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Boundary Domination of Line and Middle Graph of Wheel Graph Families

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ABSTRACT

Let $G = (V, E)$ be a connected graph. A subset S of $V(G)$ is called a boundary dominating set if every vertex of $V - S$ is boundary dominated by some vertex of S. The minimum taken over all boundary dominating sets of a graph G is called the boundary domination number of G and is denoted by $\gamma_b(G)$. We define the boundary domatic number in graphs. Exact values of of Wheel Graph Families are obtained and some other interesting results are established.

Keywords

Boundary dominating set, Boundary domination number, Boundary domatic number

1. INTRODUCTION

For graph-theoretical terminology and notations not defined here we follow Buckley [2] and Haynes et al.[4]. Let G be a nontrivial connected graph. The distance between two vertices u and v is the length of a shortest path joining them. The eccentricity $e(u)$ of a vertex u is the distance to a vertex farthest from u. A vertex v is called an eccentric vertex of u if $e(u) = d(u, v)$. A vertex v is an eccentric vertex of G if v is an eccentric vertex of some vertex of G. Consequently if v is an eccentric vertex of u and w is a neighbor of v, then $d(u, w) \leq d(u, v)$. A vertex v may have this property, however, without being an eccentric vertex of u. Let G be a simple graph $G = (V, E)$ with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. For $i \neq j$, a vertex v_i is a boundary vertex of v_j if $d(v_j, v_t) \leq d(v_j, v_i)$ for all $v_t \in N(v_i)$ [3].
A vertex v is called a boundary neighbor of u if v is a nearest boundary of u. If $u \in V$, then the boundary neighbourhood of u denoted by $N_b(u)$ is defined as $N_b(u) = \{v \in V : d(u, w) \leq d(u, v)$ for all $w \in N(u)\}$. The cardinality of $N_b(u)$ is denoted by $deg_b(u)$ in G. The maximum and minimum boundary degree of a vertex in G are denoted respectively by $\Delta_b(G)$ and $\delta_b(G)$. That is $\Delta_b(G) = \max_{u \in V} |N_b(u)|$, $\delta_b(G) = \min_{u \in V} |N_b(u)|$.
A vertex u boundary dominate a vertex v if v is a boundary neighbor of u. KM. Kathiresan, G. Marimuthu and M. Sivanandha Saraswathy [5] introduced the concept of Boundary domination in graphs. Puttaswamy and Mohammed Alatif [6] introduced the concept of Boundary edge domination in graphs. All graphs considered in this paper are finite and contains no loops and no multiple edges. For a real number x; $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.
Line graph $L(G)$ of a graph G is defined with the vertex set $E(G)$, in which two vertices are adjacent if and only if the corresponding edges are adjacent in G.
Middle graph $M(G)$ of a graph G is defined with the vertex set $V(G) \cup E(G)$, in which two elements are adjacent if and only if either both are adjacent edges in G or one of the elements is a vertex and the other one is an edge incident to the vertex in G. We need the following theorems.

THEOREM 1. *[6] If G is a connected graph of size $m \geq 3$, then $\lceil \frac{m}{\Delta_b^{'}+1} \rceil \leq \gamma_b^{'}(G) \leq m - \Delta_b^{'}(G)$.*

THEOREM 2. *[6] For any (n, m)-graph G, $\gamma^{'}(G) + \gamma_b^{'}(G) \leq m + 1$.*

THEOREM 3. *For any gear graph G_n with $n > 3$, $\gamma(G_n) = \lceil \frac{n}{2} \rceil + 1$.*

THEOREM 4. *[1] For any helm graph H_n with $n > 3$, $\gamma(H_n) = n$.*

THEOREM 5. *For any connected graph G, $d_b(G) \leq \lfloor \frac{n}{\gamma_b(G)} \rfloor$.*

2. RESULTS

2.1 Boundary Domination In Graphs

DEFINITION 6. *A subset S of $V(G)$ is called a boundary dominating set if every vertex of $V - S$ is boundary dominated by some vertex of S. The minimum taken over all boundary dominating sets of a graph G is called the boundary domination number of G and is denoted by $\gamma_b(G)$, $\gamma_b^{'}(G)$ for the line graph of G and $\gamma_b(M(G))$ for the middle graph of G.*

2.1.1 Wheel Graph. The wheel graph W_n on $n + 1$ vertices is defined as $W_n = C_n + K_1$ where C_n is $n - cycle$. Let $V(W_n) = \{v_i : 1 \leq i \leq n\} \cup \{v\}$ and $E(W_n) = \{e_i = v_i v_{i+1}, 1 \leq i \leq n$, subscripts modulo $n\} \cup \{e_i^{'} = vv_i, 1 \leq i \leq n\}$, where v is an external vertex adjacent to every other vertex.

