# Relative Asymptotic Regularity and Fixed Points in Fuzzy 2-Metric Spaces

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## ABSTRACT
In this paper, we established fixed point theorems for two and three self-maps of a complete fuzzy 2-metric space. The contractive definition is a generalization of Hardy-Rogers and the commuting condition of Jungck is replaced by the concept of weakly commuting. The notion of relative asymptotic regularity of a sequence in a fuzzy 2-metric space is introduced and fixed point theorems for two and three self-mappings of a complete fuzzy 2-metric space is proved. Further, a result for a pair of weakly commuting mappings and relative asymptotically regular sequence is presented in complete fuzzy 2-metric space.

## Keywords
Fixed point, weakly commuting mapping, asymptotically regular mapping.

## AMS Subject Classification
54H25; 47H10; 47S40; 54A40

## 1. INTRODUCTION
The notion of fuzzy set was introduced by Zadeh $[15]$ in $1965$ To use the concept in topology and analysis, many authors have extensively introduced the theory of fuzzy sets and applications. The fuzzy metric space was introduced by Kramosil and Michalek $[10]$ in $1975$. Grabiec $[16]$ stated the contraction principle in fuzzy metric spaces in $1988$. Moreover, George and Veeramani $[2]$ modified the notion of fuzzy metric spaces with the help of t-norms in $1994$. Gahler $[19]$ introduced 2-metric spaces in a series of his papers. Sharma, Sharma and Ieski $[12]$ investigated for the first time, contractive type mappings in 2-metric spaces. In 1982 Fisher $[6]$ studied some related fixed point theorems on two metric spaces. Since many authors such as Aliouche and Fisher $[1]$, R. K. Namdeo, S. Jain, B. Fisher $[18]$, Telci $[17]$ and others proved some related fixed point theorems in 2-metric and fuzzy metric spaces. S. Sharma $[21]$ proved a common fixed point theorem for three mappings in fuzzy 2-metric spaces. We are following the notion of relative asymptotic regularity of a sequence in a metric space is introduced by Rhoades-Sessa-Khan-Khan $[4]$ in $1984$.

In this paper, we prove fixed point theorems for three self maps in fuzzy 2-metric space and we use the asymptotic regularity of sequence in a fuzzy 2-metric space.

## 2. PRELIMINARIES
### 2.1. Definition
**(**Schweizer and Sklar $[22]$**)** A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if $*$ satisfies the following conditions

**[B.1]** $*$ is commutative and associative

**[B.2]** $*$ is continuous

**[B.3]** $a*1=a \ \forall\ a\in[0,1]$

**[B.4]** $a*b\le c*d$ whenever $a\le c, b\le d$ and $a,b,c,d\in[0,1]$.

**2.2. Definition**
**(**A. George and P. Veeramani $[2]$**)** the $3-$tuple $(X,M,*)$ is called a fuzzy metric space if $X$ is an arbitrary non –empty set, $*$ is a continuous t-norm and $M$ is a fuzzy metric in $X^2\times[0,\infty]\to[0,1]$, satisfying the following conditions: for all $x,y,z\in X$, and $t,s>0$.

**[FM.1]** $M(x,y,0)=0$

**[FM.2]** $M(x,y,t)=1\ \forall\ \ t>0$ if and only if $x=y$.

**[FM.3]** $M(x,y.t)=M(y,x,t)$

**[FM.4]** $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$

**[FM.5]** $M(x,y,\square):[0,\infty]\to[0,1]$ is left continuous

**[FM.6]** $\lim_{n\to\infty} M(x,y,t)=1.$

**2.3. Definition (**A. George and P. Veeramani $[2]$**)** Let $(X,M,*)$ be a fuzzy metric space and let a sequence $\{x_n\}$ in $X$ is said to be converge to $x\in X$ if $\lim_{n\to\infty} M(x_n,x,t)=1,$ for each $t>0$.

