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# Asymptotic Behavior of some Rational Difference Equations

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## ABSTRACT
In this difference equation, Stability, Periodicity, boundedness, global Stability.

We investigate some qualitative behavior of the solutions of the difference equation

$x_{n+1} = ax_{n-\ell} + \sum_{i=0}^{k} \alpha_i x_{n-i} / \sum_{i=0}^{k} \beta_i x_{n-i}$ where the the initial conditions $x_{-r}, x_{-r+1}, \ldots, x_0$ are arbitrary positive real numbers such that $r = \max\{\ell, k\}$ where $i, r \in \{0, 1, \ldots\}$ and $a, \alpha_i, \beta_i$ are positive constants.

## Keywords
difference equation, Stability, Periodicity, boundedness.

## 1. INTRODUCTION
In this paper we deal with some properties of the solutions of the difference equation

$$x_{n+1} = ax_{n-\ell} + \frac{\sum_{i=0}^{k} \alpha_i x_{n-i}}{\sum_{i=0}^{k} \beta_i x_{n-i}}, \quad n = 0, 1, 2, \ldots, (1.1)$$

where the the initial conditions $x_{-r}, x_{-r+1}, \ldots, x_0$ are arbitrary positive real numbers such that $r = \max\{\ell, k\}$ where $i, r \in \{0, 1, \ldots\}$ and $a, \alpha_i, \beta_i$ are positive constants. There is a class of nonlinear difference equations, known as the rational difference equations, each of which consists of the ratio of two polynomials in the sequence terms in the same from. there has been a lot of work concenring the global asympototic of solutions of rational difference equations [2], [3], [4], [7], [8], [11] and [12].

Many reseaches have investigated the behavior of the solution of difference equation for example:

Kulenovic1 et al.[13] has studied the global asymptotic stability of solutions of the equation

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1}}{\gamma x_n + \delta x_{n-1}}.$$

M. Saleh et al.[15] investigated the periodic character and the global stability of all positive solutions of the equation

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}.$$

Our aim in this paper is to extend and generalize the work in [1], [6], [10], [13], [15], [16] and [17]. That is, we will investigate the global behavior of (1.1) including the asymptotical stability of equilibrium points, the existence of bounded solution, the existence of period two solution and investigate the oscillation property of the recursive sequence of Eq. (1.1).

Now we recall some well-known results, which will be useful in the investigation of (1.1) and which are given in [9].

Let I be an interval of real numbers and let

$$F : I^{k+1} \to I,$$

where $F$ is a continuous function. Consider the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, 2, \ldots, \text{(1.2)}$$

with the initial condition $x_{-k}, x_{-k+1}, \ldots, x_0 \in I.$

**Definition 1** (Equilibrium Point)
A point $\bar{x} \in I$ is called an equilibrium point of Eq. (1.2) if

$$\bar{x} = f(\bar{x}, \bar{x}, \ldots, \bar{x}).$$

That is, $x_n = \bar{x}$ for $n \geq 0,$ is a solution of Eq. (1.2), or equivalently, $\bar{x}$ is a fixsed point of $f$.

**Definition 2** (Stability)
Let $\bar{x} \in (0, \infty)$ be an equilibrium point of Eq. (1.2). Then

i) An equilibrium point $\bar{x}$ of Eq. (1.2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x_{-r}, x_{-r+1}, \ldots, x_0 \in (0, \infty)$ with $|x_{-r} - \bar{x}| + |x_{-r+1} - \bar{x}| + \ldots + |x_0 - \bar{x}| < \delta,$ then $|x_n - \bar{x}| < \varepsilon$ for all $n \geq -r$ .

ii) An equilibrium point $\bar{x}$ of Eq. (1.2) is called locally asymptotically stable if $\bar{x}$ is locally stable and there exists

$\gamma > 0$ such that, if $x_{-r}, x_{-r+1}, ..., x_0 \in (0,\infty)$ with $\left|x_{-r} - \bar{x}\right| + \left|x_{-r+1} - \bar{x}\right| + ... + \left|x_0 - \bar{x}\right| < \gamma,$ then

$$\lim_{n \to \infty} x_n = \bar{x}.$$

iii) An equilibrium point $\bar{x}$ of Eq. (1.2) is called a global attractor if for every $x_{-r}, x_{-r+1}, ..., x_0 \in (0,\infty)$ we have

$$\lim_{n \to \infty} x_n = \bar{x}.$$

iv) An equilibrium point $\bar{x}$ of Eq. (1.2) is called globally asymptotically stable if $\bar{x}$ is locally stable and a global attractor.

v) An equilibrium point $\bar{x}$ of Eq. (1.2) is called unstable if $\bar{x}$ is not locally stable.

