Approximation of Asymptotic Expansion of Wavelets

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ABSTRACT

In this paper, the approximation of asymptotic expansion for the wavelet is obtained and the approximation of asymptotic expansions for generalized Mexican hat wavelet and the wavelet corresponding to m^{th} order cardinal $B-$spline are obtained. Estimates of present investigations are based on Wong's method.

General Terms

Wavelet, Approximation, Basic wavelet

Keywords

First order cardinal $B-$spline, asymptotic approximation, m^{th} order cardinal $B-$spline, asymptotic expansion, wavelet transform, generalized mexican hat wavelet, Mellin transform

1. INTRODUCTION

At first in 1989, Wong [1] studied the asymptotic approximation of certain integrals.Wavelet approximation plays an important role in Mathematics, Computer science and Technology . Approximation of asymptotic expansion of wavelet has been studied by Sweldens and Piessens [2],Pathak and Pathak [3],Wong [4] and Ashurov and Butaev [5] etc. But till now no work seems to have been done to obtain the approximation of asymptotic expansion of generalized Mexican hat wavelet and wavelet corresponding to m^{th} order cardinal B-spline. In an attempt to make an advance study in this direction, in this paper, the estimates for asymptotic expansion of generalized Mexican hat wavelet and m^{th} order cardinal B-spline wavelets are determined. These estimates are new, better and sharper than all previously known estimates.

2. DEFINITIONS

If a function $\psi \in L^2(R)$ satisfies the "admissibility" condition:

$$C_\psi =: \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\omega)\right|^2}{|\omega|} d\omega < \infty, \tag{1}$$

then ψ is called a "basic wavelet". Relative to every basic wavelet ψ, the integral wavelet transform (IWT) of a function $f \in L^2(R)$ is defined by

$$(W_\psi f)(b,a) =: |a|^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(t)\overline{\psi\left(\frac{t-b}{a}\right)} dt \tag{2}$$

where $a, b \in R$ with $a \neq 0$, (Chui [6]).
By setting

$$\psi_{b;a} =: |a|^{-\frac{1}{2}}\, \psi\left(\frac{t-b}{a}\right) \tag{3}$$

the IWT defined in (2) can be written as

$$(W_\psi f)(b,a) = \langle f, \psi_{b;a}\rangle . \tag{4}$$

Using Parsewall identity, it can also be written as

$$\begin{aligned}(W_\psi f)(b,a) &= \frac{1}{2\pi}\left\langle \hat{f}, \hat{\psi}_{b;a}\right\rangle \\ &= \frac{a\,|a|^{-\frac{1}{2}}}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\overline{e^{-ib\omega}\hat{\psi}(a\omega)}d\omega \\ &= \frac{a\,|a|^{-\frac{1}{2}}}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{ib\omega}\overline{\hat{\psi}}(a\omega)d\omega, \end{aligned} \tag{5}$$

Approximation of asymptotic expansion for the general integral

$$I(x) = \int_0^{\infty} F(t)h(xt)dt, \tag{6}$$

was discussed by Wong [1]. Considering the basic idea of Wong [1] related to approximation of asymptotic expansion of general integral, the derivation for the approximation of asymptotic expansion of a wavelet may be obtained.

Suppose that $F(t)$ has an asymptotic expansion as

$$F(t) \sim \sum_{k=0}^{n-1} c_k t^{k+\sigma-1} + F_n(t),\ as\ t \to 0^+, \tag{7}$$

where $0 < \sigma \leq 1, F_n(t) = \sum_{k=n}^{\infty} c_k t^{k+\sigma-1}$.

The generalized Mellin transform of h, denoted by $M[h;z]$, is defined by

$$M[h;z]=\lim_{\epsilon\to 0^+}\int_0^\infty t^{z-1}h(t)e^{-\epsilon t^p}dt. \quad (8)$$

By equation (7) and [Wong [1], p.216],

$$I(x)=\sum_{k=0}^{n-1}c_kM[h;k+\sigma]x^{-(k+\sigma)}+\delta_n(x), \quad (9)$$

where approximation or error bound δ_n is given by

$$\delta_n(x)=\lim_{\epsilon\to 0^+}\int_0^\infty F_n(t)h(xt)e^{-\epsilon t^p}dt. \quad (10)$$

The condition for satisfying the result (9) is already given by Wong ([1], theorem 6, p.217).