THEOREM 7. *For any wheel graph W_n, $\gamma_b(W_n) = 1$*

PROOF. Let W_n be a wheel graph of order $n + 1$. Since $d(v, v_1) = d(v, v_2) = ... = d(v, v_n) = 1$, then $N_b(v) = \{v_1, v_2, ..., v_n\}$, $\delta_b = \Delta_b = n$ so that $S = \{v\}$ and $|S| = 1$. Hence

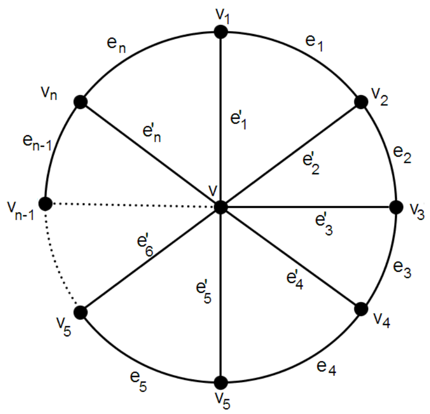

Fig. 1. Wheel graph W_n

$\gamma_b(W_n) \leq 1$. Further since $\gamma_b(W_n) \geq \lceil \frac{n+1}{\Delta_b+1} \rceil = \lceil \frac{n+1}{n+1} \rceil = 1$. Thus $\gamma_b(W_n) = 1$. □

THEOREM 8. *For a wheel graph W_n, $n \geq 3$, $\gamma'_b(W_n) = 3$.*

PROOF. Let $L(W_n)$ be the line graph of W_n of order $2n$. Since $d(e_i, e_{i+1}) \leq d(e_i, e_{i+2}), d(e_i, e_{i-1}) \leq d(e_i, e_{i-2})$ for all $e_{i+1}, e_{i-1} \in N(e_i)$ and $e_{i+2}, e_{i-2} \in N_b(e_i)$, also $d(e_i, e'_i) \leq d(e_i, e'_{i-1}), d(e_i, e'_{i+1}) \leq d(e_i, e'_{i+2})$ for all $e'_i, e'_{i+1} \in N(e_i)$ and $e'_{i-1}, e'_{i+2} \in N'_b(e_i)$ so that $\delta'_b = n-2, \Delta'_b = n$, and for $1 \leq i \leq n$, the cycle $C_3 = \{e_i, e'_i, e'_{i+1}\}$ or $\{e_{i-1}, e_i, e'_i\}$ is a boundary edge dominating set of W_n. Hence $|S| = \gamma'_b(W_n) = 3$. □

THEOREM 9. *For a wheel graph W_n, $n \geq 3$, $\gamma_b(M(W_n)) = 3$.*

PROOF. The proof is similar to the proof of Theorem 2.3. □

2.1.2 Gear Graph. The gear graph is a wheel graph with vertices added between pair of vertices of the outer cycle. The gear graph G_n has $2n+1$ vertices and $3n$ edges. Let $V(G_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v\}$ and $E(G_n) = \{e_i = v_i u_i, 1 \leq i \leq n\} \cup \{e'_i = v_i v, 1 \leq i \leq n\} \cup \{e''_i = u_i v_{i+1}, 1 \leq i \leq n$, subscripts modulo $n\}$, where v is an external vertex adjacent to every other vertex v_i for $1 \leq i \leq n$.