**Definition 3** (Permanence)
Eq. (1.2) is called permanent if there exists numbers $m$ and $M$ with $0 < m < M < \infty$ such that for any initial conditions $x_{-r}, x_{-r+1}, ..., x_0 \in (0,\infty)$ there exists a positive integer $N$ which depends on the initial conditions such that

$$m \le x_n \le M \quad \text{forall } n \ge -N.$$

**Definition 4** (Periodicity)
A sequence $\{x_n\}_{n=-r}^{\infty}$ is said to be periodic with period $p$ if $x_{n+p} = x_n$ for all $n \ge -r.$ A sequence $\{x_n\}_{n=-r}^{\infty}$ is said to be periodic with prime period $p$ if $p$ is the smallest positive integer having this property.

The linearized equation of Eq. (1.2) about the equilibrium point $\bar{x}$ is defined by the equation

$$z_{n+1} = \sum_{i=0}^{k} p_i z_{n-i}, \qquad (1.3)$$

where

$$p_i = \frac{\partial F(\bar{x}, \bar{x}, ..., \bar{x},)}{\partial x_{n-i}}, \quad i = 0,1,...,k.$$

The characteristic equation associated with Eq. (1.3) is

$$\lambda^{k+1} - p_0\lambda^k - p_1\lambda^{k-1} - ... - p_{k-1}\lambda - p_k = 0. \qquad (1.4)$$

**Theorem 1.2** [9]. Assume that $F$ is a $C^1-$ function and let $\bar{x}$ be an equilibrium point of Eq. (1.2). Then the following statements are true:

i) If all roots of Eq. (1.4) lie in the open unit disk $|\lambda| < 1$, then he equilibrium point $\bar{x}$ is locally asymptotically stable.

ii) If at least one root of Eq. (1.4) has absolute value greater than one, then the equilibrium point $\bar{x}$ is unstable.

iii) If all roots of Eq. (1.4) have absolute value greater than one, then the equilibrium point $\bar{x}$ is a source.

**Theorem 1.3** [14] Assume that $p_i \in R, i = 1,2,...,k.$ Then

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptoticcally stable of Eq. (1.5)

$$y_{n+k} + p_1 y_{n+k-1} + ... + p_k y_n = 0, \; n = 0,1,...(1.5) \; .$$

## 2. LOCAL STABILITY OF THE EQUILIBRIUM POINT OF EQ. (1.1)

In this section we investigate the local stability character of the solutions of Eq. (1.1). Eq. (1.1) has a unique nonzero equilibrium point

$$\bar{x} = a\bar{x} + \frac{\sum_{i=0}^{k} \alpha_i \bar{x}}{\sum_{i=0}^{k} \beta_i \bar{x}},$$

if $1 > a,$ of the Eq. (1) has only positive equilibrium point $\bar{x}$ is given by

$$\bar{x} = \frac{\sum_{i=0}^{k} \alpha_i}{\sum_{i=0}^{k} \beta_i (1-a)}.$$

Let

$$G = \sum_{i=0}^{k} \alpha_i \;, \quad Q = \sum_{i=0}^{k} \beta_i \;,$$

$$G^i = \sum_{\substack{j=0 \\ j \ne i}}^{k} \alpha_j \quad \text{and} \quad Q^i = \sum_{\substack{j=0 \\ j \ne i}}^{k} \beta_j \; .$$

Then, we get

$$\bar{x} = \frac{G}{Q(1-a)}.$$

Let $f : (0,\infty)^{k+1} \to (0,\infty)$ be a function defined by

$$f(u_0, u_1, ..., u_k, v) = av + \frac{\sum_{i=0}^{k} \alpha_i u_i}{\sum_{i=0}^{k} \beta_i u_i}. \qquad (2.1)$$