3. APPROXIMATION OF ASYMPTOTIC EXPANSION FOR A WAVELET

The approximation of asymptotic expansion of $(W_\psi f)(b,a)$ for fixed b and $|a|$, has been derived. Estimates of present investigations are based on Wong's method.

$$\begin{aligned}(W_\psi f)(b,a) &= \frac{a\,|a|^{-\frac{1}{2}}}{2\pi}\int_{-\infty}^\infty \hat{f}(\omega)e^{ib\omega}\bar{\hat{\psi}}(a\omega)d\omega\\ &= \frac{a\,|a|^{-\frac{1}{2}}}{2\pi}\int_0^\infty \hat{f}(\omega)e^{ib\omega}\bar{\hat{\psi}}(a\omega)d\omega\\ &+ \frac{a\,|a|^{-\frac{1}{2}}}{2\pi}\int_0^\infty \hat{f}(-\omega)e^{-ib\omega}\bar{\hat{\psi}}(-a\omega)d\omega. \quad (11)\end{aligned}$$

Assume that $\hat{f}(\omega)$ has expansion of the form

$$\hat{f}(\omega)=\sum_{k=0}^\infty c_k\omega^{k+\sigma-1}\ as\ \omega\to 0^+, \quad (12)$$

where $0<\sigma\le 1$.
Next,

$$\begin{aligned}F(\omega) &= \hat{f}(\omega)e^{ib\omega}\\ &= \left(\sum_{k=0}^\infty c_k\omega^{k+\sigma-1}\right)\left(\sum_{r=0}^\infty \frac{(ib\omega)^r}{r!}\right)\\ &= \sum_{k=0}^\infty d_k\omega^{k+\sigma-1};\ as\ \omega\to 0^+\\ &= \sum_{k=0}^{n-1} d_k\omega^{k+\sigma-1}+F_n(\omega)\ as\ \omega\to 0^+, \quad (13)\end{aligned}$$

where

$$F_n(\omega)=\sum_{k=n}^\infty d_k\omega^{k+\sigma-1}. \quad (14)$$

and

$$d_k=\sum_{r=0}^k \frac{(ib)^r}{r!}c_{k-r}. \quad (15)$$

Further assuming that

$$\bar{\hat{\psi}}(\omega)\sim e^{i\tau\omega^p}\sum_{r=0}^\infty b_r\omega^{-r-\alpha};\ \alpha>0,\ p\ge 1,\ \omega\to+\infty,\ \tau\ne 0 \quad (16)$$

and

$$\bar{\hat{\psi}}(\omega)=O(\omega^\rho);\ \omega\to 0^+,\ \rho+\sigma>0. \quad (17)$$

By equations(9) and (11),

$$\int_0^\infty e^{ib\omega}\hat{f}(\omega)\bar{\hat{\psi}}(a\omega)d\omega=\sum_{k=0}^{n-1}d_kM[\bar{\hat{\psi}}(\omega);k+\sigma]a^{-k-\sigma}+\delta_n^{(1)}(a), \quad (18)$$

where

$$\delta_n^{(1)}(a)=\lim_{\epsilon\to 0^+}\int_0^\infty F_n(\omega)\bar{\hat{\psi}}(a\omega)e^{-\epsilon\omega}d\omega. \quad (19)$$

Similarly,

$$\begin{aligned}\int_0^\infty e^{-ib\omega}\hat{f}(-\omega)\bar{\hat{\psi}}(-a\omega)d\omega \;&=\; \sum_{k=0}^{n-1}d_k(-1)^{k+\sigma+1}\\ &\quad M[\bar{\hat{\psi}}(\omega);k+\sigma]a^{-k-\sigma}+\delta_n^{(2)}(a), \quad (20)\end{aligned}$$

where

$$\delta_n^{(2)}(a)=\lim_{\epsilon\to 0^+}\int_0^\infty F_n(-\omega)\bar{\hat{\psi}}(-a\omega)e^{-\epsilon\omega}d\omega. \quad (21)$$

By equations (11), (18) and (19), $(W_\psi f)$ has been obtained as follows

$$\begin{aligned}(W_\psi f)(b,a) &= \frac{a\,|a|^{-\frac{1}{2}}}{2\pi}\sum_{k=0}^{n-1}d_k\times M\left[\bar{\hat{\psi}}(\omega);k+\sigma\right]\\ &+ (-1)^{k+\sigma+1}M\left[\bar{\hat{\psi}}(-\omega);k+\sigma\right]\times a^{-k-\sigma}\\ &+ \delta_n(a);\ as\ n\to+\infty, \quad (22)\end{aligned}$$

where approximation or error bound δ_n is given by

$$\delta_n(a)=\lim_{\epsilon\to 0^+}\int_0^\infty F_n(\omega)\bar{\hat{\psi}}(a\omega)e^{-\epsilon\omega}d\omega. \quad (23)$$