THEOREM 10. *For any gear graph G_n, $\gamma_b(G_n) = 2$.*

PROOF. Let X, Y be a bipartition of G_n, with $X = \{v_1, v_2, ..., v_n\}$ and $Y = \{u_1, u_2, ..., u_n\} \cup \{v\}$. Let $v_i \in X$. Then $d(v_i, v_j) = 2$ for all $v_j \in X - \{v_i\}; i \neq j$ and every vertex v_j in X is a boundary neighbour of v_i except v_i. Similarly $d(v, u_i) = 2$, then every vertex of $Y - \{v\}$ is a boundary neighbour of u_i except v and $\delta'_b = \Delta'_b = n$ therefore $S = \{v, v_i\}$, is a boundary dominating set of G_n for all i so that $|S| = 2$. Hence $\gamma_b(G_n) \leq 2$. Further since $\Delta_b = n, \gamma_b(G_n) \geq \lceil \frac{2n+1}{\Delta_b+1} \rceil = \lceil \frac{2n+1}{n+1} \rceil$, then $\gamma_b(G_n) \geq 2$. Hence $\gamma_b(G_n) = 2$.

□

THEOREM 11. *For a gear graph G_n, $\gamma'_b(G_n) = 3$.*

PROOF. Let $L(G_n)$ be the line graph of Gn of order $3n$. Since $d(e_i, e'_i) \leq d(e_i, e'_{i+1})$,
$d(e_i, e''_i) \leq d(e_i, e_{i+1})$ for all $e'_i, e''_i \in N(e_i)$ and $e'_{i+1}, e_{i+1} \in N_b(e_i)$, similarly $e_i, e''_i \in N'(e'_i)$ and $e_{i+1}, e''_{i+1} \in N'_b(e'_i)$ so that $\delta'_b = n+1, \Delta'_b = 2n-2$, and for $1 \leq i \leq n$, the cycle $C_3 = \{e_i, e'_i, e''_i\} = S$ is a boundary edge dominating set of G_n and $|S| = 3$. Hence $\gamma'_b(G_n) \leq 3$. Further since the collection $\{e_i, e'_i, e''_i : 1 \leq i \leq n\}$ contains n-cycles of order 3 then $|S| \geq \lceil \frac{3n}{n} \rceil = 3$ so that $\gamma'_b(G_n) \geq 3$. Thus $\gamma'_b(G_n) = 3$ □

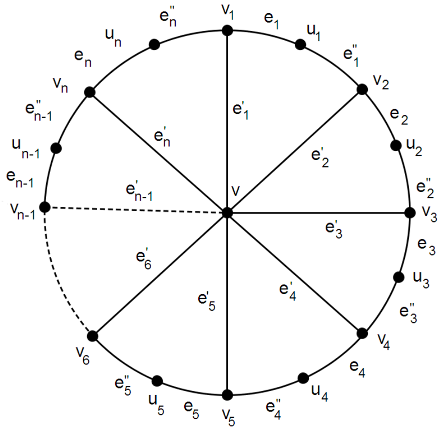

Fig. 2. G_n

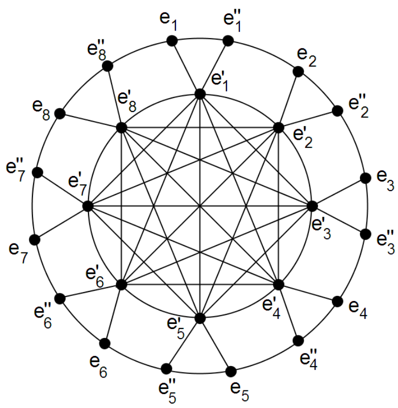

Fig. 3. $L(G_8)$

THEOREM 12. *For a gear graph G_n, $\gamma_b(M(G_n)) = \lceil \frac{n}{2} \rceil + 1$.*

PROOF. Let $M(G_n)$ be the middle graph of G_n of order $5n+1$. Since $d(e_i, e'_i) \leq d(e_i, e'_{i+1}), d(e_i, e''_i) \leq d(e_i, e_{i+1})$, for all $e'_i, e''_i \in N(e_i), e''_{i+1}$, and $e'_{i+1}, e_{i+1} \in N_b(e_i)$. Similarly $e'_i, e''_i, e_i \in N(v_i)$ and $e'_{i+1}, e''_{i+1}, u_i \in N_b(v_i)$ so that $\delta_b = n+4, \Delta_b = 3n$ and for $1 \leq i \leq n$ the set $S = \{e'_i : i = 2k+1, k < \lceil \frac{n}{2} \rceil\} \cup \{v_1\}$ is is a boundary dominating set of $M(G_n)$ and $|S| = \lceil \frac{n}{2} \rceil + 1$. Hence $\gamma_b(M(G_n)) \leq \lceil \frac{n}{2} \rceil + 1$. Further any boundary dominating set of $M(G_n)$ must contains at least one of e'_i, v_i for all i and hence $|S| \geq n \geq \lceil \frac{n}{2} \rceil + 1$ so that $\gamma_b(M(G_n)) \geq \lceil \frac{n}{2} \rceil + 1$. Thus $\gamma_b(M(G_n)) = \lceil \frac{n}{2} \rceil + 1$. □