Therefore it follows that

$$\frac{\partial f(u_0,u_1,...,u_k,v)}{\partial v}=a,$$

$$\frac{\partial f(u_0,u_1,...,u_k,v)}{\partial u_0}=\frac{\alpha_0\left(\sum_{i=1}^{k}\beta_i u_i\right)-\beta_0\left(\sum_{i=1}^{k}\alpha_i u_i\right)}{\left[\sum_{i=0}^{k}\beta_i u_i\right]^2},$$

$$\frac{\partial f(u_0,u_1,...,u_k,v)}{\partial u_1}=\frac{\alpha_1\left(\sum_{i=0,i\neq 1}^{k}\beta_i u_i\right)-\beta_1\left(\sum_{i=0,i\neq 1}^{k}\alpha_i u_i\right)}{\left[\sum_{i=0}^{k}\beta_i u_i\right]^2}$$

.
.
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$$\frac{\partial f(u_0,u_1,...,u_k,v)}{\partial u_j}=\frac{\alpha_j\left(\sum_{i=0,i\neq j}^{k}\beta_i u_i\right)-\beta_j\left(\sum_{i=0,i\neq j}^{k}\alpha_i u_i\right)}{\left[\sum_{i=0}^{k}\beta_i u_i\right]^2},$$

.
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and

$$\frac{\partial f(u_0,u_1,...,u_k,v)}{\partial u_k}=\frac{\alpha_k\left(\sum_{i=0}^{k-1}\beta_i u_i\right)-\beta_k\left(\sum_{i=0}^{k-1}\alpha_i u_i\right)}{\left[\sum_{i=0}^{k}\beta_i u_i\right]^2}.$$

Then we see that

$$\frac{\partial f(\bar{x},\bar{x},...,\bar{x},\bar{x})}{\partial v}=a,$$

$$\frac{\partial f(\bar{x},\bar{x},...,\bar{x},\bar{x})}{\partial u_0}=\frac{\left(\alpha_0\sum_{i=1}^{k}\beta_i-\beta_0\sum_{i=1}^{k}\alpha_i\right)(1-a)}{QG},$$

$$\frac{\partial f(\bar{x},\bar{x},...,\bar{x},\bar{x})}{\partial u_1}=\frac{\left(\alpha_1\sum_{i=0,i\neq 1}^{k}\beta_i-\beta_1\sum_{i=0,i\neq 1}^{k}\alpha_i\right)(1-a)}{QG},$$

.
.
.

$$\frac{\partial f(\bar{x},\bar{x},...,\bar{x},\bar{x})}{\partial u_j}=\frac{\left(\alpha_j\sum_{i=0,i\neq j}^{k}\beta_i-\beta_j\sum_{i=0,i\neq j}^{k}\alpha_i\right)(1-a)}{QG},$$

.
.
.

and

$$\frac{\partial f(\bar{x},\bar{x},...,\bar{x},\bar{x})}{\partial u_k}=\frac{\left(\alpha_k\sum_{i=0}^{k-1}\beta_i-\beta_k\sum_{i=0}^{k-1}\alpha_i\right)(1-a)}{QG}.$$

Then the linearized equation of (1.1) about $\bar{x}$ is

$$z_{n+1}=\sum_{i=0}^{k}p_i z_{n-i}. \qquad (2.2)$$

**Theorem 2.1** Assume that

$$\sum_{i=0}^{k}\left(\alpha_i Q^i-\beta_i G^i\right)<GQ.$$

Than the equilibrium point of Eq. (1.1) is locally stable.

**Proof.** It is follows by Theorem(1.3) that, Eq. (2.2) is locally stable if

$$|p_k|+...+|p_j|+...+|p_1|+|p_0|<1.$$

That is

$$|a|+\sum_{i=0}^{k}\left|\frac{\left(\alpha_i Q^i-\beta_i G^i\right)(1-a)}{QG}\right|<1.$$

If

$$\alpha_i Q^i>\beta_i G^i,$$

this implies that

$$aQG+\sum_{i=0}^{k}\left(\alpha_i Q^i-\beta_i G^i\right)(1-a)<QG.$$

Thus

$$\sum_{i=0}^{k}\left(\alpha_i Q^i-\beta_i G^i\right)<GQ.$$

Hence, the proof is completed.

