A Study of Notations and Illustrations of Axiomatic Fuzzy Set Theory

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ABSTRACT

Fuzzy logic system studies reasoning systems in which the design of precision and deception are considered in a graded fashion, in contrast with classical mathematics where only absolutely true statements are considered. Whereas, Axiomatic fuzzy logic system facilitates a significant step on how to transform the information within databases into the membership functions and their fuzzy logic operations, by taking both the fuzziness and randomness into account. In this paper, various notations and illustrations of fuzzy concepts and coherence membership functions have been studied and analyzed under the framework of Axiomatic Fuzzy set theory. Various examples are illustrated for every concept by considering the hypothetical data.

Keywords
Axiomatic Fuzzy Set structures, Axiomatic Fuzzy Set algebras, Axiomatic Fuzzy Set logic, Coherence membership functions, Fuzzy logic system.

1. INTRODUCTION
In today's enormous storage era, knowledge acquisitions and representations constitute a major knowledge engineering bottleneck. There are various methods intended at improving this problem [1]. The unification of fuzzy sets into the representations of fuzzy concepts makes it possible for us to combine the uncertainty handling and approximate reasoning capabilities with comprehensibility. In conventional fuzzy theories, the membership functions of fuzzy sets are often given by personal intuition manually and the logic operations are equipped by a kind of triangular norms as t-norms, t-conorms and s-norms which is chosen in advance and independent of the distribution of the raw data [2]. The large-scale intelligence systems in real-world applications are usually very large and complex, containing such a large number of concepts that it is impossible or difficult to define the membership functions just by expert perception [3].

In order to deal with the above issues, the authors of papers [1-4] have proposed Axiomatic Fuzzy Set (AFS) theory in which fuzzy sets membership functions and their logic operations are impersonally and automatically determined according to the distributions of original data and the semantics of the fuzzy concepts. In understanding the nature of learning approach the complement operation of the fuzzy sets has been given, thus a fuzzy logic system, which is called AFS fuzzy logic, has been developed [2]. In recent times, AFS theory has been developed and applied to fuzzy decision trees [5], fuzzy clustering analysis [6], fuzzy classifier [7, 8], formal concept analysis [9], fuzzy rough sets [10], fuzzy cognitive maps [11] and credit rating analysis [12], etc.

In AFS theory, the membership functions and their logic operations are determined by AFS structures and AFS algebras. An AFS structure is a mathematical object of triple (M, τ, X) which is a special family of combinatorics objects [8], where X is the set of objects, M is a set of some simple concepts [13] on X and τ is mathematical abstract of the complex relations of objects concerning the original data and facts such as database sub-preference relations [13], even human intuition descriptions. AFS algebra is a family of molecular lattices (i.e. completely distributive lattices) generated by sets such as X, M. With the AFS structure (M, τ, X), which can be directly obtained from the given dataset, any concepts on X can be represented by the fuzzy sets in EM, which is the AFS algebra over M. Using AFS algebra and AFS structure (M, τ, X), a great large number of complex fuzzy concepts on X and their logic operations can be implemented by the algebraic operators on few simple concepts in M. The membership functions and their logic operations are more accurate and impersonal reflections of the original data and facts than given by human intuition in other fuzzy theories, and the information contained in the original data and facts is preserved at major extent [3].

2. LITERATURE SURVEY
2.1 A review of the Axiomatic Fuzzy Set theory
In this section, were call the notations and some of the definitions of AFS theory which has been involved many mathematical theories. It is known that the AFS theory provides an effective tool to convert the information into the training examples and databases into the membership functions and their fuzzy logic operations by taking both the subjective imprecision and objective uncertainty into consideration [14-21]. AFS theory is based on AFS structure - a kind of mathematical description of the data structure, which is a special kind of combinatorics systems and AFS algebra - a kind of semantic methodology which is a family of completely distributive lattices.

2.1.1   Axiomatic Fuzzy Set Algebras
AFS algebras is defined as a family of completely distributive lattices and used to study the lattice value representations of fuzzy concepts.