The existence theorem for the formula ([1], theorem 6, p.217) has been obtained in the following form:

3.1 Theorem

If

(i) $\hat{f}^{(l)}(\omega)$ is continuous on $(-\infty,\infty)$,

(ii) for $0<\sigma\le 1$,
$\hat{f}(\omega)=\sum_{k=0}^\infty c_k\omega^{k+\sigma-1}\ as\ \omega\to 0^+$;

(iii) $\bar{\hat{\psi}}(\omega)\sim e^{i\tau\omega^p}\sum_{r=0}^\infty b_r\omega^{-r-\alpha};\quad \alpha>0,\ \omega\to+\infty,\ \tau\ne 0,\ p\ge 1$;
and

(iv) $\omega^{-\alpha}\hat{f}^{(j)}(\omega)=O(\omega^{-1-\epsilon})$, as $\omega\to\infty$, for $j=0,1,2,\cdots,l$, l being a non negative integer, $\epsilon>0$.

then,

$$\begin{aligned}(W_\psi f)(b;a) &= \frac{a\,|a|^{-\frac{1}{2}}}{2\pi}\sum_{k=0}^{n-1} d_k \times M\left[\bar{\hat{\psi}}(\omega);k+\sigma\right]\\ &+ (-1)^{k+\sigma+1} M\left[\bar{\hat{\psi}}(-\omega);k+\sigma\right] \times a^{-k-\sigma}\\ &+ \delta_n(a);\ as\ n\to+\infty, \qquad (24)\end{aligned}$$

holds with approximation or error bound δ_n given by

$$\delta_n(a) = \frac{(-1)^l}{a^l}\int_{-\infty}^{\infty} F_n^{(l)}(\omega)\left(\bar{\hat{\psi}}(a\omega)\right)^{(-l)} d\omega \qquad (25)$$

and $\sigma + n > l$.

3.2 Lemmas

For the proof of the Theorem 3.1 following lemmas are required.
Lemma 1 If the integral $\int_0^\infty f(t)dt$ exists as an improper Riemann integral then,

$$\lim_{\epsilon\to0^+}\int_0^\infty e^{-(\epsilon-x)t} f(t)dt = \int_0^\infty f(t)dt,$$

(Wong [1], p. 197).

Lemma 2 For $x > 0$ and $Re\lambda > 0$,

$$\lim_{\epsilon\to0^+}\int_0^\infty t^{\lambda-1}e^{-(\epsilon-ix)t}dt = \frac{e^{\lambda\pi i/2}\Gamma(\lambda)}{x^\lambda},$$

(Wong [1], p. 198).

Example 1 Let $f(t)$ be absolutely integrable on $[0,\delta]$ for some $\delta > 0$, and be bounded for $t \geq \delta$. Then for any $p \geq 1$,

$$\lim_{\epsilon\to0^+}\epsilon\int_0^\infty f(t)e^{-\epsilon t^p}dt = 0,$$

(Wong [1], p. 233).

3.3 Proof of Theorem 3.1

Integration by parts,

$$\begin{aligned}\int_0^\infty F_n(\omega)\bar{\hat{\psi}}(a\omega)e^{-\epsilon\omega^p}d\omega &= \left[\left(\int\bar{\hat{\psi}}(a\omega)d\omega\right)F_n(\omega)e^{-\epsilon\omega^p}\right]_0^\infty\\ &- \frac{1}{a}\int_0^\infty \bar{\hat{\psi}}^{(-1)}(a\omega)\\ &\times \frac{d}{d\omega}F_n(\omega)e^{-\epsilon\omega^p}d\omega. \qquad (26)\end{aligned}$$

By condition (iv) of Theorem 3.1 ,

$$\left[\left(\int\bar{\hat{\psi}}(a\omega)d\omega\right)F_n(\omega)e^{-\epsilon\omega^p}\right]_0^\infty = 0.$$

Then,

$$\begin{aligned}\int_0^\infty F_n(\omega)\bar{\hat{\psi}}(a\omega)e^{-\epsilon\omega^p}d\omega &= -\frac{1}{a}\int_0^\infty\bar{\hat{\psi}}^{(-1)}(a\omega)\\ &\times F_n^{(1)}(\omega)e^{-\epsilon\omega^p}d\omega\\ &+ \frac{\epsilon p}{a}\int_0^\infty\bar{\hat{\psi}}^{(-1)}(a\omega)\\ &\times F_n(\omega)\omega^{p-1}e^{-\epsilon\omega^p}d\omega. \qquad (27)\end{aligned}$$