2.1.3 Helm Graph. The helm graph H_n is the graph obtained from an $n-$wheel graph by adjoining a pendant edge at each node of the cycle. The helm graph H_n has $2n+1$ vertices and $3n$ edges and $V(G_n) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$ and $E(G_n) = \{e_i = v_iv_{i+1}, 1 \leq i \leq n-1\} \cup \{e^{'}_i = v_iv, 1 \leq i \leq n-1\} \cup \{e_i = v_iu_i, 1 \leq i \leq n-1$. where v is an external vertex adjacent to every other vertex v_i for $1 \leq i \leq n$

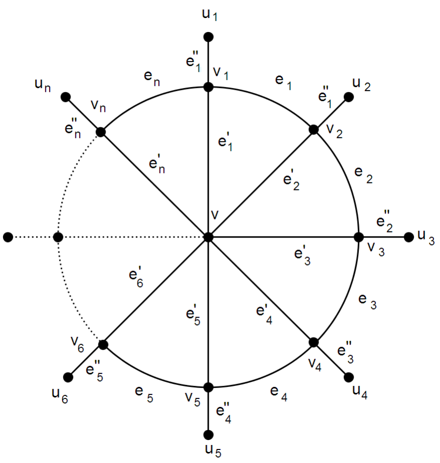

Fig. 4. H_n

THEOREM 13. *For any helm graph H_n, $\gamma_b(H_n) = 3$.*

PROOF. Let (X, Y) be a bipartition of H_n, with $X = \{v_1, v_2, ..., v_n\}$ and $Y = \{u_1, u_2, ..., u_n\} \cup \{v\}$. Let $u_i \in Y$. Since $d(v, u_i) = 2$, then every vertex of $Y - \{v\}$ is a boundary neighbour of u_i except v and $\Delta_b = n$. Similarly since $d(v_i, v) \leq d(v_i, v_{i+2})$ for all $v, v_{i+1} \in N(v_i)$, then every vertex v_j in X for $i+2 \leq j < n$ is a boundary neighbour of v_i except v_i, also v_{i+1} is a boundary neighbour of v_{i-1} except v_i, v_{i+1} and $\delta^{'}_b = 2$, so that for all i the set $S = \{v, v_i, v_{i+1}\}$ is a boundary dominating set of H_n where $S = \{v, v_1, v_2\}$ or $\{v, v_2, v_3\}$ or $...\{v, v_{n-1}, v_n\}$ and $|S| = 3$. Hence $\gamma_b(H_n) = 3$. □

THEOREM 14. *For any helm graph H_n,*

$$\gamma^{'}_b(H_n) = \begin{cases} 2 & \textit{if } n = 3 \textit{ or } 4 \\ 3 & \textit{otherwise} \end{cases} .$$

.

PROOF. The result is obvious if $n = 3$ or 4. Suppose $n \geq 5$, Since $d(e_i, e^{'}_i) \leq d(e_i, e^{'}_{i+2}), d(e_i, e^{''}_i) \leq d(e_i, e_{i+2})$ for all $e^{'}_i, e^{''}_i \in N(e_i)$ and $e^{'}_{i+2}, e_{i+2} \in N_b(e_i)$, similarly $e_i, e^{''}_i \in N^{'}(e^{'}_i)$ and $e_{i+1}, e^{''}_{i+1} \in N^{'}_b(e^{'}_i)$ so that $\delta^{'}_b = n+2, \Delta^{'}_b = 2n-3$, and for $1 \leq i \leq n$, the set $S = \{e^{'}_i, e_i, e^{'}_{i+1}\}$ is a boundary edge dominating set of H_n and $|S| = 3$. Hence $\gamma^{'}_b(H_n) \leq 3$. Further since the collection $\{e^{'}_i, e_i, e^{'}_{i+1} : 1 \leq i \leq n\}$ contains n-cycles of order 3 then $|S| \geq \lceil \frac{3n}{n} \rceil = 3$ so that $\gamma^{'}_b(H_n) \geq 3$. Thus $\gamma^{'}_b(H_n) = 3$ □