Definition 1([1])
Let $X_1, X_2, \ldots, X_n, M$ be a non-empty set. Then the set $EX_1 X_2 \ldots X_n M$ is defined as

$$EM^{\#} = \left\{ \sum_{i \in I} \left(\prod_{m \in A_i} m \right) \mid A_i \subseteq M, i \in I, I \text{ is a nonempty index set} \right\}$$

where Σ *and* Π denotes a disjunction and conjunction. For $\sum_{i \in I} (\prod_{m \in A_i} m)$, which is a formal sum of $(\prod_{m \in A_i} m), A_i \subseteq$

$M, i \in I$, is the disjunctions of conjunctions represented by $\prod_{m \in Ai} m's$(i.e., the disjunctive normal form a formula representing a concept).

Example 1

Let $X = \{p_1, p_2, p_3, p_4, p_5\}$ be a set of 5 persons, $M = \{m_1, m_2, \ldots, m_8\}$ be a set of attributes where m_1 = Bengali, m_2 = Marathi, m_3 = Assame, m_4 = male, m_5 = female, m_6 = Professor, m_7 = Engineer, m_8 = Doctor, i.e., each feature/attribute is a crisp concept are shown in table 1.

Table 1: Crisp attributes descriptions

	Bengali	Marathi	Assame	Male	Female	Prof.	Eng.	Doctor
p_1	Y	N	N	Y	N	N	Y	Y
p_2	N	Y	N	N	Y	Y	N	N
p_3	N	Y	N	Y	N	N	Y	N
p_4	Y	N	N	Y	N	Y	N	N
p_5	N	N	Y	Y	N	Y	Y	N

For example, $\gamma = m_1 m_5 + m_3 m_8 + m_6$ this translates as "bengali female" or "Assame doctor" or "professor". (the "+" denotes here a disjunction of concepts) while M may be a set of fuzzy or Boolean (two – valued) concepts, every $\sum_{i \in I}(\prod_{m \in A_i} m)$, $| A_i \subseteq M, i \in I$ has a well – defined meaning.

Definition 2 ([1])

Let M be a non-empty set. A binary relation R on EM is defined as follows:

for, $\sum_{i \in I}(\prod_{m \in A_i} m)$, $\sum_{j \in J}(\prod_{m \in B_j} m) \in EM$
and $[\sum_{i \in I}(\prod_{m \in A_i} m)] R [\sum_{j \in J}(\prod_{m \in B_j} m)] \Leftrightarrow$

     (i) $\forall A_i (i \in I), \exists B_h (h \in J)$ such that $A_i \subseteq B_h$
     (ii) $\forall B_j (j \in J), \exists A_k (k \in I)$ such that $B_j \subseteq A_k$

It is obvious that R is an equivalence relation. The quotient set $EM^{\#}/R$ is represented by EM.

The notation,

$\sum_{i \in I}(\prod_{m \in Ai} m) = \sum_{i \in J}(\prod_{m \in Bj} m)$ means that $\sum_{i \in I}(\prod_{m \in Ai} m)$ and $\sum_{i \in J}(\prod_{m \in Bj} m)$ are equivalent under equivalence relation R. Thus the semantics they represent are equivalent.

In example 1, for $\xi = m_3 m_8 + m_1 m_4 + m_2 m_5 m_7 + m_1 m_4 m_8$

$$\zeta = m_3 m_8 + m_1 m_4 + m_2 m_5 m_7 \in EM$$

By definition 2, this implies that $\xi = \zeta$.

Theorem 1([1])

Let M be a non-empty set. Then $(EM, \vee, \wedge)$ forms a completely distributive lattice under the binary compositions $\vee$ and $\wedge$ defined as follows.
For any, $\sum_{i \in I}(\prod_{m \in A_i} m)$, $\sum_{j \in J}(\prod_{m \in B_j} m) \in EM$.

$$\sum_{i \in I}(\prod_{m \in A_i} m) \vee (\sum_{j \in J}(\prod_{m \in B_j} m) = \sum_{k \in I \cup J}(\prod_{m \in C_k} m)$$

$$\sum_{i \in I}(\prod_{m \in A_i} m) \wedge (\sum_{j \in J}(\prod_{m \in B_j} m) =$$
$$\sum_{i \in I, j \in J}(\prod_{m \in A_i \cup B_j} m),$$

where for any $k \in I \cup J$ (the disjoint union of I and J, i.e., every element in I and every element in J are always regarded as different elements in $I \cup J$), $C_k = A_k$ if $k \in I$, and $C_k = B_k$ if $k \in J$.

