

Folding of Cantor String

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ABSTRACT

In recent years, folding of various objects have been generated using different approaches. The classical Cantor set is an interesting mathematical construction with links to several areas of analysis and topology. The purpose of this paper is to represent the folding of Cantor string (compliment of Cantor set) using direct folding and folding by cut methods. Moreover, the results governing these types of folding are obtained.

Keywords

Cantor set, Cantor string, Folding methods, Retraction.

1. INTRODUCTION

Cantor set is a classical example of perfect subset of the closed interval $[0, 1]$, which has the same cardinality as the real line but whose Lebesgue measure is zero[10]. It was discovered in 1875 by Henry John Stephen Smith [33] and first introduced by German mathematician George Cantor (1845-1918) that become known as Cantor ternary set [3-8]. Later on, Devil's and other researchers gave graphical representation of Cantor set in the form of staircases [28-30].

The folding of a manifold was, firstly introduced by Robertson in 1977 [31]. When a sheet of paper is crumpled in the hands and then crushed flat against a desk-top, the pattern so formed is governed by certain simple rules. These rules are generalized to theorems on folding manifolds isometrically into one another which has been examined independently by Robertson [1977] and Sewell [1973]. In 1986, the topological character of the manifolds has been introduced by Robertson et al. [32]. Later on, in a series of papers, Ghoul [15-20], Ghoul et al. [12-14] jointly with others carried further analysis and generalizations of manifolds (see also [24]). Also, for more properties and applications of manifolds in analysis one may refer to [1, 9, 20-23, 25-27]. The limit of folding of a manifold is defined in [22].

Lapidus and van Frankenhuijsen [30, 31] introduced the concept of fractal string and established the geometric zeta function, zeta function, spectra of fractal string and the complex dimension of the fractal string. Further in 2008, Lapidus [32] suggested that fractal string and their quantization may be related to aspects of string theory. In last few decades, M. L. Lapidus, jointly with other researchers

generalized and introduced the various properties of fractal string.

In 2012, Attiya [2] studied the folding of hyperbolic manifold using mathematical results. Further, in 2012, the types of retractions of one dimensional manifolds and the isometric and topological folding have been studied by A.E. El-Ahmady et al.[11].

In this paper, we study the folding of Cantor string using two different approaches of folding. The limit of folding of a Cantor string is also discussed. Section 2 deals with some basic definition pertaining to the folding of a manifold, Cantor set and Cantor string. In Section 3, the main result of our study has been presented.

2. PRELIMINARIES

Definition 1. Let M be a non-empty (second-countable) Hausdorff topological space such that:

(i) M is the union of open subsets U_α and each U_α is equipped with a homeomorphism x_α taking U_α to an open set in \mathbb{R}^n , i.e.;

$$x_\alpha : U_\alpha \rightarrow x_\alpha(U_\alpha) \subset \mathbb{R}^n$$

(ii) If $U_\alpha \cap U_\beta = W \neq \emptyset$, the sets $x_{\alpha^{-1}}(W)$ and $x_{\beta^{-1}}(W)$ are open sets in \mathbb{R}^n then the overlap map

$$x_{\beta^{-1}} \circ x_{\alpha} : x_{\alpha}(W) \rightarrow x_{\beta}(W) \quad x_{\beta} x_{\alpha}^{-1} : x_{\alpha}(W) \rightarrow x_{\beta}(W)$$

is a smooth map, see **Fig.1**, where the map $x_{\beta}^{-1} \circ x_{\alpha}$ from an open set in \mathbb{R}^n to \mathbb{R}^n is smooth if all partial derivatives of all orders of each component of the map exist everywhere, where the map is defined.

Each pair (U_α, x_α) is called a chart on M , and the collection $A = \{(U_\alpha, x_\alpha)\}$ of charts is called (smooth) atlas on M . The space M taken together with the atlas A will be called a smooth manifold of dimension n or smooth n -manifold or C^∞ n -manifold.