THEOREM 15. *For a helm graph H_n, $\gamma_b(M(H_n)) = \lceil \frac{n}{2} \rceil + 1$.*

PROOF. The proof is similar to the proof of Theorem 2.7. □

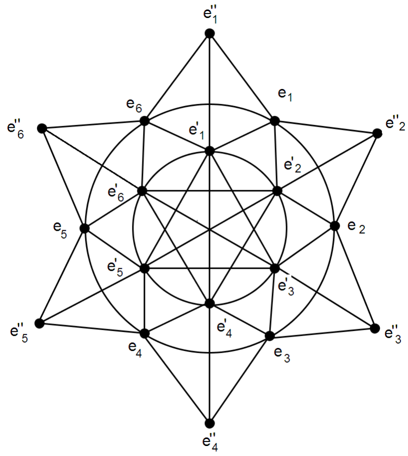

Fig. 5. $L(H_6)$

2.2 Boundary Domatic Number

The maximum order of a partition of the vertex set V of a graph G into dominating sets is called the domatic number of G and is denoted by $d(G)$. For a survey of results on domatic number and their variants we refer to Zelinka [7]. In this section we present a few basic results on the boundary domatic number of a graph.

DEFINITION 16. *Let $G = (V, E)$ be a connected graph. The maximum order of a partition of V into boundary dominating sets of G is called the boundary domatic number of G and is denoted by $d_b(G)$.*

THEOREM 17. , $d_b(W_n) = d_b(G_n) = d_b(H_n) = 1$.

THEOREM 18. *For a wheel graph W_n, $n \geq 3$, $d^{'}_b(W_n) = \lceil \frac{2n}{3} \rceil$.*

PROOF. By the definition of line graph, $V(L(W_n)) = E(W_n) = \{e_i = v_iv_{i+1}, 1 \leq i \leq n$, subscripts modulo $n\}$ $\cup\{e^{'}_i = vv_i, 1 \leq i \leq n\}$. Let

$$C = \{e_ie^{'}_ie^{'}_{i+1} : i = 3(k-1), 1 \leq k \leq \lceil \frac{2n}{3} \rceil\}.$$

and

$$C^{'} = \{e_ie_{i+1}e^{'}_{i+1} : i = 3k-2, 1 \leq k \leq \lceil \frac{2n}{3} \rceil\}.$$

be a collection of 3-cycles of $L(W_n)$. Clearly the cycles of C and $C^{'}$ are vertex disjoint and if $V(C)$ and $V(C^{'})$ denotes the set of vertices belonging to the cycles of C and $C^{'}$ respectively then $V(C) \cap V(C^{'}) = \phi$. Hence $d^{'}_b(W_n) \geq |C| + |C^{'}| = 2\lceil \frac{n}{3} \rceil$. If $n \equiv 0$ or 1 $(mod\ 3)$, then $2\lceil \frac{n}{3} \rceil = \lceil \frac{2n}{3} \rceil$ and $d^{'}_b(W_n) \geq \lceil \frac{2n}{3} \rceil$.
If $n \equiv 2(mod\ 3)$, then $\lceil \frac{2n}{3} \rceil = 2\lceil \frac{n}{3} \rceil + 1$. In this case $e^{'}_{n-2}, e^{'}_{n-1}, e_{n-2}, e_{n-1} \notin V(C) \cup V(C^{'})$ and the set $\{e^{'}_{n-2}, e^{'}_{n-1}, e_{n-2}\}$ induces a 3-cycle. Hence if $n \equiv 2(mod\ 3)$ $d^{'}_b(W_n) \geq 2\lceil \frac{n}{3} \rceil + 1 = \lceil \frac{2n}{3} \rceil$. Therefore in both the cases $d^{'}_b(W_n) \geq \lceil \frac{2n}{3} \rceil$. Also since $V(L(W_n)) = 2n$ and $\gamma^{'}_b(W_n) = 3$, we have $d^{'}_b(W_n) \leq \frac{2n}{\gamma^{'}_b} \leq \lceil \frac{2n}{3} \rceil$. Hence $d^{'}_b(W_n) = \lceil \frac{2n}{3} \rceil$. □

THEOREM 19. *For a wheel graph W_n and its middle graph $M(W_n)$,*

$$d_b(M(W_n)) = \begin{cases} 2 & if\, n = 3, \\ n & otherwise \end{cases}.$$

.