For example,

The new attributes α, β and γ can be represented by the elements in EM as $\alpha = \{m_1 m_4\} + \{m_2 m_5 m_6\}$,

$$\beta = \{m_1 m_4\} + \{m_2 m_5 m_6\} + \{m_1 m_4 m_8\} \ and$$

$$\gamma = \{m_5 m_6\} + \{m_5 m_8\}$$

$\alpha = (m_1 m_4) + (m_2 m_5 m_6)$ which translates the meaning that: "Bengali males or Marathi female professors";

$\beta = (m_1 m_4) + (m_2 m_5 m_6) + (m_1 m_4 m_8)$ which translates the meaning that: "Bengali males or Marathi female professors or Bengali male Doctors";

$\gamma = (m_5 m_6) + (m_5 m_8)$ which translates the meaning that: "Female Professors or Female Doctors".

In general, M is a set of fuzzy or crisp attributes, for $\sum_{i \in I} A_i \in EM$, A_i expresses a new attribute generated by the "and", "or" of all attributes in $A_i, i \in I$. Even if there are fuzzy attributes in M, $\sum_{i \in I} A_i$ has definite semantic signification like crisp attributes α, β and γ which are discussed above. Representing attributes generated by the attributes in M in the form $\sum_{i \in I} A_i \in EM$ can not only preserve the semantic signification, but also avoid to define the fuzzy logic operators "and", "or" in advance.

By Definition 2,

$$\beta = (m_1 m_4) + (m_2 m_5 m_6) + (m_1 m_4 m_8)$$
$$= (m_1 m_4) + (m_2 m_5 m_6) = \alpha$$

From Table 1, it verifies that for $x \in X$, x satisfies α if and only if it satisfies β. In other words, term $\{m_1 m_4 m_8\}$ in β is redundant. In general, M is a set of fuzzy or crisp concepts, for $\alpha, \beta \in EM$. If $\alpha = \beta$, then the ordinary fuzzy sets (i.e., the membership degrees in the interval [0,1]) or L-fuzzy sets (i.e., the membership degrees in a lattice L) representing fuzzy concepts α and β should be identical. In the following, each element in EM denote it as a fuzzy concept or crisp concept. The semantic significations of concepts "α or γ" and "α and γ" can be simply denoted as $\alpha \vee \gamma$ and $\alpha \wedge \gamma$ respectively with (1), (2), therefore

$$\alpha \vee \gamma = \{m_1 m_4\} + \{m_2 m_5 m_6\} + \{m_5 m_6\} + \{m_5 m_8\}$$
$$= \{m_5 m_6\} + \{m_5 m_8\} + \{m_1 m_4\}$$

$$\alpha \vee \gamma = \{m_1 m_4\} + \{m_2 m_5 m_6\} + \{m_5 m_6\} + \{m_5 m_8\}$$
$$= \{m_5 m_6\} + \{m_5 m_8\} + \{m_1 m_4\}$$

In general, for any fuzzy concepts $\alpha, \beta \in EM$, to define their membership functions and fuzzy logic operators, fuzzy concepts $\alpha \vee \beta$ and $\alpha \wedge \beta$ are the fuzzy logic "or", "and" of α, β, respectively. $\alpha \geq \beta$ in lattice EM implies that no matter how to define the membership functions for α, β, the membership degree of any x belonging to α should be larger

than or equal to that of x belonging to β. For M a set of few fuzzy or crisp concepts, a great number of fuzzy concepts can be expressed by the elements in EM and the fuzzy logic operations of them can be implemented by the operations $\vee$, $\wedge$ of the completely distributive lattice $(EM, \vee, \wedge)$.