**2.4. Definition (**A. George and P. Veeramani $[2]$**)** A sequence $\{x_n\}$ in $X$ is called Cauchy sequence if $\lim_{n\to\infty} M(x_n,x_{n+p},t)=1,$ for each $t>0,$ and $p=1,2,3,\cdots$

**2.5. Definition (**A. George and P. Veeramani $[2]$**)** A fuzzy metric space $(X,M,*)$ is said to be complete if every Cauchy sequence in $X$ is convergent in $X$.

A fuzzy metric space in which every Cauchy sequence is convergent is called complete. It is called compact if every sequence contains a convergent subsequence.

**2.6. Definition (**A. George and P. Veeramani [2]**)** A self mapping $T: X \to X$ is called fuzzy contractive mapping if $M(Tx,Ty,t) > M(x,y,t)$ for each $x \neq y \in X$, and $t > 0$.

**2.7. Definition** A binary operation

$*:[0,1]\times[0,1]\times[0,1]\to[0,1]$ is called a continuous t-norm if $([0,1],*)$ is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 \le a_2 * b_2 * c_2$ when $a_1 \le a_2, b_1 \le b_2, c_1 \le c_2$ for $a_1, a_2, b_1, b_2$ and $c_1, c_2$ are in $[0,1]$.

**2.8. Definition** The 3-tuple $(X, M, *)$ is called a fuzzy 2-metric space if $X$ is an arbitrary set, $*$ is continuous t-norm and $M$ is a fuzzy set in $X^3 \times [0,\infty)$ satisfying the following conditions; for all $x,y,z,u \in X$ and $t_1,t_2,t_3 > 0$

**[FM-1]** $M(x,y,z,0) = 0$

**[FM-2]** $M(x,y,z,t) = 1$ and when at least two of the three points are equal

**[FM-3]** $M(x,y,z,t) = M(x,z,y,t) = M(y,z,x,t)$

(Symmetry about three variables)

**[FM-4]** $$M(x,y,z,t_1+t_2+t_3) \ge M(x,y,u,t_1) * M(x,u,z,t_2) * M(u,y,z,t_3)$$

(This corresponds to tetrahedron inequality in 2-metric space)

The function value $M(x,y,z,t)$ may be interpreted as the probability the area of triangle is less than 1.

**[FM-5]** $M(x,y,z,\cdot):[0,1)\to[0,1]$ is left continuous.

**2.9. Definition** Let $(X, M, *)$ be a fuzzy 2-metric space, then

**(I)** A sequence $\{x_n\}$ in fuzzy 2-metric space $X$ is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, a, t) = 1$ for all $a \in X$ and $t > 0$.

**(II)** A sequence $\{x_n\}$ in fuzzy 2-metric space $X$ is called Cauchy sequence, if

$\lim_{n\to\infty} M(x_{n+p}, x_n, a, t) = 1$ for all $a \in X$ and $t > 0, p > 0$.

**(III)** A fuzzy 2-metric space in which every Cauchy sequence is convergent is said to be complete.

**2.10. Definition** A function $M$ is continuous in fuzzy 2-metric space if and only if whenever $x_n \to x, y_n \to y$ then $\lim_{n\to\infty} M(x_n, y_n, a, t) = M(x,y,a,t)$ for all $a \in X$ and $t > 0$.

**2.11. Definition** Two mapping $P$ and $S$ in fuzzy 2-metric space $X$ are weakly commuting if and only if $M(PSu, SPu, a, t) \ge M(Pu, Su, a, t)$ for all $u, a \in X$ and $t > 0$.

**2.11. Definition** Let $(X, M, *)$ be a fuzzy 2-metric space and $P$ and $S$ be self-maps of $X$. A sequence $\{x_n\}$ in $X$ is said to be asymptotically regular with respect to $\{P, S\}$ if $\lim_{n\to\infty} M(Px_n, Sx_n, a, t) = 1$ for all $a \in X$ and $t > 0$.

