Discrete Wavelets Associated with DUNKL Operator on Real Line

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ABSTRACT
Using convolution theory of the dunkl transform, discrete dunkl wavelet transform is defined. A reconstruction formula for the discrete dunkl wavelet is obtained. Important properties of the discrete dunkl wavelet are presented. Frames and Riesz basis involving dunkl wavelets are studied.

Keywords
Dunkl transform, wavelet transform, Dunkl operator.

AMS Subject Classifications: 42C40, 65T60, 44A35,65R10

1. INTRODUCTION
The wavelet transform of a function $f \in L^2(R)$ with respect to the wavelet

$\psi \in L^2(R)$ is $\psi \in L^2(R)$ defined by

$$\left(W_\psi f\right)(b,a) = \int_{-\infty}^{\infty} f(t)\overline{\psi_{b,a}(t)}\,dt,\ a,b \in R, a > 0, \qquad (1)$$

where $\psi_{b,a}(t) = a^{-1/2}\psi\left(\frac{t-b}{a}\right).$ (2)

In terms of translation τb defined by

$$\tau_b\psi(t) = \psi(t-b),\quad \mathrm{b} \in \mathrm{R}$$

and dilation Da defined by

$$D_a\psi(t) = a^{-1/2}\,\psi\left(\frac{t}{a}\right),\quad \mathrm{a} > 0.$$

we can write $\psi_{b,a}(t) = \tau_b\ D_a\ \psi(t)$ (3)

From (1) and (3) it is clear that wavelet transform of the function f on R is an integral transform for which the kernel is the dilated translate of ψ.

We can also express (1) as the convolution:

$$\left(W_\psi f\right)(\mathrm{b,a}) = \left(f * \mathrm{g}_{\mathrm{o,a}}\right)(\mathrm{b}) \qquad (4)$$

where $g(t) = \overline{\psi(-t)}.$

For $\alpha \geq -1/2$ and $\lambda \in C$, the initial value problem

$$\Lambda_\alpha(f)(x) = \lambda f(x),\quad f(0) = 1,\ x \in R, \qquad (5)$$

where

$$\Lambda_\alpha(f)(x) = \frac{d}{dx}f(x) + \frac{2\alpha+1}{x}\left(\frac{f(x)-f(-x)}{2}\right).$$

called Dunkl Operator has a unique solution $E_\alpha(\lambda x)$, called Dunkl kernel and given by

$$E_\alpha(\lambda x) = j_\alpha(i\lambda x) + \frac{\lambda x}{2(\alpha+1)}\, j_{\alpha+1}(i\lambda x),\quad x \in R, \qquad (6)$$

where j_α is the normalized Bessel function of the first kind and order α defined by

$$j_\alpha(z) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n}}{n!\Gamma(n+\alpha+1)},\ z \in C. \qquad (7)$$

We can write for $x \in R$ and $\lambda \in C$

$$E_\alpha(-i\lambda x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}\int_{-1}^{1}\left(1-t^2\right)^{\alpha-1/2}(1-t)e^{i\lambda xt}dt \qquad (8)$$

Let $\alpha > -1/2$ be a fixed number and μ_α be the weighted Lebesgue measure on R, given by

$$d\mu_\alpha(x) := \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1}|x|^{2\alpha+1}dx. \qquad (9)$$

For every $1 \leq p \leq \infty$, we denote by $L_{p,\alpha} = L_p(d\mu_\alpha)$, the space of complex-valued functions f, measurable on R such that

$$\|f\|_{p,\alpha} = \left(\int_R |f(x)|^p \, d\mu_\alpha(x)\right)^{1/p} < \infty \quad if \quad p \in [1, \infty)$$

(10)

The Dunkl kernel gives rise to an integral transform, called Dunkl transform on R, which was introduced and studied in [5].

The Dunkl transform F_α of a function $f \in L_{1,\alpha}(R),$ is given by

$$F_\alpha f(\lambda) = \hat{f}(\lambda) = \int_R E_\alpha(-i\lambda x) f(x) \, d\mu_\alpha(x) \quad ; \lambda \in R$$

(11)

An inversion formula for this transform is given by

$$F_\alpha^{-1}\left(\hat{f}(\lambda)\right) = \left(\hat{f}(\lambda)\right)^{\vee} = f(x) = \int_R E_\alpha(i\lambda x) \hat{f}(\lambda) \, d\mu_\alpha(\lambda)$$

(12)

An Parseval formula for this transform is given by

$$\int_{-\infty}^{\infty} f(x) g(x) dx = \int_{-\infty}^{\infty} \hat{f}(\lambda) \hat{g}(\lambda) \, d\mu_\alpha(\lambda)$$

(13)

2. DUNKL TRANSLATION AND CONVOLUTION

In this section following [5] we define Dunkl translation and associated convolution and discuss their important properties.