2.1.2. Axiomatic Fuzzy Set Structures
In this work, the fuzzy set of X denote that all kinds of representing forms for fuzzy concepts. The fuzzy sets and crisp subsets on X can be explained as the following:

For a fuzzy set ζ on universe of discourse X, any $x \in X$, either x belongs to ζ at some degree or does not belong to ζ at all. While for a crisp subset A of X, any $x \in X$, either x belongs to A or does not belong to A at all. Based on this proclamation that, both a fuzzy set and a crisp subset on X can be represented by a binary relation R on X through comparing the degrees of each pair x, y in X belonging to the concept without require to classify their membership degrees.

Definition 3([1])

Let ζ be any concept on the universe of discourse X. R_ζ is called a binary relation (i.e., $R_\zeta \subset X \times X$) of ζ if R_ζ satisfies: $x, y \in X, (x, y) \in R_\zeta \Leftrightarrow x$ belongs to concept ζ at some degree and the degree of x belonging to ζ is larger than or equals to that of y, or x belongs to concept ζ at some degree and y does not at all.

Even though fuzzy concepts are indistinct, the binary relations corresponding to them are crisp subset of $X \times X$. And the notation that $(x, x) \in R_\zeta$ means that x belongs to concept ζ at some degree and $(x, x) \notin R_\zeta$ means that does not belong to concept at all.

Definition 4([14])

Let X be a set and R be a binary relation on X. R is called a preference relation on X if it satisfies the following conditions:

1. $\forall x \in X, (x, x) \in R$.
2. $if (x, y) \in R, (y, z) \in R, then (x, z) \in R, x, y, z \in X$.
3. $for \ any \ x, y \in X, either (x, y) \in R \ or \ (y, x) \in R$.

Although preference relations are very simple and have very good mathematical properties, it is too strict to represent ordinary concepts. So, it can be given that the following definition, which is mathematical abstraction of the simple concepts.

Definition 5 ([14])

Let X be a set and R be a binary relation on X. R is called a sub-preference relation on X if for $x, y, z \in X, x = y$, R satisfies the following conditions:

1. $if (x, y) \in R, then (x, x) \in R$;
2. $if (x, x) \in R \ and \ (y, y) \notin R, then (x, y) \in R; if (x, y), (y, z) \in R, then (x, z) \in R$;
3. $if (x, x) \in R \ and \ (y, y) \in R, then \ either (x, y) \in R \ or \ (y, x) \in R$.

A concept ζ is called simple concept on X if R_ζ is a sub-preference relation. Otherwise ζ is called complex concept on X.

Definition 6 ([14])

Let m be a simple concept on X and $\rho_m: X \to R^+ = [0, \infty)$. ρ_m is called a weighted function of simple concept m if ρ_m satisfies the following:

1. $\rho_m = 0 \Leftrightarrow (x, x) \notin R_m, x \in X$;
2. $(x, y) \in R_m \Rightarrow \rho_m(x) \geq \rho_m(y), x, y \in X$

The sequence systems express the information by information-based criteria such as human perception-based information, gain ratio, symmetric uncertainty, and order are regularized to be AFS structures by two axioms.

Definition 7([1, 17, 23, 24])

Let X, M be two sets and 2^M be the power set of M $\tau : X \times X \to 2^M$. (M, τ, X) is called an AFS structure if τ satisfies the following conditions:

$AX1: \forall (x_1, x_2) \in X \times X, \tau(x_1, x_2) \subseteq \tau(x_1, x_1)$;
and
$AX2: \forall (x_1, x_2), (x_2, x_3) \in X \times X, \tau(x_1, x_2) \cap \tau(x_2, x_3) \subseteq \tau(x_1, x_3)$

In addition, X is called universe of discourse, M is called an attribute set and τ is called a structure.

Let X be a data set of objects and M be a set of simple concepts on X, for any $x, y \in X, \tau(x, y)$ are subsets of M, $\tau(x, y) = \{ m | m \in M, (x, y) \in R_m \}$, where R_m is the binary relation of simple concept m i.e., for any $m \in \tau(x, y)$, the degree of x belonging to the concept m is greater than or equal to y i.e., $\gamma_m(x) \geq \gamma_m(y)$.

