Some Properties of Cartesian Product Graphs of Cayley Graphs with Arithmetic Graphs

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ABSTRACT
Nathanson was the pioneer in introducing the concepts of Number Theory, particularly, the "Theory of Congruences" in Graph Theory. Thus he paved the way for the emergence of a new class of graphs, namely "Arithmetic Graphs". Cayley graphs are another class of graphs associated with the elements of a group. If this group is associated with some arithmetic function then the Cayley graph becomes an Arithmetic graph.

Graph product is a fundamental tool with rich *applications* in both graph theory and theoretical computer science. The extensive literature on products that has evolved over the years presents a wealth of profound and beautiful results.

In this paper, results related to some properties of Cartesian product graphs of Euler totient Cayley graphs with Arithmetic V_n graphs are determined.

Keywords
Euler totient Cayley graph, Arithmetic V_n graph, Cartesian product graph.

AMS (MOS) Subject Classification: 6905c

1. INTRODUCTION

EULER TOTIENT CAYLEY GRAPH $G(Z_n, \varphi)$ AND ITS PROPERTIES

Madhavi [1] introduced the concept of Euler totient Cayley graphs and studied some of its properties. She gave methods of enumeration of disjoint Hamilton cycles and triangles in these graphs. Sujatha [2] studied some cyclic structures of Euler totient Cayley graphs.

For any positive integer n, let $Z_n = \{0,1,2, \ldots \ldots n-1\}$ be the residue classes modulo n. Then $(Z_n, \oplus)$, where, $\oplus$ is addition modulo n, is an abelian group of order n. The number of positive integers less than n and relatively prime to n is denoted by $\varphi(n)$ and is called Euler totient function. Let S denote the set of all positive integers less than n and relatively prime to n. That is $S = \{r / 1 \le r < n \text{ and } \mathrm{GCD}\,(r, n) = 1\}$. Then $|S| = \varphi(n)$.

Now define Euler totient Cayley graph as follows.

For each positive integer n, let Z_n be the additive group of integers modulo n and let S be the set of all integers less than n and relatively prime to n. The Euler totient Cayley graph $G(Z_n, \varphi)$ is defined as the graph whose vertex set V is given by $Z_n = \{0,1,2, \ldots . n-1\}$ and the edge set is $E = \{(x, y) / x - y \in S \text{ or } y - x \in S\}$.

Clearly as proved by Madhavi, the Euler totient Cayley graph $G(Z_n, \varphi)$ is

1. a connected, simple and undirected graph,
2. $\varphi\,(n)$ - regular and has $\frac{n.\varphi(n)}{2}$ edges,
3. Hamiltonian,
4. Eulerian for $n \ge 3$,
5. bipartite if n is even and complete graph if n is a prime.

ARITHMETIC V_n GRAPH

Vasumathi [3] introduced the concept of Arithmetic V_n graphs and studied some of its properties.

Let n be a positive integer such that $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots \ldots . p_k^{\alpha_k}$. Then the Arithmetic V_n graph is defined as the graph whose vertex set consists of the divisors of n and two vertices u, v are adjacent in V_n graph if and only if GCD $(u, v) = p_i$, for some prime divisor p_i of n.

In this graph the vertex 1 becomes an isolated vertex. Hence consider the Arithmetic graph V_n without vertex 1 because the contribution of this isolated vertex is nothing when the properties of these graphs and enumeration of some domination parameters are studied.

Clearly, V_n graph is a connected graph. Because if n is a prime, then V_n graph consists of a single vertex. Hence it is a connected graph. In other cases, by the definition of adjacent in V_n, there exist edges between prime number vertices and their prime power vertices and also to their prime product vertices. Therefore, each vertex of V_n is connected to some vertex in V_n.

CARTESIAN PRODUCT GRAPHS

The Cartesian product of graphs is a straight forward and natural construction. According to Imrich and Klavzar [4] Cartesian products of graphs were defined in 1912 by Whitehead and Russell [5]. These products were repeatedly rediscover later, notably by Sabidussi [6] in 1960.

Cartesian product graphs can be recognized efficiently, in time $O(m \log n)$ for a graph with m edges and n vertices [7]. For more details, refer [8] and [9].

Let G_1 and G_2 be two simple graphs with their vertex sets as $V_1 = \{u_1, u_2, \ldots \ldots\}$ and $V_2 = \{v_1, v_2, \ldots \ldots\}$ respectively. Then the Cartesian product of these two graphs denoted by $G_1 \square G_2$ is defined to be a graph whose vertex set is $V_1 \times V_2$, where $V_1 \times V_2$ is the Cartesian product of the sets V_1 and V_2 and any two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \times G_2$ are adjacent if

(i) $u_1 = u_2$ and $v_1 v_2 \in E(G_2)$ or

(ii) $u_1 u_2 \in E(G_1)$ and $v_1 = v_2$.