## 3. MAIN RESULT

**Theorem 3.1** Let $P, S$ be self mappings of a complete fuzzy 2-metric space $X$ satisfying

(i) $$M(Px,Py,a,t) \ge a_1 M(Sx,Px,a,\tfrac{t}{5}) + a_2 M(Sy,Py,a,\tfrac{t}{5}) + a_3 M(Sx,Py,a,\tfrac{t}{5}) + a_4 M(Sy,Px,a,\tfrac{t}{5}) + a_5 M(Sx,Sy,a,\tfrac{t}{5})$$

for all $x, y, a \in X$

where the $a_i = a_i(x,y) \ge 0 \quad (i = 1,\cdots,5)$

(ii) $\{P, S\}$ is a weekly commuting pair

(iii) There exists a sequence $\{x_n\}$ in $X$ which is asymptotically regular with respect to $\{P, S\}$

(iv) $S$ and $P$ are continuous

Then $P$ and $S$ have a unique common fixed point.

**Theorem 3.2** Let $P, S, T$ be self mappings of a complete fuzzy 2-metric space $X$ satisfying

(3.1) $$M(Px,Py,a,t) \ge a_1 M(Sx,Px,a,\tfrac{t}{4}) + a_2 M(Tx,Px,a,\tfrac{t}{4}) + a_3 M(Sy,Py,a,\tfrac{t}{4}) + a_4 M(Ty,Py,a,\tfrac{t}{4})$$ for all $x, y, a \in X$

where $a_1, a_2, a_3, a_4$ are non-negative bounded numbers such that $a_2 + a_3 \le 1$ and $a_3 + a_4 \le 1$.

(3.2) $\{P, S\}$ and $\{P, T\}$ are weakly commuting pairs .

(3.3) There exists a sequence $\{x_n\}$ in $X$ which is asymptotically regular with respect to $\{P, S\}$ and $\{P, T\}$ both.

(3.4) $S$ and $T$ are continuous

if $M$ is continuous then $P, S$ and $T$ have a unique common fixed point.

**Proof** Let $\{x_n\}$ satisfying $(3.2)$ , then from $(3.1)$

$$M(Px_n,Px_m,a,t) \ge a_1 M(Sx_n,Px_n,a,\tfrac{t}{4}) + a_2 M(Tx_n,Px_n,a,\tfrac{t}{4}) + a_3 M(Sx_m,Px_m,a,\tfrac{t}{4}) + a_4 M(Tx_m,Px_m,a,\tfrac{t}{4})$$

making $m, n \to \infty$

$$\lim_{n\to\infty} M(Px_n, Px_m, a, t) = 1$$

and this is true for every $a \in X$ . Hence $\{Px_n\}$ is Cauchy sequence and so convergent. Call the limit $z$ . Also

$$M(Qx_n,z,a,t) \ge M(Qx_n,Px_n,a,\tfrac{t}{3}) + M(Px_n,z,a,\tfrac{t}{3}) + M(Qx_n,Px_n,z,\tfrac{t}{3}) \to 1 \text{ as } n\to\infty.$$

So, $Qx_n \to z.$ Similarly $Tx_n \to z.$

The continuity of $S$ and $T$ implies

$SPx_n \to Sz,\ S^2x_n \to Sz,\ STx_n \to Sz,\ TPx_n \to Tz,$
$T^2x_n \to Tz$ and $TSx_n \to Tz.$

From (3.1)

(3.5) $M(RSx_n, PTx_n, a, t) \geq a_1 M(S^2x_n, PSx_n, a, \frac{t}{4})$
$+ a_2 M(TSx_n, PSx_n, a\frac{t}{4}) + a_3 M(STx_n, PTx_n, a, \frac{t}{4})$
$+ a_4 M(T^2x_n, PTx_n, a, \frac{t}{4})$

$$M(PSx_n, Sz, a, t) \geq M(PSx_n, SPx_n, a, \frac{t}{3}) + M(SPx_n, Sz, a, \frac{t}{3})$$
$$+ M(PSx_n, SPx_n, Sz, \frac{t}{3})$$
$$\geq M(Sx_n, Px_n, a, \frac{t}{3}) + M(SPx_n, Sz, a, \frac{t}{3})$$
$$+ M(Sx_n, Px_n, Sz, \frac{t}{3}) \to 1 \text{ as } n \to \infty.$$

