0.0021)	0.0723, 0.0751 (0.0027)
	9	0.3384, 0.3548 (0.0164)	0.3374, 0.3597 (0.0222)	0.0746, 0.0771 (0.0025)	0.0744, 0.0779 (0.0034)
	10	0.3548, 0.3761 (0.0212)	0.3554, 0.3819 (0.0265)	0.0772, 0.0805 (0.0033)	0.0772, 0.0814 (0.0041)
9	**10**	0.2212, 0.2318 (0.0105)	0.2212, 0.2347 (0.0135)	0.0618, 0.0637 (0.0019)	0.0618, 0.0642 (0.0024)

Table 3: 90% and 95% BPI for future upper record observations

r	*s*	σ =0.5		σ =0.1	
		90% BPI (Width)	95% BPI (Width)	90% BPI (Width)	95% BPI (Width)
7	**8**	0.2711, 0.3454 (0.0742)	0.2705, 0.3622 (0.0917)	0.0692, 0.0826 (0.0133)	0.0691, 0.0852 (0.0161)
	9	0.2783, 0.3808 (0.1025)	0.2756, 0.3983 (0.1226)	0.0706, 0.0880 (0.0174)	0.0701, 0.0905 (0.0203)
	10	0.2883, 0.4064 (0.1181)	0.2838, 0.4232 (0.1393)	0.0725, 0.0916 (0.0190)	0.0717, 0.0937 (0.0219)
8	**9**	0.2882, 0.3538 (0.0656)	0.2876, 0.3686 (0.0810)	0.0714, 0.0830 (0.0116)	0.0713, 0.0854 (0.0141)
	10	0.2946, 0.3855 (0.0908)	0.2922, 0.4008 (0.1086)	0.0726, 0.0880 (0.0153)	0.0722, 0.0902 (0.0180)

Table 4: 90% and 95% BPI for future upper record observations x_s when unknown *θ* and *σ*.

r	*s*	*θ* = 0.0155 and *σ* = 0.225		*θ* = 0.0155 and *σ* = 0.586	
		90% BPI (Width)	95% BPI (Width)	90% BPI (Width)	95% BPI (Width)
7	**8**	0.0177, 0.0281 (0.0104)	0.0176, 0.0313 (0.0136)	0.1074, 0.1429 (0.0354)	0.1071, 0.1522 (0.0450)
	9	0.0185, 0.0351 (0.0165)	0.0182, 0.0394 (0.0212)	0.1106, 0.1646 (0.0539)	0.1047, 0.1763 (0.0716)
	10	0.0196, 0.0415 (0.0218)	0.0191, 0.0469 (0.0278)	0.1154, 0.1844 (0.0690)	0.1133, 0.1986 (0.0853)
8	**9**	0.0185, 0.0275 (0.0090)	0.0184, 0.0305 (0.0117)	0.1167, 0.1511 (0.0343)	0.1164, 0.1600 (0.0435)
	10	0.0192, 0.0335 (0.0142)	0.0189, 0.0372 (0.0182)	0.1132, 0.1720 (0.0588)	0.1140, 0.1832 (0.0692)

Table 5: 90% and 95% BPI for future progressive type II censoring observations x_s when unknown *θ* and *σ*.

r	*s*	*θ* = 0.0155 and *σ* = 0.726		*θ* = 0.0155 and *σ* = 0.184	
		90% BPI (Width)	95% BPI (Width)	90% BPI (Width)	95% BPI (Width)
7	**8**	0.0085, 0.0155 (0.007)	0.0085, 0.0174 (0.0089)	0.0020, 0.0047 (0.0026)	0.0020, 0.0054 (0.0034)
	9	0.0095, 0.0261 (0.0166)	0.0091, 0.0300 (0.0208)	0.0023, 0.0088 (0.0064)	0.0022, 0.0105 (0.0082)
	10	0.0115, 0.0403 (0.0288)	0.0107, 0.0740 (0.0633)	0.0030, 0.014 (0.0114)	0.0027, 0.0173 (0.0145)
8	**9**	0.0107, 0.0245 (0.0137)	0.0106, 0.0281 (0.0175)	0.0021, 0.0067 (0.0046)	0.0020, 0.0080 (0.0059)
	10	0.0123, 0.0389 (0.0266)	0.0112, 0.0449 (0.0331)	0.0026, 0.0117 (0.0091)	0.0024, 0.0141 (0.0116)

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