Example 2

Let $X = \{x_1, x_2, \dots, x_{10}\}$ be a set of 10 persons and $M = \{m_1, m_2, \dots, m_{10}\}$ be the set of fuzzy concepts on X and the elements of M are analyzed as simple concepts, where $m_1 =$ age, $m_2 =$ high height, $m_3 =$ weight, $m_4 =$ high salary, $m_5 =$ more wealth, $m_6 =$ male, $m_7 =$ female, $m_8 =$ Eyecolor black, $m_9 =$ Eyecolorblue, $m_{10} =$ Eyecolor brown, where the attributes are described by real numbers such as age, height, weight, salary, wealth and gender is described as Boolean values and the eye color such as black, blue, brown are described as in order relations i.e., $x = y$ in the order relation implies that the degrees of x and y belonging to the attribute are equal, as a substitute of x and y being the same element of X. The samples $x_1, x_2, \dots, x_{10}$ are shown in Table 2.

From Table 2, it can verify that each concept $m \in M$ is a simple concept and the structure τ can be defined as follows.

For any $x_i, x_j \in X, \ i \neq j$

$\tau(x_i, x_j) = \{m/m \ M, x_i \ possesses \ attribute \ m \ at \ some \ degree\}$

$\tau(x_i, x_j) = \{m/m \in M, x_i \geq_m x_j\}$

where $x_i \geq_m x_j$ implies that x_i possesses attribute m in some degree and the degree of x_i belonging to m is larger than or equal to that of x_j.

For example,

$$\tau(x_1, x_1) = \{m_1, m_2, m_3, m_4, m_6, m_8, m_9, m_{10}\}$$

$$\tau(x_4, x_7) = \{m_1, m_2, m_6, m_9, m_{10}\}$$

For example, the numbers in the column "Eye color" implies the order relation as ($>$)

Eye color black

$$m_8 : x_4 > x_1 > x_3 = x_7 > x_2 = x_9 > x_6 > x_5 = x_8 = x_{10};$$

Table 2: Representation of the samples with the attribute values

Sample X	Age m_1	Height m_2	Weight m_3	Salary m_4	Wealth m_5	Male m_6	Female m_7	Eye color black m_8	Eye color blue m_9	Eye color brown m_{10}
x_1	30	1.9	90	1	0	y	n	2	5	3
x_2	20	1.3	32	0	1	n	y	4	3	1
x_3	54	1.6	67	200	35	n	y	3	4	2
x_4	83	1.7	73	20	80	y	n	1	6	4
x_5	34	1.4	54	15	2	y	n	6	1	4
x_6	37	1.6	80	80	28	n	y	5	2	2
x_7	45	1.7	78	350	89	y	n	3	4	2
x_8	70	1.65	70	30	48	y	n	6	1	4
x_9	60	1.82	83	25	100	n	y	4	3	1
x_{10}	10	1.12	21	0	0	n	y	6	1	4

Eye color blue
$$m_9 : x_5 = x_8 = x_{10} > x_6 > x_2 = x_9 > x_3 = x_7 > x_1 > x_4;$$

Eye color brown
$$m_{10} : x_2 = x_9 > x_3 = x_6 = x_7 > x_1 > x_4 = x_5 = x_8 = x_{10}$$

In this order, $x_i > x_j$ implies that the eye color of x_i is closer to the eye color of black than the eye color of an individual x_j. Similarly $x_i = x_j$ implies that the eye color of x_i is same as the eye color of x_j.

The AFS structure *(M, τ, X)* is the mathematical abstraction of the complex relationships between universe of discourse X and the attribute set M. This implies that the information contained in Table 2 are represented by *(M, τ, X)*.

3. COHERENCE MEMBERSHIP FUNCTIONS OF FUZZY SETS

The membership functions are consistent with both the AFS logic system *(M, ∨, ∧)* in the sense of the underlying semantics and the distribution of the data.

Definition 8 ([1])
Let (M, τ, X) be an AFS structure of a data set X. For $x \in X, A \subseteq M$, the set $A^\tau(x) \subseteq X$ is defined as follows.

$$A^\tau(x) = \{y \mid y \in X, \tau(x, y) \supseteq A\}$$
$$= \{y \in X \mid x \geq_m y \text{ for any } m \in A\}$$

$A^\tau(x)$ is the set of all elements in data set X whose degree of elements are belonging to concept $\prod_{m \in Ai} m$ are less than or equal to x.