So $PSx_n \to Sz.$ Similarly $PTx_n \to Tz.$ Further

(3.6) $M(STx_n, TSx_n, a, t) \geq M(STx_n, PSx_n, a, \frac{t}{3})$
$+ M(STx_n, TSx_n, PSx_n, \frac{t}{3}) + M(PSx_n, TSx_n, a, \frac{t}{3})$

$$M(STx_n, TSx_n, a, t) \geq M(STx_n, PSx_n, a, \frac{t}{3})$$
$$+ M(STx_n, TSx_n, PSx_n, \frac{t}{3}) + M(PSx_n, PTx_n, a, \frac{t}{9})$$
$$+ M(PTx_n, TSx_n, a, \frac{t}{9}) + M(PSx_n, TSx_n, PTx_n, \frac{t}{9})$$

Using (3.5) and making $n \to \infty$,(3.6) obtains

$$M(Sz, Tz, a, t) \geq \frac{1}{3} + \frac{1}{3} + a_2 M(Tz, Sz, a, \frac{t}{36})$$
$$+ a_3 M(Sz, Tz, a, \frac{t}{36}) + \frac{1}{9} + \frac{1}{9}$$
$$\geq \frac{2}{3} + \frac{2}{9} + (a_2 + a_3) M(Tz, Sz, a, \frac{t}{36}) \to 1$$

therefore $Sz = Tz.$

From (3.1)

$$M(PTx_n, Pz, a, t) \geq a_1 M(STx_n, PTx_n, a, \frac{t}{4})$$
$$+ a_2 M(T^2x_n, PTx_n, a, \frac{t}{4}) + a_3 M(Sz, Pz, a, \frac{t}{4})$$
$$+ a_4 M(Tz, Pz, a, \frac{t}{4})$$

Making $n \to \infty$

$$M(Tz, Pz, a, t) \geq (a_3 + a_4) M(Tz, Pz, a, \frac{t}{4}) .$$

Yielding $Pz = Tz.$

Again from (3.1)

$$M(PPz, Pz, a, t) \geq a_1 M(SPz, PPz, a, \frac{t}{4})$$
$$+ a_2 M(TPz, PPz, a, \frac{t}{4}) + a_3 M(Sz, Pz, a, \frac{t}{4})$$
$$+ a_4 M(Tz, Pz, a, \frac{t}{4})$$
$$= a_1 M(SPz, Psz, a, \frac{t}{4}) + a_2 M(TPz, Ptz, a, \frac{t}{4})$$
$$\geq a_1 M(Sz, Pz, a, \frac{t}{4}) + a_2 M(Tz, Pz, a, \frac{t}{4}) \to 1$$

(Since $\{P, S\}$ and $\{P, T\}$ are weakly commuting pairs)

Hence $PPz = Pz$ i.e. $Pz$ is a fixed point of $P$ . Set $Pz = u.$ Then $PSz = PPz = Pz = u$ and

$$M(Su, u, a, t) = M(SPz, u, a, t)$$
$$\geq M(SPz, PSz, a, \frac{t}{3}) + M(SPz, Psz, u, \frac{t}{3})$$
$$+ M(PSz, u, a, \frac{t}{3})$$
$$\geq M(Sz, Pz, a, \frac{t}{3}) + M(Pz, Sz, a, \frac{t}{3})$$
$$+ M(u, u, a, \frac{t}{3}) \to 1$$

So $Su = u.$ Similarly $Tu = u.$ Thus $Pu = Su = Tu = u.$

To prove the uniqueness of $u$ as a common fixed point of $P, S$ and $T$ . Let $v$ be another common fixed point of $P, S$ and $T$ . Then

$$M(Pu, Pv, a, t) \geq a_1 M(Su, Pu, a, \frac{t}{4}) + a_2 M(Tu, Pu, a, \frac{t}{4})$$
$$+ a_3 M(Sv, Pv, a, \frac{t}{4}) + a_4 M(Tv, Pv, a, \frac{t}{4}) \to 1$$

Hence $u = v.$

## 4. CONCLUSION

In this paper, we have proved some fixed point theorems for three self maps in fuzzy 2-metric space and we have used the asymptotic regularity of sequence in a fuzzy 2-metric space.

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