2. RESULTS

Let G_1 be an Euler Totient Cayley graph and G_2 be an Arithmetic V_n graph. Then G_1 and G_2 are simple graphs as they have no loops and multiple edges. Hence by the definition of adjacency in Cartesian product, $G_1 \square G_2$ is also a simple graph.

Now investigate some properties of $G_1 \square G_2$.

Theorem 2.1: The degree of a vertex in the Cartesian product graph $G_1 \square G_2$ is given by

$$deg_{G_1 \square G_2}(u_i, v_j) = deg_{G_1}(u_i) + deg_{G_2}(v_j)$$

where $u_i \in V_1$ and $v_j \in V_2$.

Proof: By the definition of Cartesian product, vertex (u_i, v_j) in $G_1 \square G_2$ is adjacent to all the vertices of the sets $\{u_i \times N_{G_2}(v_j)\}$ and $\{N_{G_1}(u_i) \times v_j\}$ where $N_{G_1}(u_i)$ denotes the open neighbourhood set of u_i in the graph G_1 and $N_{G_2}(v_j)$ denotes the open neighbourhood set of v_j in G_2.

So $N_{G_1 \square G_2}(u_i, v_j) = \{u_i \times N_{G_2}(v_j)\} \cup \{N_{G_1}(u_i) \times v_j\}$

Further $|N_{G_1}(u_i)| = deg_{G_1}(u_i)$ and

$$|N_{G_2}(v_j)| = deg_{G_2}(v_j).$$

$$|N_{G_1 \square G_2}(u_i, v_j)| = deg_{G_1 \square G_2}(u_i, v_j).$$

Now

$$|N_{G_1 \square G_2}(u_i, v_j)| = |\{u_i \times N_{G_2}(v_j)\}| + |\{N_{G_1}(u_i) \times v_j\}|$$

$$= deg_{G_2}(v_j) + deg_{G_1}(u_i).$$

Hence $deg_{G_1 \square G_2}(u_i, v_j) = deg_{G_1}(u_i) + deg_{G_2}(v_j)$. ■

Remark: Since graph G_1 is a $\varphi(n)$- regular graph, we have $deg_{G_1}(u_i) = \varphi(n)$, for any i. Hence we can write $deg(u_i, v_j) = \varphi(n) + deg(v_j)$.

Theorem 2.2: $G_1 \square G_2$ is a simple finite graph without isolated vertices.

Proof: Since G_1 and G_2 are simple finite graphs, by the definition of Cartesian product it follows that $G_1 \square G_2$ is also a simple finite graph.

Since G_1 is a graph without isolated vertices for all values of n

$deg_{G_1}(u_i) \neq 0$ for any i. G_2 is a single vertex graph if n is a prime. Otherwise G_2 is graph without isolated vertices.

So $deg_{G_2}(v_j) = 0$ if n is a prime and $deg_{G_2}(v_j) \neq 0$ otherwise. Hence by Theorem 2.1, $deg_{G_1 \square G_2}(u_i, v_j) \neq 0$ for any i, j.

Thus $G_1 \square G_2$ admits no isolated vertices. ■

Theorem 2.3: The number of vertices and edges in $G_1 \square G_2$ is given respectively by

1. $|V(G_1 \square G_2)| = |V(G_1)|\,|V(G_2)|$.
2. $|E(G_1 \square G_2)| = |V(G_1)|\,|E(G_2)| + |V(G_2)|\,|E(G_1)|$

Proof: Let p_1, p_2, p denote the number of vertices and q_1, q_2, q denote the number of edges of graphs G_1, G_2 and $G_1 \square G_2$ respectively. By the definition of Cartesian product, it follows that $p = p_1 . p_2$.

i.e., $|V(G_1 \square G_2)| = |V(G_1)|\,|V(G_2)|$.

Also $|E(G_1)| = q_1 = \frac{1}{2}\sum_{i \in V_1} deg(u_i)$

and $|E(G_2)| = q_2 = \frac{1}{2}\sum_{j \in V_2} deg(v_j)$

Now

$$|E(G_1 \square G_2)| = q = \frac{1}{2}\sum_{i,j} \deg(u_i, v_j)$$

$$= \frac{1}{2}\left\{\sum_{i,j}[deg(u_i) + deg(v_j)]\right\} \quad \text{(By Theorem 2.1)}$$

$$= \frac{1}{2}\left\{\left(\sum_{i,j} deg(u_i)\right) + \left(\sum_{i,j} deg(v_j)\right)\right\}$$

$$= \frac{1}{2}\left\{\sum_j\left(\sum_i deg(u_i)\right) + \sum_i\left(\sum_j deg(v_j)\right)\right\}$$

$$= \frac{1}{2}\left\{\sum_j(2q_1) + \sum_i(2q_2)\right\}$$

$$= \frac{1}{2}\left\{p_2(2q_1) + p_1(2q_2)\right\}$$

$$= p_1 q_2 + p_2 q_1$$

$$= |V(G_1)|\,|E(G_2)| + |V(G_2)|\,|E(G_1)|$$ ■

Now examine the property of connectivity in Cartesian product of these graphs.