**Example 2.1** Consider the difference equation

$$x_{n+1}=0.5x_{n-1}+\frac{0.125x_n+0.25x_{n-1}}{0.5x_n+0.5x_{n-1}},$$

where

$$k=1,\ell=1,a=0.5,\alpha_0=0.125,\alpha_1=0.25,$$
$$\beta_0=0.5,\beta_1=0.5.$$

Figure(2.1), shows that the equilibrium point of Eq. (1.1) has locally stable, with initial data $x_{n-1} = 0.1, x_0 = 6.1.$

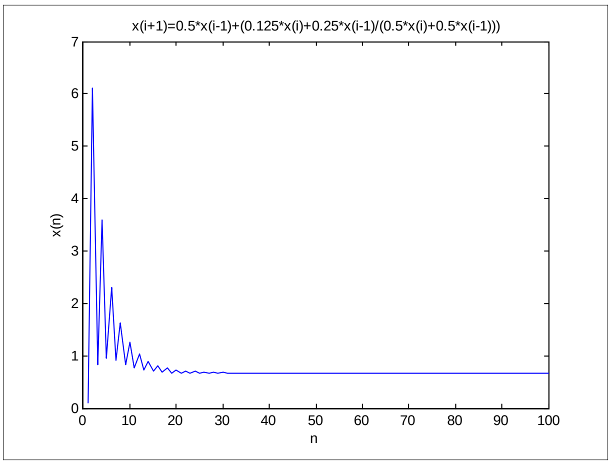


**Fig 2.1**

## 3. PERIODIC SOLUTIONS OF EQ. (1.1)

In this section we investigate the periodic character of the positive solutions of Eq. (1.1).

**Theorem 3.1** Eq. (1.1) has positive prime priod two solution If

$$(i)\ \ \ell - odd, k - odd\ \ and \quad (\delta-\gamma)(\mu-\lambda)(1-a) > 4\gamma, \tag{3.1}$$

$$(ii)\ \ \ell - odd, k - even\ \ and \quad (\delta-\gamma)(\mu-\lambda)(1-a) > 4\gamma,$$

$$(iii)\ \ \ell - even, k - odd\ \ and \quad (\delta-\gamma)(\mu-\lambda)(1+a) > 4(\delta a\mu + \lambda\gamma), \tag{3.2}$$

$$(iv)\ \ \ell - even, k - even\ \ and \quad (\delta-\gamma)(\mu-\lambda)(1+a) > 4(\delta a\mu + \lambda\gamma).$$

**Proof.** For case(i) assume that there exists a prime period-two solution

$$\ldots, p, q, p, q, \ldots$$

of (1.1). Let $x_n = q, x_{n+1} = p.$ Since $\ell - odd, k - odd$ we have $x_{n-\ell} = p, x_{n-k} = p$ . Thus, from Eq. (1.1), we get

$$p = ap + \frac{\alpha_0 q + \alpha_1 p + \ldots + \alpha_k p}{\beta_0 q + \beta_1 p + \ldots + \beta_k p},$$

and

$$q = aq + \frac{\alpha_0 p + \alpha_1 q + \ldots + \alpha_k q}{\beta_0 p + \beta_1 q + \ldots + \beta_k q}.$$

Let

$$\alpha_0 + \alpha_2 + \ldots + \alpha_{k-1} = \gamma,$$

and

$$\alpha_1 + \alpha_3 + \ldots + \alpha_k = \delta,$$

and let

$$\beta_0 + \beta_2 + \ldots + \beta_{k-1} = \mu,$$

and

$$\beta_1 + \beta_3 + \ldots + \beta_k = \lambda.$$

Than

$$p = ap + \frac{\gamma q + \delta p}{\mu q + \lambda p},$$

and

$$q = aq + \frac{\gamma p + \delta q}{\mu p + \lambda q}.$$

Than

$$\mu pq + \lambda p^2 = a\mu pq + a\lambda p^2 + \gamma q + \delta p, \tag{3.3}$$

and

$$\mu pq + \lambda q^2 = a\mu pq + a\lambda q^2 + \gamma p + \delta q. \tag{3.4}$$