For instance, by comparing the features values of x_3 and other samples are shown in Table 2.

$$\{m_1\}^\tau(x_3) = \{x_1, x_2, x_3, x_5, x_6, x_7, x_{10}\}$$

$$\{m_3\}^\tau(x_3) = \{x_2, x_3, x_5, x_{10}\}$$

$$\{m_1, m_3\}^\tau(x_3) = \{x_2, x_3, x_5, x_{10}\}$$

For $\zeta \in EM$, let $\mu_\zeta : X \to [0, 1]$ be the membership function of the concept ζ. $\{\mu_\zeta(x) \mid \zeta \in EM\}$ is called a set of coherence membership functions of the AFS fuzzy logic system (EM, ∨, ∧) and the AFS structure *(M, τ, X)*, if the following conditions are satisfied.

1. *for $\alpha, \beta \in EM$, if $\alpha \leq \beta$ in lattice (EM,∧,∨), then $\mu_\alpha(x) \leq \mu_\beta(x)$ for any $x \in X$;*

2. *for $x \in X, \eta = \sum_{i \in I}(\prod_{m \in A_i} m) \in EM$, if $A_i^\tau(x) = \emptyset$ for all $i \in I$ then $\mu_\eta(x) = 0$;*

3. *for $x, y \in X, A \subseteq M, \eta = \prod_{m \in A} m \in EM$, if $A^\tau(x) \subseteq A^\tau(y)$, then $\mu_\eta(x) \leq \mu_\eta(y)$; if $A^\tau(x) = X$ then $\mu_\eta(x) = 1$.*

Proposition 1
Let M be a set of simple concepts on X and *(M, τ, X)* be an AFS structure defined as in definition 8. Let $\{\mu_\zeta(x) \mid \zeta \in EM\}$ be a set of coherence membership functions of (EM, ∨, ∧) and algebraic structure *(M, τ, X)*. Then for any $\delta, \beta \in EM$, any $x \in X$,

$$\mu_{\alpha \vee \beta}(x) \geq \max\{\mu_\alpha(x), \mu_\beta(x)\},$$

$$\mu_{\alpha \wedge \beta}(x) \leq \min\{\mu_\alpha(x), \mu_\beta(x)\}$$

Proof:
In lattice (EM, ∨, ∧), for any $\alpha, \beta \in EM$, the $\alpha \vee \beta \geq \alpha$, $\alpha \vee \beta \geq \beta$ and $\alpha \wedge \beta \leq \alpha, \alpha \wedge \beta \leq \beta$. By using the condition 1 of Definition 8, for any $x \in X$, one has

$$\mu_{\alpha \vee \beta}(x) \geq \mu_\alpha(x), \ \mu_{\alpha \wedge \beta}(x) \geq \mu_\beta(x),$$

$$\mu_{\alpha \wedge \beta}(x) \leq \mu_\alpha(x), \ \mu_{\alpha \wedge \beta}(x) \leq \mu_\beta(x)$$

This implies that the inequality is proved.

Theorem 2

Let M be a set of simple concepts on data set X and (M, τ, X) be an AFS structure of a data set X. Let S be a σ-algebra over X such that for any $m \in M$ and any $x \in X$, $\{m\}^\tau(x) \in S$. For each simple concept $\gamma \in M$, let $\mathcal{M}_\gamma$ be a measure over S with $0 \le \mathcal{M}_\gamma(U) \le 1$ for all $U \in S$ and $\mathcal{M}_\gamma(X) = 1$. $\{\mu_\zeta(x) \mid \zeta \in EM\}$ is a set of coherence membership functions of $(EM, \vee, \wedge)$ and (M, τ, X), if for each concept $\zeta = \sum_{i \in I}(\prod_{m \in A_i} m) \in EM$, $\mu_\zeta : X \to [0,1]$ is defined as follows: for any $x \in X$

$$\mu_\zeta(x) = \sup_{i \in I}(\textstyle\prod_{\gamma \in A_i} \mathcal{M}_\gamma(A_i^\tau(x))) \qquad (2.1)$$
or
$$\mu_\zeta(x) = \sup_{i \in I}(\inf_{\gamma \in A_i} \mathcal{M}_\gamma(A_i^\tau(x))) \qquad (2.2)$$