PROOF. The result is obvious if $n = 3$, Suppose $n \geq 4$ by the definition of middle graph $V(M(G)) = V(G) \cup E(G)$, and since $|V(M(W_n))| = 3n + 1, \gamma_b(M(W_n)) = 3$, then $d_b(M(W_n)) \leq \frac{3n+1}{\gamma_b} \leq \lfloor \frac{3n+1}{3} \rfloor \leq n$. Further let $C = \{P_i = v_i e_i^{'} e_{i+2}^{'} : 1 \leq i \leq n\}$ be the collection of paths of $M(W_n)$. Clearly the paths of C are vertex disjoint and $|C| = n$, then $d_b(M(W_n)) \geq n$. Hence $d_b(M(W_n)) = n$. □

THEOREM 20. *For a gear graph G_n, $d_b^{'}(G_n) = n$.*

PROOF. let $L(G_n)$ be a line graph of gear graph of order $3n$, since $\gamma_b^{'}(G_n) = 3$, it follows that $d_b^{'}(G_n) \leq \lfloor \frac{3n}{\gamma_{cb}^{'}} \rfloor = \lfloor \frac{3n}{3} \rfloor = n$. To prove the reverse inequality, let $\Gamma = \{e_i, e_i^{'}, e_i^{''} : 1 \leq i \leq n\}$ be a partition of the set of cycles of $L(G_n)$. It is clear that the cycles of Γ are vertex disjoint and $|\Gamma| = n$ therefore $d_b^{'}(G_n) \geq n$. Hence $d_b^{'}(G_n) = n$. □

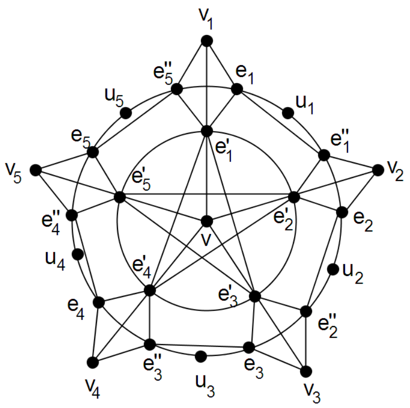

Fig. 6. $M(G_5)$

THEOREM 21. *For a gear graph G_n,*

$$d_b^{'}(M(G_n)) = \begin{cases} n+1 & if\, n \leq 5 \\ n & if\, n = 6\ or\ 7 \\ 4 & if\, n = 8\ or\ 9 \\ 2 & otherwise \end{cases}$$

.

PROOF. The result is obvious if $n \leq 9$. In otherwise, by the definition of middle graph, $V(M(G_n)) = V(G_n) \cup E(G_n), |V(M(G_n))| = 5n + 1$ in which the set $\{e_i^{'} : 1 \leq i \leq n\} \cup \{v\}$ induces a clique K_{n+1} of order $n + 1$ and for each $i, (1 \leq i \leq n)$, the set of vertices $\{e_i^{''}, e_{i+1}^{'}, e_{i+1}, v_{i+1}$: subscript modulo $n\}$ induce a clique of order 4. Also Since $deg_b(u_i) = 4$ and $|N_b(u_i) : 1 \leq i \leq n| = 4n$, then $d_b^{'}(M(G_n) \leq \lceil \frac{5n+1}{4n} \rceil = 2$. To prove the reverse inequality, we consider the following cases.

Case 1 n is even.
Let $S_1 = \{v_i : i = 2k+1, 0 \leq k < \lfloor \frac{n}{3} \rfloor\} \cup \{e_{n-3}, e_{n-2}^{'}, e_{n-2}^{''}\}$ and $S_2 = \{v_i : i = 2k, 1 \leq k \leq \lfloor \frac{n}{3} \rfloor\} \cup \{e_{n-2}, e_{n-1}^{'}, e_{n-1}^{''}\}$. Clearly $\{S_1, S_2\}$ is a boundary domatic partition of $M(G_n)$ so that $d_b^{'}(M(G_n) \geq 2$.