Subtracting (3.3) from (3.4) gives

$$\lambda(1-a)(p^2 - q^2) = (\delta - \gamma)(p - q).$$

Since $p \neq q$ , we have

$$p + q = \frac{\delta - \gamma}{\lambda(1-a)}. \tag{3.5}$$

Also, since $p$ and $q$ are positive, $(\delta-\gamma), \lambda(1-a)$ shuold be positive. Again, adding (3.3) and (3.4) yields

$$2\mu pq + \lambda(p^2 + q^2) = 2a\mu pq + a\lambda(p^2 + q^2) + \gamma(p+q) + \delta(p+q). \tag{3.6}$$

It follows by (3.5), (3.6) and the relation

$$p^2 + q^2 = (p+q)^2 - 2pq, \quad \forall p, q \in \mathrm{R},$$

that

$$pq = \frac{\gamma(\delta-\gamma)}{\lambda^2(1-a)^3(\mu-\lambda)}. \tag{3.7}$$

It is clear now, from Eq. (3.5) and Eq. (3.7) that $p$ and $q$ are the two distinct roots of the quadratic equation

$$t^2 - \left(\frac{\delta-\gamma}{\lambda(1-a)}\right)t + \frac{\gamma(\delta-\gamma)}{\lambda^2(1-a)^3(\mu-\lambda)} = 0,$$

and so

$$\left(\frac{\delta-\gamma}{\lambda(1-a)}\right)^2 - \frac{4\gamma(\delta-\gamma)}{\lambda^2(1-a)^3(\mu-\lambda)} > 0,$$

which is equivalent to

$$(\delta-\gamma)(\mu-\lambda)(1-a) > 4\gamma.$$

The proof follows by induction. The cases where (ii), holds is similar and will be omitted.

For case (iii) assume that there exists a prime period-two solution

$$\ldots, p, q, p, q, \ldots$$

of (1.1). Let $x_n = q, x_{n+1} = p.$ Since $\ell - even, k - odd$ we have $x_{n-\ell} = q, x_{n-k} = p$ . Thus, from Eq. (1.1), we get

$$p = aq + \frac{\alpha_0 q + \alpha_1 p + ... + \alpha_k p}{\beta_0 q + \beta_1 p + ... + \beta_k p},$$

and

$$q = ap + \frac{\alpha_0 p + \alpha_1 q + ... + \alpha_k q}{\beta_0 p + \beta_1 q + ... + \beta_k q}.$$

Let where

$$\alpha_0 + \alpha_2 + ... + \alpha_{k-1} = \gamma,$$

and

$$\alpha_1 + \alpha_3 + ... + \alpha_k = \delta,$$

and let

$$\beta_0 + \beta_2 + ... + \beta_{k-1} = \mu,$$

and

$$\beta_1 + \beta_3 + ... + \beta_k = \lambda.$$

Than

$$p = aq + \frac{\gamma q + \delta p}{\mu q + \lambda p},$$

and

$$q = ap + \frac{\gamma p + \delta q}{\mu p + \lambda q}.$$

Than

$$\mu pq + \lambda p^2 = a\mu q^2 + a\lambda pq + \gamma q + \delta p, \tag{3.8}$$

and

$$\mu pq + \lambda q^2 = a\mu p^2 + a\lambda pq + \gamma p + \delta q. \tag{3.9}$$

Subtracting (3.8) from (3.9) gives

$$(\lambda + a\mu)(p^2 - q^2) = (\delta - \gamma)(p - q).$$

Since $p \neq q$ , we have

$$p + q = \frac{\delta - \gamma}{\lambda + a\mu}. \tag{3.10}$$

Also, since $p$ and $q$ are positive, $(\delta - \gamma)$ shuold be positive. Again, adding (3.8) and (3.9) yields

$$2\mu pq + \lambda(p^2 + q^2) = a\mu(p^2 + q^2) + 2a\gamma pq + \gamma(p+q) + \delta(p+q). \tag{3.11}$$