Proof:

Let $\quad \alpha = \sum_{i \in I}(\prod_{m \in A_i} m), \qquad \beta = \sum_{j \in J}(\prod_{m \in B_j} m) \in EM$ and $\alpha \le \beta$ in lattice $(EM, \vee, \wedge)$. By Theorem 1, for any A_i $(i \in I)$, there exist $B_h(h \in J)$ such that $A_i \supseteq B_h$. Then $A_i^\tau(x) \subseteq B_h^\tau(x)$ for any $x \in X$. Thus for any $i \in I$,

$$
\begin{aligned}
\textstyle\prod_{\gamma \in A_i} \mathcal{M}_\gamma(A_i^\tau(x)) &\le \textstyle\prod_{\gamma \in A_i} \mathcal{M}_\gamma((B_h^\tau(x)) \\
&\le \textstyle\prod_{\gamma \in B_h} \mathcal{M}_\gamma(B_h^\tau(x)) \\
&\le \mu_\beta(x).
\end{aligned}
$$

which implies that,

$$\mu_\zeta(x) = \sup_{i \in I}(\inf_{\gamma \in A_i} \mathcal{M}_\gamma(A_i^\tau(x)) \le \mu_\beta(x)$$

Thus condition 1 of Definition 8 holds.
Since $\mathcal{M}_\gamma(\emptyset) = 0$, for any simple concept $\gamma \in M$ hence condition 2 of Definition 8 holds.

For $x, y \in X, A \subseteq M, \quad \eta = \prod_{m \in A} m \in EM$, if $A^\tau(x) \subseteq A^\tau(y)$, then for any $\gamma \in A$,

$$\mathcal{M}_\gamma(A_i^\tau(x)) \le \mathcal{M}_\gamma A_i^\tau(y)).$$

This implies that $\mu_\eta(x) \le \mu_\eta(y)$. Furthermore, since $\mathcal{M}_\gamma(X) = 1$, hence condition 3 of Definition 8 holds. Therefore $\{\mu_\zeta(x) \mid \zeta \in EM\}$ is the set of coherence membership functions of $(EM, \vee, \wedge)$ and (M, τ, X). As for each $\gamma \in M$ and for any $U \in S$, $0 \le \mathcal{M}_\gamma(U) \le 1$ and $\mathcal{M}_\gamma(U) = 1$, the functions defined by (2.1) or (2.2) are coherence membership functions. In real world applications, the measure $\mathcal{M}_\gamma$ can be constructed according to the semantic meaning of the simple concept γ and may have various versions depending on the identification of the problem.

In general, $\mathcal{M}_\gamma(A_\tau(x))$ measures the degree of set $A_\tau(x)$ supporting the claim : "x belongs to ". The coherence applied to membership functions of fuzzy concepts in EM to denote the membership functions with respect to the semantic interpretations expressed by the fuzzy concepts, the logic relationships among the fuzzy concepts in Axiomatic Fuzzy Set logic systems $(EM, \vee, \wedge)$. AFS theory presents a far more flexible and potential framework for achieving and representation of human knowledge which is suitable for studying large-scale intelligence systems in real applications.

4. CONCLUSION

Axiomatic Fuzzy Set theory proposes a future flexible and powerful framework for acquisition and illustration of human knowledge which is appropriate for studying large-scale intelligence systems in real-time applications. Axiomatic Fuzzy set theory approach is considered as the knowledge representation of the system, it helps in compact with data sets of mixed data type features processed in a uniform way. In addition, AFS theory with the properties of AFS algebra and AFS structures, the membership functions and their logic operations for any model can be achieved commonly based on the original statistical data and by using the axiomatic theory coherence membership function its focal point on an innovative design of rule extraction. The design helps to deal with cases where it meets imbalanced classes. The modification and pruning methods help to generate a echo and efficient fuzzy rule-based classifier.

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