To define Dunkl convolution $*_\alpha$ we need to introduce a special type of translation, called Dunkl translation. For this purpose we need the basic function

$$W_\alpha(x, y, z) = \left(1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x}\right) \Delta_\alpha(x, y, z)$$

(14)

Where

$$\sigma_{x,y,z} = \begin{cases} \dfrac{x^2 + y^2 + z^2}{2xy}, if & x, y \in R \setminus 0 \\ 0 & otherwise \end{cases}.$$

And Δ_α is the Bessel kernel given by

$$\Delta_\alpha(x, y, z) = \begin{cases} d_\alpha \dfrac{\left(\left[(|x|+|y|)^2 - z^2\right]\left[z^2 - (|x|-|y|)^2\right]\right)^{\alpha - 1/2}}{|xyz|^{2\alpha}}, & if \quad |z| \in A_{xy}, \\ 0 & otherwise, \end{cases}$$

where

$$d_\alpha = \left(\Gamma(\alpha+1)\right)^2 / \left(2^{\alpha-1}\sqrt{\pi}\,\Gamma(\alpha + \frac{1}{2})\right),$$

and

$$A_{x,y} = \left(\big||x| - |y|\big|, |x| + |y|\right).$$

Also

$$\int_R |W_\alpha(x, y, z)| \, d\mu_\alpha(z) \le 4.$$

(15)

The Dunkl translation $\tau_x f(y)$ of $f \in L_{p,\alpha}(R)$, $1 \le p < \infty$ is defined as follows

$$\tau_x f(y) = f(x, y) = \int_R f(z) W_\alpha(x, y, z) d\mu_\alpha(z).$$

(16)

Lemma 1. For all $x \in R$ **and** $f \in L_{p,\alpha}(R)$, $p \ge 1$

$$\|\tau_x f\|_{p,\alpha} \le 4\|f\|_{p,\alpha}. \quad (17)$$

$$(\tau_x f)^{\wedge}(\lambda) = E_\alpha(i\lambda x) f^{\wedge}(\lambda). \quad (18)$$

Let $p, q, r \in [1, \infty[$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. Then Dunkl convolution of $f \in L_{p,\alpha}(R)$ and $g \in L_{q,\alpha}(R)$ is defined by

$$f *_\alpha g(x) = \int_R \tau_x f(-y) g(y) d\mu_\alpha(y).$$

(19)

Lemma 2. Let $p, q, r \in [1, \infty[$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, $f \in L_{p,\alpha}(R)$ and $g \in L_{q,\alpha}(R)$. Then convolution $f *_\alpha g(x)$ satisfies the following norm inequality

(i) $$\|f *_\alpha g\|_{r,\alpha} \le 4\|f\|_{p,\alpha}\|g\|_{q,\alpha}, \quad (20)$$

Moreover for all $f \in L_{1,\alpha}(R)$ and $g \in L_{2,\alpha}(R)$, we have

(ii) $$(f *_\alpha g)^{\wedge}(\lambda) = f^{\wedge}(\lambda) g^{\wedge}(\lambda) \quad (21)$$

3. DUNKL WAVELET TRANSFORM

For a function $\psi \in L_{p,\alpha}(R)$, define the dilation Da is given by

$$D_a\psi(x) = \psi(ax), \quad a \in R. \quad (22)$$

Using the Dunkl translation and the above dilation, the Dunkl wavelet $\psi_{b,a}(x)$ is defined as follows

$$\psi_{b,a}(x) = \tau_b D_a \psi(x) = \tau_b \psi(ax)$$

$$= \tau_b \psi(ax)$$

$$= \int_R \psi(az)\, W_\alpha(b,x,z)\, d\mu_\alpha(z) \quad , \quad b \in R.$$

(23)

The integral is convergent by virtue of (18). Now, using the wavelet $\psi_{b,a}$ the Dunkl wavelet transform (DWT) of $f \in L_{q,\alpha}$, $\frac{1}{p} + \frac{1}{q} = 1,$ is defined as follows:

$$(D_\psi f)\ (b,a) = \langle f(x),\ \psi_{b,a}(x) \rangle$$

$$= \int_{-\infty}^{\infty} f(x)\, \overline{\psi_{b,a}(x)}\, d\mu_\alpha$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) \overline{\psi(az)}\, W_\alpha(b,x,z) d\mu_\alpha(z) d\mu_\alpha(x)$$

(24)

Provided the integral is convergent. Since by (17) and (18) $\psi_{b,a}(x) \in L_{p,\alpha}(R)$ whenever $\psi \in L_{p,\alpha}(R)$. By virtue of Lemma, the integral is convergent for $f \in L_{q,\alpha}$, $\frac{1}{p} + \frac{1}{q} = 1$.