It follows by (3.10), (3.11) and the relation

$$p^2 + q^2 = (p+q)^2 - 2pq, \quad \forall p, q \in \mathrm{R},$$

that

$$pq = \frac{(\delta - \gamma)(\delta a\mu + \lambda\gamma)}{(1+a)(\mu - \lambda)(\lambda - a\mu)^2}. \tag{3.12}$$

It is clear now, from Eq. (3.10) and Eq. (3.12) that $p$ and $q$ are the two distinct roots of the quadratic equation

$$t^2 - \left(\frac{\delta - \gamma}{\lambda + a\mu}\right)t + \frac{(\delta - \gamma)(\delta a\mu + \lambda\gamma)}{(1+a)(\mu - \lambda)(\lambda - a\mu)^2} = 0,$$

and so

$$\left(\frac{\delta - \gamma}{\lambda + a\mu}\right)^2 - \frac{4(\delta - \gamma)(\delta a\mu + \lambda\gamma)}{(1+a)(\mu - \lambda)(\lambda - a\mu)^2} > 0,$$

which is equivalent to

$$(\delta - \gamma)(\mu - \lambda)(1+a) > 4(\delta a\mu + \lambda\gamma).$$

The proof follows by induction. The cases where (iv), holds is similar and will be omitted. Now the proof is completed.

**Example 3.1** Consider the difference equation

$$x_{n+1} = 0.5x_{n-1} + \frac{0.002x_n + 0.5x_{n-1}}{0.9x_{n-2} + 0.2x_{n-1}},$$

$$k - odd = 1, \ell - odd, a = 0.5, \alpha_0 = 0.002,$$
$$\alpha_1 = 0.5, \beta_0 = 0.9, \beta_1 = 0.2.$$

Figure(3.1), shows that Eq. (1.1) which is periodic with period two $x_{-1} = 0.7, x_0 = 5.3.$. Where the initial data satisfies condition(3.1) of Theorem(3.1) (seeTable 3.1)

| $n$ | $x(n)$ | $n$ | $x(n)$ | $n$ | $x(n)$ | $n$ | $x(n)$ | $n$ | $x(n)$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.7000 | 17 | 0.0392 | 33 | 0.0255 | 49 | 0.0253 | 65 | 0.0253 |
| 2 | 5.3000 | 18 | 4.7454 | 34 | 4.8832 | 50 | 4.8862 | 66 | 4.8863 |
| 3 | 0.4319 | 19 | 0.0337 | 35 | 0.0254 | 51 | 0.0253 | 67 | 0.0253 |
| 4 | 4.4801 | 20 | 4.7954 | 36 | 4.8844 | 52 | 4.8863 | 68 | 4.8863 |
| 5 | 0.2773 | 21 | 0.0303 | 37 | 0.0254 | 53 | 0.0253 | 69 | 0.0253 |
| 6 | 4.1959 | 22 | 4.8286 | 38 | 4.8852 | 54 | 4.8863 | 70 | 4.8863 |
| 7 | 0.1832 | 23 | 0.0283 | 39 | 0.0253 | 55 | 0.0253 | 71 | 0.0253 |
| 8 | 4.1877 | 24 | 4.8501 | 40 | 4.8856 | 56 | 4.8863 | 72 | 4.8863 |
| 9 | 0.1241 | 25 | 0.0271 | 41 | 0.0253 | 57 | 0.0253 | 73 | 0.0253 |
| 10 | 4.3000 | 26 | 4.8638 | 42 | 4.8859 | 58 | 4.8863 | 74 | 4.8863 |
| 11 | 0.0866 | 27 | 0.0264 | 43 | 0.0253 | 59 | 0.0253 | 75 | 0.0253 |
| 12 | 4.4425 | 28 | 4.8724 | 44 | 4.8861 | 60 | 4.8863 | 76 | 4.8863 |
| 13 | 0.0629 | 29 | 0.0259 | 45 | 0.0253 | 61 | 0.0253 | 77 | 0.0253 |
| 14 | 4.5715 | 30 | 4.8778 | 46 | 4.8862 | 62 | 4.8863 | 78 | 4.8863 |
| 15 | 0.0482 | 31 | 0.0257 | 47 | 0.0253 | 63 | 0.0253 | 79 | 0.0253 |
| 16 | 4.6725 | 32 | 4.8811 | 48 | 4.8862 | 64 | 4.8863 | 80 | 4.8863 |