It is proved by Wilfried Imrich and Sandi Klavzar [10] that the Cartesian product of two graphs is connected if and only if both the graphs are connected.

Since the graphs G_1 and G_2 are connected, the following result is an immediate consequence.

Theorem 2.4: $G_1 \square G_2$ is a connected graph.

Theorem 2.5: $G_1 \square G_2$ is a complete graph, if n is a prime.

Proof: Suppose *n* is a prime. Then Euler totient Cayley graph G_1 is a complete graph and Arithmetic V_n graph G_2 is a single vertex graph K_1. Hence by the definition of Cartesian product, $G_1 \square G_2$ becomes a complete graph. ■

It is known that a graph is bipartite if and only if it contains no odd cycles.

To examine the property of bipartite of $G_1 \square G_2$, recall the following result given by Sabidussi.

Result: A Cartesian product graph is bipartite if and only if each of its factors is bipartite.

Assume that G_1 and G_2 are bipartite. By the definition of Cartesian product, each cycle in $G_1 \square G_2$ has edges either from G_1 or from G_2 (but not both). Since G_1 and G_2 are bipartite, these edges form an even cycle in G_1 or an even cycle in G_2. So the number of edges of the cycle in $G_1 \square G_2$ is even. Hence there is no odd cycles in $G_1 \square G_2$. Hence $G_1 \square G_2$ is bipartite.

Conversely if $G_1 \square G_2$ is bipartite then there are no odd cycles in $G_1 \square G_2$. Since both G_1 and G_2 are subgraphs of $G_1 \square G_2$, it follows that there are no odd cycles in G_1 and there are no odd cycles in G_2. Hence they are bipartite.

We now examine for what values of n, the Cartesian product

$G_1 \square G_2$ is a bipartite graph?

By the above result given by Sabidussi, bipartition of graph $G_1 \square G_2$ depends on the bipartition of both the graphs G_1 and G_2.

As proved by Madhavi, Euler Totient Cayley Graph G_1 is not bipartite for odd values of n and it is bipartite for even values of n. This implies that Cartesian product graph $G_1 \square G_2$ may bipartite only for even values of n and not for its odd values.

Theorem 2.6: Let n be an even number such that $n > 2$, $n = 2^\alpha$ or $n = 2p$ where p is a prime. Then the Cartesian product graph $G_1 \square G_2$ is a bipartite graph.

Proof: Suppose n is an even number such that $n > 2$, $n = 2^\alpha$ or $n = 2p$ where p is a prime. Then Euler totient Cayley graph G_1 is a bipartite graph.

Now it has to be proved that G_2 is also a bipartite graph by showing that G_2 contains no odd cycles. The proof follows in two cases.

Case 1: Suppose $n = 2^\alpha$.

In this case Arithmetic graph G_2 contains the vertices $2, 2^2, 2^3, \ldots\ldots, 2^\alpha$. Since GCD $(2^i, 2^j) \neq 2$ for any $i, j > 1$, there exists no edge between any two powers of 2. The only edges are between 2 and its powers. Hence odd cycles cannot occur in G_2.

Case 2: Suppose $n = 2p$ where p is a prime.

In this case Arithmetic graph G_2 has the vertices $2, p$ and $2p$. Then by the definition of edges in G_2, there are edges between 2 and $2p$ since GCD $(2, 2p) = 2$ and p and $2p$ since GCD $(p, 2p) = p$. Since p being an odd prime, we have GCD $(2, p) = 1$. This implies that there is no edge between the vertices 2 and p of G_2. Thus G_2 has no odd cycle.

Thus in either of the cases, G_2 has no odd cycle. And hence it is a bipartite graph. Therefore G_1 and G_2 are bipartite graphs if the even number $n > 2$ is of the form 2^α or $2p$ which implies that the Cartesian product graph $G_1 \square G_2$ is a bipartite graph. ■

Theorem 2.7: $G_1 \square G_2$ is not a bipartite graph, if the even number n is neither in the form 2^α nor $2p$.

Proof: Suppose n is an even number such that $n \neq 2^\alpha$ or $n \neq 2p$ where p is a prime.