4. THE DISCRETE DUNKL WAVELET TRANSFORM

In the continuous Dunkl wavelet transform (25), if we discretize only the dilation parameter a by assuming that aj = 2-j, j ∈ Z, and the translation parameter b is allowed to vary over all of R, then the transform so obtained is called semi-discrete Dunkl wavelet transform. If we discretize the translation parameter b also by restricting it to the discrete set of points:

$$b_{j,k} = \frac{k}{2^j} b_0,\ j \in \mathbf{Z}, k \in \mathbf{N}_0,$$

where b0 > 0 is a fixed constant, we get the discrete Dunkl wavelet transform. We shall use the notation:

$$\psi_{b_0;j,k}(t) = \psi_{b_{j,k};a_j}(t) = \psi(2^{-j}t,\ 2^{j}k\ b_0).$$

(25)

Then the discrete Dunkl wavelet transform of any $f \in L_{2,\alpha}(R)$ can be expressed as

$$(D_\psi\ f)\ (b_{j,k,a_j}) = < f, \psi_{b_0;j,k} >,\ j \in \mathbf{Z}, k \in \mathbf{N}_0.$$

(26)

The stability condition for this reconstruction takes the form

$$A\|f\|_2^2 \ \le \sum_{K \in N_0} |< f, \psi_{b_0;j,k} >|^2 \ \le \ B\|f\|_2^2,\ f \in L_{2,\alpha}(R)$$

(27)

In what follows we assume that $\psi \in L_{1,\alpha} \cap L_{2,\alpha}$ satisfies, the so called, "stability condition"

$$A \le \sum_{j=-\infty}^{\infty} |\hat{\psi}(2^{-j}\lambda)|^2 \le B \text{ a.e.}$$

(28)

for certain positive constants A and B, $0 < A \le B < \infty$.

The function $\psi \in L_{1,\alpha} \cap L_{2,\alpha}$ satisfying (29) is called dyadic wavelet. Using the definition (25) we define the semi-discrete Dunkl wavelet transform of any $f \in L_{1,\alpha} \cap L_{2,\alpha}$ by

$$(D_j^\psi\ f)\ (b) = (D_\psi f)\ (b, 2^{-j})$$

$$= \int_{-\infty}^{\infty} f(t) \overline{\psi_{b,2^{-j}}(t)}\ d\mu_\alpha(t)$$

(29)

$$= \int_{-\infty}^{\infty} f(t) \overline{\psi(2^{-j}t, b)}\ d\mu_\alpha(t) \quad (30)$$

$$= (f * \overline{\psi}_j)_{j \in Z},$$

(31)

where $\psi_j(z) = \psi(2^j z),\ j \in Z$.

Theorem1. Assume that the semi-discrete Dunkl wavelet transform of any $f \in L_{1,\alpha} \cap L_{2,\alpha}$ is defined by (32). Let us define another wavelet □ * by means of its Dunkl transform,

$$\hat{\psi}^*(\lambda) = \frac{\hat{\psi}(\lambda)}{\sum_{k=-\infty}^{\infty} |\hat{\psi}(2^{-k}\lambda)|^2}.$$

Then

$$f(t) = \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} (D_j^\psi f)\ (b) \left(\hat{\psi}^*\ (2^{-j}\lambda)\ E_\alpha(i\lambda t) \right)^\vee (b) d\mu_\alpha(b).$$

(32)

Proof. For any $f \in L_{1,\alpha} \cap L_{2,\alpha}$ we have

$$\sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} \left(D_j^\psi f \right)(b) \left(\hat{\psi}^*(2^{-j}\lambda)\ E_\alpha(i\lambda t) \right)^\nu (b)\ d\mu\ (b)$$

$$= \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} \left(D_j^\psi f \right)^\wedge(\lambda)\ \hat{\psi}^*(2^{-j}\lambda)\ E_\alpha(i\lambda t)\ d\mu_\alpha(\lambda)$$

$$= \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\lambda)\ \overline{\hat{\psi}(2^{-j}\lambda)}\ \hat{\psi}^*\ (2^{-j}\lambda)\ E_\alpha(i\lambda t) d\ \mu_\alpha(\lambda)$$

$$= \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\lambda)\ \overline{\hat{\psi}(2^{-j}\lambda)}\ \frac{\hat{\psi}(2^{-j}\lambda)}{\sum_{k=-\infty}^{\infty}\left|\hat{\psi}(2^{-k}2^{-j}\lambda)\right|^2} E_\alpha(i\lambda t) d\ \mu_\alpha\ (\lambda)$$

$$= \int_{-\infty}^{\infty} \hat{f}(\lambda)\ E_\lambda(i\lambda t)\ d\ \mu_\alpha\ (\lambda)$$

$= f(t)$.