**Table 3.1**

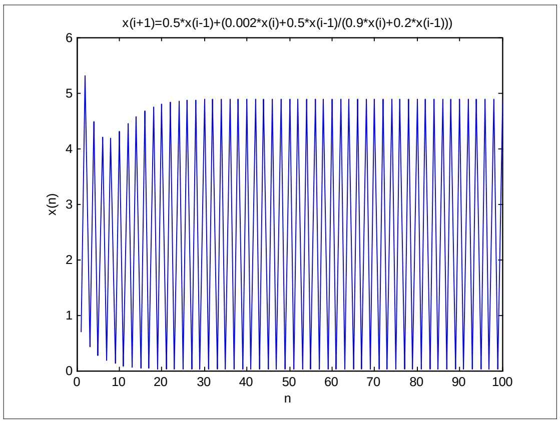


**Fig3.2**

**Example 3.2** Consider the difference equation

$$x_{n+1} = 0.005x_{n-2} + \frac{0.1x_n + 5x_{n-1}}{9x_{n-2} + 0.02x_{n-1}},$$

where
$$k - odd = 1, \ell - even, a = 0.005, \alpha_0 = 0.1,$$
$$\alpha_1 = 5, \beta_0 = 9, \beta_1 = 0.02.$$

Figure(3.2), shows that Eq. (1.1) which is periodic with period two. Where the initial data satisfies condition(3.2) of Theorem(3.1) $x_{n-2} = 6.7, x_{-1} = 3.9, x_0 = 2.9.$

(seeTable 3.2)

| $n$ | $x(n)$ | $n$ | $x(n)$ | $n$ | $x(n)$ | $n$ | $x(n)$ | $n$ | $x(n)$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 6.7000 | 17 | 4.6513 | 33 | 4.6290 | 49 | 4.6290 | 65 | 4.6290 |
| 2 | 3.9000 | 18 | 0.5565 | 34 | 0.5523 | 50 | 0.5523 | 66 | 0.5523 |
| 3 | 2.9000 | 19 | 4.6174 | 35 | 4.6290 | 51 | 4.6290 | 67 | 4.6290 |
| 4 | 1.0684 | 20 | 0.5519 | 36 | 0.5523 | 52 | 0.5523 | 68 | 0.5523 |
| 5 | 1.6253 | 21 | 4.6209 | 37 | 4.6290 | 53 | 4.6290 | 69 | 4.6290 |
| 6 | 0.5417 | 22 | 0.5515 | 38 | 0.5523 | 54 | 0.5523 | 70 | 0.5523 |
| 7 | 1.7153 | 23 | 4.6276 | 39 | 4.6290 | 55 | 4.6290 | 71 | 4.6290 |
| 8 | 0.3550 | 24 | 0.5521 | 40 | 0.5523 | 56 | 0.5523 | 72 | 0.5523 |
| 9 | 2.6942 | 25 | 4.6297 | 41 | 4.6290 | 57 | 4.6290 | 73 | 4.6290 |
| 10 | 0.3512 | 26 | 0.5523 | 42 | 0.5523 | 58 | 0.5523 | 74 | 0.5523 |
| 11 | 4.2277 | 27 | 4.6295 | 43 | 4.6290 | 59 | 4.6290 | 75 | 4.6290 |
| 12 | 0.4824 | 28 | 0.5524 | 44 | 0.5523 | 60 | 0.5523 | 76 | 0.5523 |
| 13 | 4.8260 | 29 | 4.6291 | 45 | 4.6290 | 61 | 4.6290 | 77 | 4.6290 |
| 14 | 0.5593 | 30 | 0.5523 | 46 | 0.5523 | 62 | 0.5523 | 78 | 0.5523 |
| 15 | 4.7622 | 31 | 4.6290 | 47 | 4.6290 | 63 | 4.6290 | 79 | 4.6290 |
| 16 | 0.5656 | 32 | 0.5523 | 48 | 0.5523 | 64 | 0.5523 | 80 | 0.5523 |