By Lemmas (1), (2) and example (1)

$$lim_{\epsilon\to0^+}\frac{\epsilon p}{a}\int_0^\infty \bar{\hat{\psi}}^{(-1)}(a\omega)F_n(\omega)\omega^{p-1}e^{-\epsilon\omega^p}d\omega = 0$$

Thus,

$$\delta_n^{(1)} = -\frac{1}{a}\lim_{\epsilon\to0^+}\int_0^\infty\bar{\hat{\psi}}^{(-1)}(a\omega)F_n^{(1)}(\omega)e^{-\epsilon\omega^p}d\omega. \qquad (28)$$

Repeating this process l times,

$$\begin{aligned}\delta_n^{(1)} &= \frac{(-1)^l}{(a)^l}\lim_{\epsilon\to0^+}\int_0^\infty\bar{\hat{\psi}}^{(-l)}(a\omega)F_n^{(l)}(\omega)e^{-\epsilon\omega^p}d\omega\\ &= \frac{(-1)^l}{(a)^l}\int_0^\infty\bar{\hat{\psi}}^{(-l)}(a\omega)F_n^{(l)}(\omega)d\omega\\ &\quad \textit{by Lemma 1}. \qquad (29)\end{aligned}$$

Similarly

$$\begin{aligned}\int_0^\infty F_n(-\omega)\bar{\hat{\psi}}(-a\omega)e^{-\epsilon\omega^p}d\omega &= -\frac{1}{(a)}\int_0^\infty\bar{\hat{\psi}}^{(-1)}(-a\omega)\\ &\times \frac{d}{d\omega}\left(F_n(-\omega)e^{-\epsilon\omega^p}\right)d\omega.\\ &= \frac{1}{-a}\int_0^\infty\bar{\hat{\psi}}^{(-1)}(-a\omega)\\ &\times F_n^{(1)}(-\omega)e^{-\epsilon\omega^p}d\omega\\ &+ \frac{\epsilon p}{a}\int_0^\infty\bar{\hat{\psi}}^{(-1)}(-a\omega)\\ &\times F_n(-\omega)\omega^{p-1}e^{-\epsilon\omega^p}d\omega. (30)\end{aligned}$$

Since,

$$lim_{\epsilon\to0^+}\frac{\epsilon p}{a}\int_0^\infty\bar{\hat{\psi}}^{(-1)}(a\omega)F_n(-\omega)\omega^{p-1}e^{-\epsilon\omega^p}d\omega = 0,$$

therefore

$$\begin{aligned}\delta_n^{(2)} &= \frac{(-1)^l}{(a)^l}\lim_{\epsilon\to0^+}\int_0^\infty\bar{\hat{\psi}}^{(-l)}(-a\omega)F_n^{(l)}(-\omega)e^{-\epsilon\omega^p}d\omega\\ &= \frac{(-1)^l}{(a)^l}\int_0^\infty\bar{\hat{\psi}}^{(-l)}(-a\omega)F_n^{(l)}(-\omega)d\omega. \qquad (31)\end{aligned}$$

Combining equations 29 and 31

$$\begin{aligned}\delta_n(a) &= \delta_n^{(1)}(a) + \delta_n^{(2)}(a)\\ &= \frac{(-1)^l}{(a)^l}\lim_{\epsilon\to0^+}\int_0^\infty\bar{\hat{\psi}}^{(-l)}(a\omega)F_n^{(l)}(\omega)e^{-\epsilon\omega^p}d\omega\\ &+\frac{(-1)^l}{(a)^l}\lim_{\epsilon\to0^+}\int_0^\infty\bar{\hat{\psi}}^{(-l)}(-a\omega)F_n^{(l)}(-\omega)e^{-\epsilon\omega^p}d\omega\\ &= \frac{(-1)^l}{(a)^l}\int_0^\infty\bar{\hat{\psi}}^{(-l)}(a\omega)F_n^{(l)}(\omega)d\omega\\ &+\frac{(-1)^l}{(a)^l}\int_0^\infty\bar{\hat{\psi}}^{(-l)}(-a\omega)F_n^{(l)}(-\omega)d\omega\\ &= \frac{(-1)^l}{(a)^l}\int_{-\infty}^\infty\bar{\hat{\psi}}^{(-l)}(a\omega)F_n^{(l)}(\omega)d\omega.\end{aligned}$$

Thus, Theorem 3.1 is completely established.