The above theorem leads to the following definition of dyadic dual.

Definition1. A function $\tilde{\psi} \in L_{2,\alpha}(R)$ is called a dyadic dual of a dyadic wavelet □ □ , if every f $\in L_{2,\alpha}(\mathrm{R})$ can be expressed as

$$f(t) = \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} \left(D_j^{\psi} f\right)(b)\left(\tilde{\psi}(2^{-j}\lambda) E_\alpha(i\lambda t)\right)^{\vee}(b) d\mu(b)$$

. (33)

Theorem2. Assume that the discrete Dunkl wavelet transform of any f $\in L_{2,\alpha}(\mathrm{R})$ is defined by (27) and stability condition (28) holds. Let T be a linear operator on $L_{2,\alpha}(\mathrm{R})$ defined by

$$Tf = \sum_{\substack{j \in Z \\ K \in N_0}} < f, \psi_{b_0;j,k} > \psi_{b_0;j,k} \quad (34)$$

Then

$$f = \Sigma < f, \psi_{b_0;j,k} > \psi_{b_0}^{j,k}, \quad (35)$$

where $\psi_{b_0}^{j,k} = T^{-1}\psi_{b_0;j,k}; j \in Z$.

Proof. From the condition (28), it follows that the operator defined by (35) is a one-one bounded linear operator. Set g = T f, $f \in L_{2,\alpha}(\mathrm{R})$

Then, we have

$$< Tf, f > = \sum_{\substack{j \in Z \\ K \in N_0}} |< f, \psi_{b_0;j,k} >|^2 .$$

Therefore,

$$A \| T^{-1}g \|_2^2 = A \| f \|_2^2 \leq$$
$$< Tf, f > = < g, T^{-1}g > \leq \| g \|_2 \| T^{-1}g \|_2 ,$$

so that

$$\| T^{-1}g \| \leq \frac{1}{A} \| g \|_2 .$$

Hence every f ∈ $L_{2,\alpha}$(R) an be reconstructed from its discrete Dunkl wavelet transform given by (27). Thus

$$f = T^{-1}T f = \sum_{\substack{j \in Z \\ k \in N_0}} < f, \psi_{b_0;j,k} > T^{-1}\psi_{b_0;j,k} . \quad (36)$$

Finally, set

$$\psi_{b_0}^{j,k} = T^{-1}\psi_{b_0;j,k};\ j \in \mathbf{Z},\ k \in \mathbf{N}_0 .$$

Then, the reconstruction (37) can be expressed as follows:

$$f = \sum_{\substack{j \in Z \\ k \in N_0}} < f, \psi_{b_0;j,k} > \psi_{b_0}^{j,k} .$$

5. FRAMES AND RIESZ BASIS IN $L_{2,\alpha}(\mathrm{R})$

In this section, using $\psi_{b_0;j,k}$ a frame is defined and Riesz basis of $f \in L_{2,\alpha}(\mathrm{R})$ is studied.

Definition 2. A function $\psi \in L^2(\mu)$ is said to generate a frame $\{\psi_{bo;j,k}\}$ *of* $f \in L_{2,\alpha}(\mathrm{R})$ with sampling rate b0 if (28) holds for some positive constants A and B. If A = B, then the frame is called a tight frame.

Definition3. A function $\psi \in L_{2,\alpha}(\mathrm{R})$ is said to generate a Riesz basis $\{\psi_{b_0;j,k}\}$ with sampling rate b0 if the following two properties are satisfied.

(i) The linear span $\left\langle \psi_{b_0;j,k} : j \in N_0 \right\rangle$ is dense in $\psi \in L_{2,\alpha}(\mathrm{R})$ (37)

(ii) There exist positive constants A and B, with $0 < A \leq B < \infty$ such that

$$A\left\|\{c_{j,k}\}\right\|_{\ell^2}^2 \leq \left\|\sum_{\substack{j \in N_0 \\ k \in N_0}} c_{j,k}\ \psi_{b_0;j,k}\right\|_2^2 \leq B\left\|\{c_{j,k}\}\right\|_{\ell^2}^2 \quad (38)$$

for all $\{c_{j,k}\} \in \ell^2\left(N_0^2\right)$. Here A and B are called the Riesz bounds of $\{\psi_{b0;j,k}\}$.