Since n being an even number, Euler totient Cayley graph G_1 is a bipartite graph. Since the even number n is not in the form 2^α and $2p$, it can be written as

$n = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots\ldots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are odd primes and $\alpha_i \geq 1$. Then G_2 contains three distinct vertices $2, 2p_i, 2p_j$ with GCD $(2, 2p_i) = 2$, GCD $(2, 2p_j) = 2$, and GCD $(2p_i, 2p_j) = 2$. This implies that these vertices are connected by edges. So, G_2 contains an odd cycle and hence it is not bipartite.

Now, G_1 is a bipartite graph and G_2 is not a bipartite graph implies that $G_1 \square G_2$ is not a bipartite graph. ■

3. ILLUSTRATIONS

Let $n = 4$.

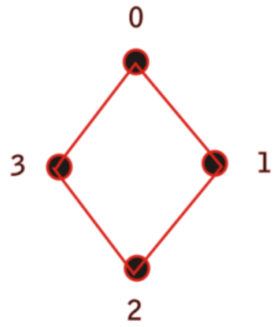


Fig 1

$$G_1 = G(Z_4, \varphi)$$

Fig 2

$$G_2 = G(V_4)$$

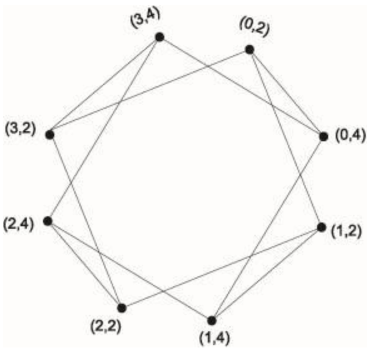


Fig 3

$$G_1 \square G_2$$

Let $n = 6$

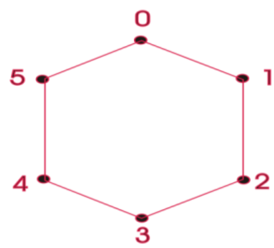


Fig 4

$$G_1 = G(Z_6, \varphi)$$

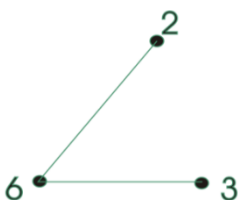


Fig 5

$$G_2 = G(V_6)$$

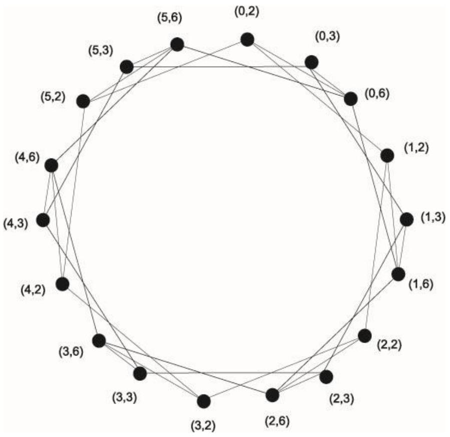


Fig 6, $G_1\square\, G_2$

Let $n = 11$

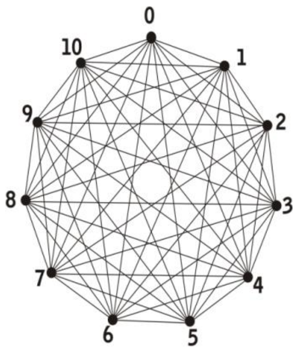


Fig 7

$$G_1 = G(Z_{11}, \varphi)$$

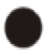
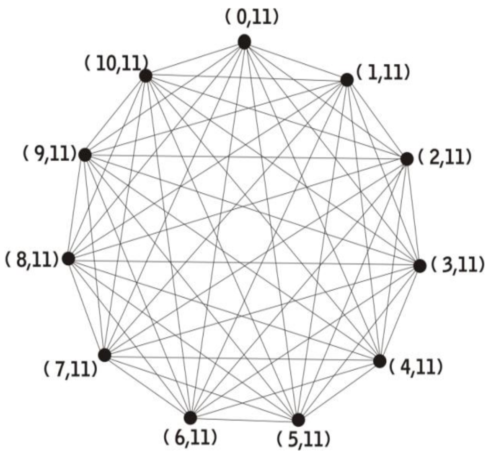


Fig 8



$$G_2 = G(V_{11})$$

Fig 9, $G_1\square\, G_2$

4. CONCLUSION

Graph Theory is young but rapidly maturing subject. Its basic concepts are simple and can be used to express problems from many different subjects. The purpose of this work is to familiarize the reader with the Cartesian product graph of Euler Totient Cayley graph with Arithmetic V_n graph.

It is useful other Researchers for further studies of other properties of these product graphs and their relevance in both combinatorial problems and classical algebraic problems.

5. ACKNOWLEDGMENTS

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