4. GENERALIZED MEXICAN HAT WAVELET

In this section the approximation of asymptotic expansions for generalised Mexican hat wavelet has been determined in the following form

4.1 Theorem

If $\hat{f}(\omega)$ satisfies conditions of Theorem 3.1 then

$$\begin{aligned}(W_\psi f)(b,a) &= -\frac{a|a|^{-\frac{1}{2}}}{2\pi}\sum_{k=0}^{n-1} d_k \frac{1}{2\alpha^{\frac{k+\sigma+2}{2}}} \\ &\times \Gamma\left(\frac{k+\sigma+2}{2}\right)(1+(-1)^{k+\sigma-1})a^{-k-\sigma} \\ &+ \delta_n(a), \end{aligned} \tag{32}$$

holds with approximation

$$\begin{aligned}\delta_n(a) &= \frac{(-1)^{l+1}|a|^{-\frac{1}{2}}}{a^{l-1}2\pi}\int_{-\infty}^{\infty} F_n^{(l)}\left((a\omega)^2 e^{-\alpha(a\omega)^2}\right)^{(-l)} d\omega \\ &= O\left(\int_{-\infty}^{\infty} F_n^{(l)}\left((a\omega)^2 e^{-\alpha(a\omega)^2}\right)^{(-l)} d\omega.\right). \end{aligned} \tag{33}$$

Proof:. The Gaussian function, denoted by g_α, is defined by

$$g_\alpha(t) = \frac{1}{2\sqrt{\pi\alpha}} e^{-\frac{t^2}{4\alpha}}, \quad \alpha > 0, \tag{34}$$

(Chui [6], pg. 50).

The generalized Mexican hat wavelet is defined by

$$\psi(t) = -\frac{d^2}{dt^2}(g_\alpha(t)) = -\frac{1}{8\alpha^2\sqrt{\pi\alpha}}(t^2-2\alpha)e^{-\frac{t^2}{4\alpha}}. \tag{35}$$

The Fourier transform of Gaussian function is given as

$$\hat{g_\alpha}(\omega) = e^{-\alpha\omega^2}.$$

Using the well known result $\hat{f}^{(n)}(\omega) = (i\omega)^n \hat{f}(\omega)$,

$$\begin{aligned}\hat{\psi}(\omega) &= \omega^2 e^{-\alpha\omega^2} \\ &= O(\omega^2), \ \ as \to 0^+. \end{aligned} \tag{36}$$

Since

$$\hat{f}(\omega) = O(e^{\delta\omega^2}), \ \omega \to \infty \ \ for\ some \ \ \delta > 0 \tag{37}$$

therefore,

$$F(\omega) = e^{ib\omega}\hat{f}(\omega) = O(e^{\delta\omega^2}), \ \omega \to \infty. \tag{38}$$

Thus,

$$\begin{aligned}\int_0^\infty \hat{f}(\omega)e^{ib\omega}\bar{\hat{\psi}}(a\omega)d\omega &= \sum_{k=0}^{n-1} d_k M[-\omega^2 e^{-\alpha\omega^2}; k+\sigma]a^{-k-\sigma} \\ &+ \delta_n^{(1)}(a), \end{aligned} \tag{39}$$

where

$$\begin{aligned}M[\omega^2 e^{-\alpha\omega^2}; k+\sigma] &= \int_0^\infty \omega^{k+\sigma+1}e^{-\alpha\omega^2}d\omega \\ &= \frac{1}{2\alpha^{\frac{k+\sigma+2}{2}}}\Gamma\left(\frac{k+\sigma+2}{2}\right) \end{aligned} \tag{40}$$

and

$$\delta_n^{(1)} = \lim_{\epsilon\to 0^+}\int_0^\infty F_n(\omega)(-(a\omega)^2 e^{-\alpha(a\omega)^2})e^{-\epsilon\omega}d\omega. \tag{41}$$

From equations (39) and (40),

$$\begin{aligned}\int_0^\infty \hat{f}(\omega)e^{ib\omega}\bar{\hat{\psi}}(a\omega)d\omega &= \sum_{k=0}^{n-1} d_k \frac{1}{2\alpha^{\frac{k+\sigma+2}{2}}}\Gamma\left(\frac{k+\sigma+2}{2}\right) \\ &\times \ a^{-k-\sigma} + \delta_n^{(1)}(a). \end{aligned} \tag{42}$$