Case 2 n is odd.
Let $S_1 = \{v_i : i = 2k+1, 0 \leq k \leq \lfloor \frac{n}{3} \rfloor\} \cup \{e_{n-3}, e_{n-2}^{'}, e_{n-2}^{''}\}$ and $S_2 = \{v_i : i = 2k, 1 \leq k \leq \lceil \frac{n}{3} \rceil\} \cup \{e_{n-2}, e_{n-1}^{'}, e_{n-1}^{''}\}$. Clearly $\{S_1, S_2\}$ is a boundary domatic partition of $M(G_n)$ so that $d_b^{'}(M(G_n) \geq 2$. Thus $d_b^{'}(M(G_n) = 2$

□

THEOREM 22. *For a helm graph H_n, $d_b^{'}(H_n) = n$.*

PROOF. The proof is similar to the proof of Theorem 2.15. □

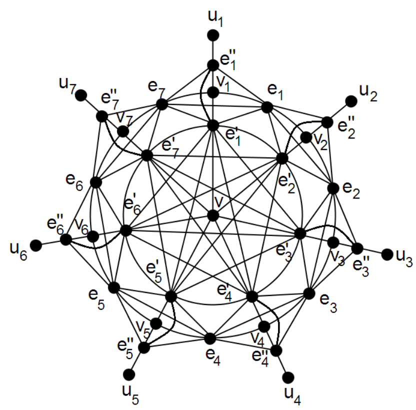

Fig. 7. $M(H_7)$

THEOREM 23. *For a helm graph H_n,*

$$d_b^{'}(M(H_n)) = \begin{cases} 5 & if\, n = 5 \\ 3 & if\, n = 7\ or\ 9 \\ 2 & otherwise \end{cases}.$$

.

PROOF. The result is obvious if $n = 5, 7$ or 9. In otherwise, by the definition of middle graph, $V(M(H_n)) = V(H_n) \cup E(H_n), |V(M(H_n))| = 5n + 1$ in which for each $i, (1 \leq i \leq n)$, the set of vertices $\{e_i, e_{i+1}, e_{i+1}^{'}, e_{i+1}^{''}, v_{i+1}$: subscript modulo $n\}$ induce a clique of order 5. Also $\{e_i^{'} : 1 \leq i \leq n\} \cup \{v\}$ induces a clique of order $n + 1$ (say K_{n+1}). Since $deg_b(u_i) = 4$ and $|N_b(u_i) : 1 \leq i \leq n| = 3n + 1$, then $d_b^{'}(M(H_n) \leq \lceil \frac{5n+1}{3n+1} \rceil = 2$. To prove the reverse inequality, we consider the following cases.

Case 1 n is even.
Let $S_1 = \{e_i : i = 2k+1, 0 \leq k \leq \lceil \frac{n}{2} \rceil + 1\} \cup \{e_{n-1}^{'}, v_n\}$ and $S_2 = \{e_i : i = 2k, 1 \leq k \leq n-2\} \cup \{v_1, e_n^{'}\}$. Clearly $\{S_1, S_2\}$ is a boundary domatic partition of $M(H_n)$ so that $d_b^{'}(M(H_n) \geq 2$.

Case 2 n is odd.
Let $S_1 = \{e_i : i = 2k+1, 0 \leq k \leq \lceil \frac{n}{2} \rceil + 2\} \cup \{u_{n-2}, e_{n-1}^{'}, v_n\}$ and $S_2 = \{e_i : i = 2k, 1 \leq k \leq n-3\} \cup \{u_{n-1}, e_n^{'}, v_1\}$. Clearly $\{S_1, S_2\}$ is a boundary domatic partition of $M(H_n)$ so that $d_b^{'}(M(H_n) \geq 2$. Thus $d_b^{'}(M(H_n) = 2$

□

3. CONCLUSION

In this paper we computed the exact value of the boundary domination number and the boundary domatic number for the Wheel Graph Families, line graph of Wheel Graph Families and middle of Wheel Graph Families .

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