**Fig3.2**

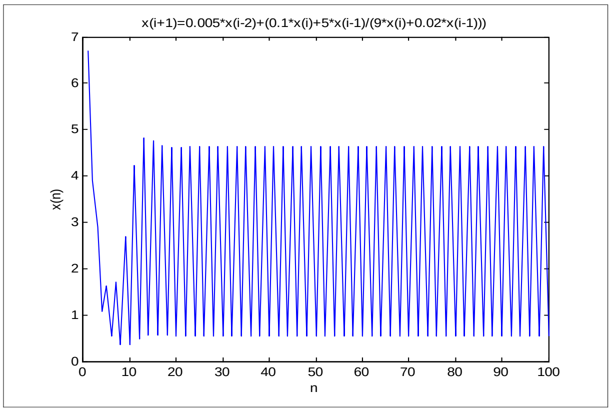


**Fig3.3**

## 4. BOUNDED SOLUTIONS OF EQ. (1.1)

Our aim in this section we investigate the boundedness of the positive solutions of Eq. (1.1).

**Theorem 4.1** For Eq. (1.1) every solution is bounded if $1 > a.$

**Proof.** Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq. (1.1). It follows from Eq. (1.1) that

$$\begin{aligned} x_{n+1} &= ax_{n-\ell} + \frac{\sum_{i=0}^{k}\alpha_i x_{n-i}}{\sum_{i=0}^{k}\beta_i x_{n-i}} \\ &= ax_{n-\ell} + \frac{\alpha_0 x_n}{\sum_{i=0}^{k}\beta_i x_{n-i}} + \frac{\alpha_1 x_{n-1}}{\sum_{i=0}^{k}\beta_i x_{n-i}} + \ldots + \frac{\alpha_k x_{n-k}}{\sum_{i=0}^{k}\beta_i x_{n-i}} \\ &\le ax_{n-\ell} + \frac{\alpha_0 x_n}{\beta_0 x_n} + \frac{\alpha_1 x_{n-1}}{\beta_1 x_{n-1}} + \ldots + \frac{\alpha_k x_{n-k}}{\beta_k x_{-k}} \\ &= ax_{n-\ell} + \sum \frac{\alpha_i}{\beta_i} \quad \text{for all } n \ge 1. \end{aligned}$$

By using a comparison, we can write the right hand side as follows

$$y_{n+1} = ay_{n-\ell} + \sum_{i=0}^{k}\frac{\alpha_i}{\beta_i},$$

then

$$y_n = a^n y_{-\ell} + \sum_{i=0}^{k}\frac{\alpha_i}{\beta_i},$$

and this equation is locally stable because $1 > a,$ and converges to the equilibrium point

$$\bar{y} = \frac{1}{(1-a)}\sum_{i=0}^{k}\frac{\alpha_i}{\beta_i}.$$

Therefore

$$\limsup_{n\to\infty} x_n \le \frac{1}{(1-a)}\sum_{i=0}^{k}\frac{\alpha_i}{\beta_i}.$$

Thus, for Eq. (1.1) every solutoin is bounded and the proof is completed.

**Theorem 4.2** For Eq. (1b) every solution is unbounded if $1 < a.$

**Proof.** Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq. (1.1). It follows from Eq. (1.1) then

$$x_{n+1} = ax_{n-\ell} + \frac{\sum_{i=0}^{k}\alpha_i x_{n-i}}{\sum_{i=0}^{k}\beta_i x_{n-i}} > ax_{n-\ell} \quad \text{for all } n \ge 1.$$

We can written as follows

$$y_{n+1} = ay_{n-\ell},$$

then

$$y_n = a^n y_{-\ell}, \qquad (4.1)$$

and the Eq. (4.1) is unstable because $1 < a,$ and

$$\lim_{n\to\infty} y_n = \infty.$$

Thus, the proof is completed.

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