Similarly,

$$\begin{aligned}\int_0^\infty \hat{f}(-\omega)e^{-ib\omega}\bar{\hat{\psi}}(-a\omega)d\omega &= \sum_{k=0}^{n-1}(-1)^{k+\sigma-1}d_k\frac{1}{2\alpha^{\frac{k+\sigma+2}{2}}} \\ &\times \ \Gamma\left(\frac{k+\sigma+2}{2}\right)a^{-k-\sigma} + \delta_n^{(2)}(a) \end{aligned} \tag{43}$$

where

$$\delta_n^{(2)} = \lim_{\epsilon\to 0+}\int_0^\infty F_n(-\omega)(-(a\omega)^2 e^{-\alpha(a\omega)^2})e^{-\epsilon\omega}d\omega. \tag{44}$$

By equations (1), (42) and (43),

$$\begin{aligned}(W_\psi f)(b,a) &= \frac{a|a|^{-\frac{1}{2}}}{2\pi}\sum_{k=0}^{n-1} d_k\frac{1}{2\alpha^{\frac{k+\sigma+2}{2}}} \\ &\times \ \Gamma\left(\frac{k+\sigma+2}{2}\right)(1+(-1)^{k+\sigma-1})a^{-k-\sigma} \\ &+ \ \delta_n(a), \end{aligned} \tag{45}$$

where

$$\begin{aligned}\delta_n(a) &= \delta_n^{(1)}(a) + \delta_n^{(2)}(a) \\ &= \lim_{\epsilon\to 0^+}\int_0^\infty F_n(\omega)(a\omega)^2 e^{-\alpha(a\omega)^2}e^{-\epsilon\omega}d\omega \\ &+ \lim_{\epsilon\to 0^+}\int_0^\infty F_n(-\omega)(a\omega)^2 e^{-\alpha(a\omega)^2}e^{-\epsilon\omega}d\omega \\ &= \int_0^\infty F_n(\omega)(a\omega)^2 e^{-\alpha(a\omega)^2})d\omega \\ &+ \int_0^\infty F_n(-\omega)(a\omega)^2 e^{-\alpha(a\omega)^2}d\omega \\ &= \int_{-\infty}^\infty F_n(\omega)(a\omega)^2 e^{-\alpha(a\omega)^2}d\omega. \end{aligned} \tag{46}$$

Considering above steps and the proof of the Theorem (3.1) the result has been completely established.

5. THE WAVELET CORRESPONDING TO M^{TH} ORDER CARDINAL $B-SPLINE$

In this section the approximation of asymptotic expansions for m^{th} Order Cardinal $B-spline$ has been determined in the following form:

5.1 Theorem

If $\hat{f}(\omega)$ satisfies conditions of Theorem 3.1 then,

$$\begin{aligned}(W_\psi f)(b,a) &= \frac{|a|^{-\frac{1}{2}}}{2\pi a^{m-1}} i^m g(b) \\ &+ \frac{a|a|^{-\frac{1}{2}}}{2\pi}\sum_{r=1}^{m}\sum_{k=1}^{n-1} i^{m+2r}C_r^m d_k(1+(-1)^{k+\sigma+1}) \\ &\times\ e^{i\frac{k+\sigma-m}{2}}\Gamma(k+\sigma-m)(ar)^{-(k+\sigma-m)} + \delta_n(a), \end{aligned} \quad (47)$$

holds with approximation

$$\begin{aligned}\delta_n(a) &= \frac{i^m(-1)^l|a|^{-\frac{1}{2}}}{2\pi a^{m+l}}\int_{-\infty}^{\infty} F_n^{(l)}(\omega)\left(\frac{e^{ira\omega}}{(\omega)^m}\right)^{(-l)} d\omega \\ &= O\left(\int_{-\infty}^{\infty} F_n^{(l)}(\omega)\left(\frac{e^{ira\omega}}{(\omega)^m}\right)^{(-l)} d\omega\right)\end{aligned}$$

where n is the smallest positive integer such that $\sigma + n > l$.
Proof: The m^{th} order cardinal B-spline

$$\begin{aligned}N_m(t) &= N_1 * N_1 * N_1 * ...N_1 \\ & \quad (N_1 \textit{ convolutes itself } m \textit{ times}) \\ &= \int_0^1 N_{m-1}(t-x)dx, \quad m = 2,3,\cdots,\end{aligned} \quad (48)$$

where

$$N_1(x) = \begin{cases} 1, & x \in [0,1); \\ 0, & \textit{otherwise}, \end{cases}$$

Chui ([6], p.56)