Theorem3. Let $\psi \in L_{2,\alpha}(\mathrm{R})$ and b0 >0, then the following two statements are equivalent.

(i) $\{\psi_{bo;j,k}\}$ is a Riesz basis of $\in L_{2,\alpha}(\mathrm{R})$;

(ii) $\{\psi_{b_0;j,k}\}$ is a frame of $L_{2,\alpha}(\mathrm{R})$ and is also an l^2 linearly independent family in the sense that if

$$\Sigma\ \psi_{b_0;j,k} c_{j,k} = 0 \text{ and } \{c_{j,k}\} \in \ell^2, \text{ then } c_{j,k} = 0.$$

Furthermore, the Riesz bounds and frame bounds agree.

Proof. It follows from (39) that any Riesz basis is l^2- linearly independent. Let $\{\psi_{bo;j,k}\}$ be a Riesz basis with Reisz bounds A and B, and consider the "Matrix operator"

$$M = [\gamma_{\ell,m,j,k}]_{(\ell,m),(j,k)\in N_0 \times N_0},$$

where the entries are defined by

$$\gamma_{\ell,m,j,k} = \langle \psi_{b_0;\ell,m}, \psi_{b_0;j,k} \rangle. \quad (39)$$

Then from (39), we have

$$A\|\{c_{j,k}\}\|^2_{\ell^2} \le \sum_{\ell,m,j,k} c_{\ell,m}\ \gamma_{\ell,m;\ell,k} c_{j,k} \le B\|\{c_{j,k}\}\|^2_{\ell^2}$$

so that M is positive definite. We denote the inverse of M by

$$M^{-1} = [\mu_{\ell,m,j,k}]_{(\ell,m),(j,k)\in N_0^2}, \quad (40)$$

which means that both

$$\sum_{\gamma,s} \mu_{\ell,m,r,s}\gamma_{r,s,j,k} = \delta_{\ell,j}\ \delta_{m,k}\quad \ell,m,j,k \in N_0 \quad (41)$$

and

$$B^{-1}\|\{c_{j,k}\}\|^2_{\ell^2} \le \sum_{\ell,m,j,k} c_{\ell,m}\ \mu_{\ell,m}\ \mu_{\ell,m,j,k}\bar{c}_{j,k} \le A^{-1}\|\{c_{j,k}\}\|^2_{\ell^2} \quad (42)$$

are satisfied. This allows us to introduce

$$\psi^{\ell,m}(x) = \sum_{j,k} \mu_{\ell,m,j,k}\psi_{bo;j,k}(x). \quad (43)$$

Clearly, $\psi^{\ell,m} \in L_{2,\alpha}(\mathrm{R})$ and it follows from (40) and (42) that

$$\langle \psi^{\ell,m};\psi_{b_0;j,k} \rangle = \delta_{\ell,j}\ \delta_{m,k}\quad \ell,m,j,k \in N_0$$

which means that $\{\psi^{\ell,m}\}$ is the basis of $L_{2,\alpha}(\mathrm{R})$ which is dual to $\{\psi_{b_0;j,k}\}$.

Furthermore, from (42) and (44); we conclude that

$$\langle \psi^{\ell,m}, \psi^{j,k} \rangle = \mu_{\ell,m,j,k}$$

and the Riesz bounds of $\{\psi^{\ell,m}\}$ are B^{-1} and A^{-1}

In particular, for any $f \in L_{2,\alpha}(\mathrm{R})$ we may write

$$f(x) = \sum_{j,k} \langle f, \psi_{b_0;j,k} \rangle \psi^{j,k}(x)$$

and

$$B^{-1}\sum_{j,k}|\langle f,\psi_{b_0;j,k}\rangle|^2 \le \|f\|_2^2 \le A^{-1}\sum_{j,k}|\langle f,\psi_{b_0;j,k}\rangle|^2. \quad (44)$$

Since, (45) is equivalent to (28) therefore, statement (i) implies statement (ii). To prove the converse part, we recall Theorem 2 and we have for any $g \in L_{2,\alpha}(\mathrm{R})$ and $f = T^{-1}g$,

$$g(x) = \sum \langle f, \psi_{b_0,j;k} \rangle \psi_{b_0 j;k}.$$

Also, by the l^2 linear independence of $\{\psi_{b_0 j;k}\}$, this representation is unique. From the Banach-Steinhaus and open mapping theorem it follows that $\{\psi_{b_0,j;k}\}$ is Riesz basis of $L_{2,\alpha}(\mathrm{R})$

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