Then

$$\begin{aligned}\bar{\hat{N}}_m(\omega) &= \left(\frac{1-e^{-i\omega}}{i\omega}\right)^m \\ &= \left(\frac{i}{\omega}\right)^m \left(1-e^{i\omega}\right)^m \\ &= \left(\frac{i}{\omega}\right)^m \sum_{r=0}^{m}(-1)^r C_r^m e^{i\omega r} \\ &= \left(\frac{i}{\omega}\right)^m + (i)^m\sum_{r=1}^{m}(-1)^r C_r^m \frac{e^{i\omega r}}{\omega^m} \\ &= \left(\frac{i}{\omega}\right)^m + \sum_{r=1}^{m} i^{m+2r} C_r^m \frac{e^{i\omega r}}{\omega^m}.\end{aligned} \quad (49)$$

Since,

$$\lim_{\omega\to 0^+}\bar{\hat{N}}_m(\omega) = \lim_{\omega\to 0^+} e^{im\frac{\omega}{2}}\left(\frac{sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^m = 1,$$

therefore

$$\bar{\hat{N}}_m(\omega) = O(1) \quad as \quad \omega \to 0^+. \quad (50)$$

By equations.(6), (2) and (49),

$$\begin{aligned}(W_\psi f)(b,a) &= \frac{a\,|a|^{-\frac{1}{2}}}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{ib\omega} \\ &\times \left(\left(\frac{i}{a\omega}\right)^m + \sum_{r=1}^{m} i^{m+2r}C_r^m\frac{e^{ia\omega r}}{a^m\omega^m}\right)d\omega \\ &= \frac{|a|^{-\frac{1}{2}}}{2\pi a^{m-1}} i^m \int_{-\infty}^{\infty}\frac{\hat{f}(\omega)e^{ib\omega}}{\omega^m}d\omega \\ &+ \frac{a|a|^{-\frac{1}{2}}}{2\pi}\sum_{r=1}^{m} i^{m+2r}C_r^m\int_{-\infty}^{\infty}\hat{f}(\omega)e^{ib\omega}\frac{e^{ia\omega r}}{a^m\omega^m}d\omega \\ &= \frac{|a|^{-\frac{1}{2}}}{2\pi a^{m-1}} i^m g(b) + \frac{a|a|^{-\frac{1}{2}}}{2\pi}\sum_{r=1}^{m} i^{m+2r}C_r^m \\ &\times \int_0^{\infty}\hat{f}(\omega)e^{ib\omega}\frac{e^{iar\omega}}{a^m\omega^m}d\omega \\ &+ \int_0^{\infty}\hat{f}(-\omega)e^{-ib\omega}\frac{e^{-ai\omega r}}{(-a)^m\omega^m}d\omega,\end{aligned} \quad (51)$$

where

$$g(b) = \int_{-\infty}^{\infty}\frac{\hat{f}(\omega)e^{ib\omega}}{\omega^m}d\omega.$$

The generalized Mellin transform formula of e^{it} is given by

$$\begin{aligned}M[e^{it};z] &= \lim_{\epsilon\to 0^+}\int_0^{\infty} t^{z-1}e^{it}e^{-\epsilon t}dt \\ &= \lim_{\epsilon\to 0^+}\int_0^{\infty} t^{z-1}e^{-t(\epsilon-i)}dt \\ &= \lim_{\epsilon\to 0^+}\int_0^{\infty}\left(\frac{x}{\epsilon-i}\right)^{z-1}e^{-x}\frac{1}{(\epsilon-i)}dx, \\ &= \lim_{\epsilon\to 0^+}\frac{1}{(\epsilon-i)^z}\int_0^{\infty}x^{z-1}e^{-x}dx \\ &= \lim_{\epsilon\to 0^+}\frac{1}{(\epsilon-i)^z}\Gamma z \\ &= \frac{1}{(-i)^z}\Gamma z \\ &= e^{i\pi\frac{z}{2}}\Gamma z, \ \ i = e^{\frac{i\pi}{2}}.\end{aligned} \quad (52)$$

By (Wong [1], p.192,)

$$M[\frac{e^{ir\omega}}{\omega^m};k+\sigma] = e^{i\frac{\pi}{2}(k+\sigma-m)}\Gamma(k+\sigma-m)r^{-(k+\sigma-m)}. \quad (53)$$

By equation (7),

$$\begin{aligned}\int_0^{\infty}\hat{f}(\omega)e^{ib\omega}\left(\frac{e^{ia\omega r}}{a\omega^m}\right) &= \sum_{k=0}^{n-1} d_k M[\frac{e^{ia\omega r}}{a^m\omega^m};k+\sigma-m] \\ &\times\ a^{-(k+\sigma-m)} \\ &+ \lim_{\epsilon\to 0^+}\int_0^{\infty}F_n(\omega)\frac{e^{ia\omega r}}{a^m\omega^m}e^{-\epsilon\omega}d\omega \\ &= \sum_{k=0}^{n-1} d_k e^{i\frac{(k+\sigma-m)}{2}} \\ &\times\ \Gamma(k+\sigma-m)(ar)^{-(k+\sigma-m)} \\ &+ \lim_{\epsilon\to 0^+}\int_0^{\infty}F_n(\omega)\frac{e^{ia\omega r}}{a^m\omega^m}e^{-\epsilon\omega}d\omega\end{aligned}$$

$$. \tag{54}$$

Similarly,

$$\begin{aligned}\int_0^\infty \hat{f}(-\omega)e^{-ib\omega}\left(\frac{e^{-ia\omega r}}{(-a)^m\omega^m}\right) &= \sum_{k=0}^{n-1}(-1)^{k+\sigma+1}d_k e^{i\frac{(k+\sigma-m)}{2}} \\ &\times \Gamma(k+\sigma-m)(ar)^{-(k+\sigma-m)} \\ &+ \lim_{\epsilon\to 0^+}\int_0^\infty F_n(-\omega) \\ &\times \frac{e^{-ia\omega r}}{(-a)^m\omega^m}e^{-\epsilon\omega}d\omega \\ &. \end{aligned} \tag{55}$$

By equations (51), (54) and (55),

$$\begin{aligned}(W_\psi f)(b,a) &= \frac{|a|^{-\frac{1}{2}}}{2\pi a^{m-1}}i^m g(b) \\ &+ \frac{a|a|^{-\frac{1}{2}}}{2\pi}\sum_{r=1}^{m}\sum_{k=1}^{n-1} i^{m+2r}C_r^m d_k(1+(-1)^{k+\sigma+1}) \\ &\times e^{i\frac{k+\sigma-m}{2}}\Gamma(k+\sigma-m)(ar)^{-(k+\sigma-m)} \\ &+ \delta_n(a),\end{aligned} \tag{56}$$

where

$$\begin{aligned}\delta_n(a) &= \lim_{\epsilon\to 0^+}\int_0^\infty F_n(\omega)\frac{e^{ia\omega r}}{a^m\omega^m}e^{-\epsilon\omega}d\omega \\ &+ \lim_{\epsilon\to 0^+}\int_0^\infty F_n(-\omega)\frac{e^{-ia\omega r}}{(-a)^m\omega^m}e^{-\epsilon\omega}d\omega \\ &= \int_0^\infty F_n(\omega)\frac{e^{ia\omega r}}{a^m\omega^m}d\omega + \int_0^\infty F_n(-\omega)\frac{e^{-\hat{\bar{i}}a\omega r}}{-a^m\omega^m}d\omega \\ &= \int_{-\infty}^\infty F_n(\omega)\left(\frac{e^{ia\omega r}}{a^m\omega^m}d\omega\right).\end{aligned} \tag{57}$$

By the help of above mentioned steps and the proof of the Theorem (3.1),this result has been proved.

6. CONCLUSION

(i) Estimates for the asymptotic expansion of generalized Mexican hat wavelet and m^{th} order cardinal $B-$spline have been obtained.

(ii) The result of Pathak and Pathak ([3] Theorem(4)) is a particular case of Theorem(4.1) in this paper if $\alpha = \frac{1}{2}$.

(iii) If $m = 2$, then estimate for the asymptotic expansion of 2^{nd} order cardinal $B-$spline satisfies

$$\delta_n(a) = O\left(\int_{-\infty}^{\infty} F_n^{(l)}(\omega)e^{-iral\omega}\omega^{ml}d\omega\right),$$

where n is the smallest positive integer such that $\sigma + n > l$. This estimate may be developed independently as similar to Theorem 5.1 taking m=2.

(iv) For $m = 1, m^{th}$ order $B-$spline reduced to

$$N_1(x) = \begin{cases} 1, & x \in [0,1); \\ 0, & \textit{otherwise}, \end{cases}$$

This is the first order $B-$spline.

$$\hat{N_1}(\omega) = e^{-i\frac{\omega}{2}}\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}$$

$$\begin{aligned}\hat{N_1}(0) &= \lim_{\omega\to 0}\left(e^{-i\frac{\omega}{2}}\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right) \\ &= 1\end{aligned}$$

$N_1(t)$ is a scaling function. It is not a wavelet.The Theorem 5.1 is not applicable. Thus, the estimate for m^{th} order $B-$spline has been obtained for $m \geq 2$.

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