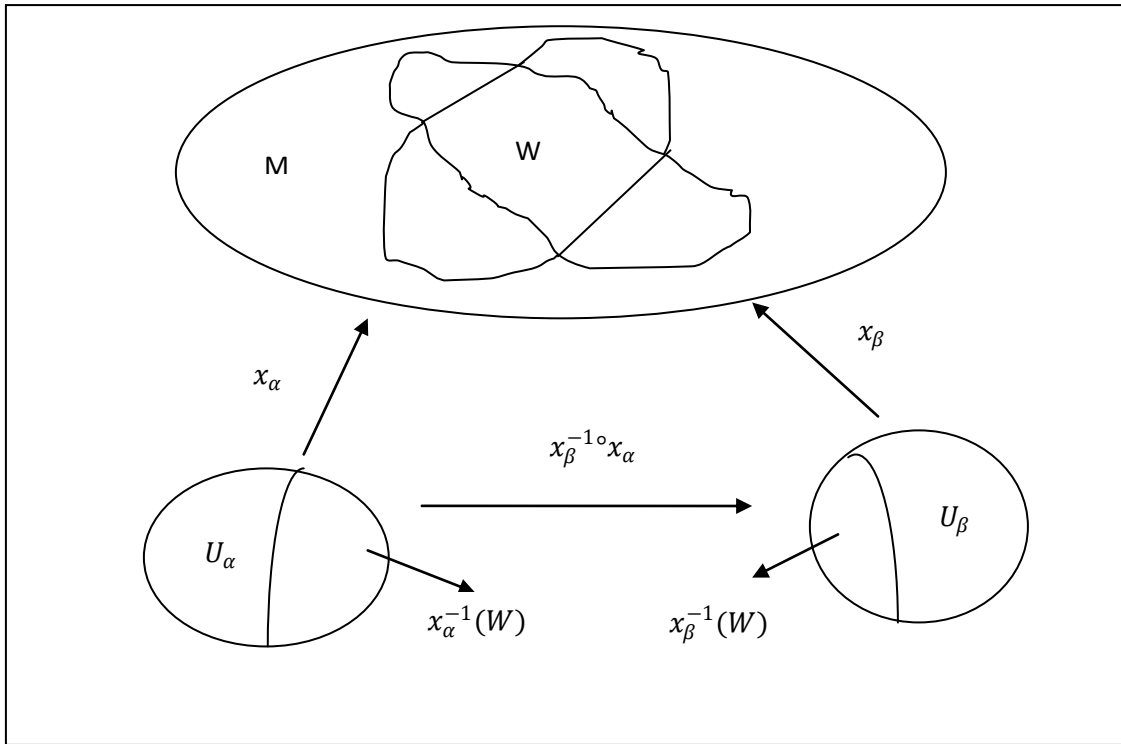


Fig. 1 Smooth mapping

Definition 2. Let M and N be two C^∞ - Riemannian manifolds of dimension m and n respectively. A map $f : M \rightarrow N$ is said to be an isometric folding of M into N if for every piecewise geodesic path $\gamma : I \rightarrow M$ the induced path $f \circ \gamma : I \rightarrow N$ is piecewise geodesic and of the same length as γ . If f does not preserve length it is called a topological folding. [34]

Definition 3. A subset A of a topological space X is a retract of X if there exist a continuous map $r : X \rightarrow A$ called a retraction such that $r(a) = a$ for any $a \in A$. [20]

Definition 4. The Cantor set C is defined as $C = \bigcap_{n=1}^{\infty} I_n$, where I_{n+1} is constructed by trisecting I_n and removing the middle third, I_0 being the closed real interval $[0, 1]$.

In 2000, Lapidus and van Frankenhuijsen introduced the concept of fractal strings. They defined it as follows:

Definition 5. A fractal string Ω is a bounded open subset of the real line \mathbb{R} . The collection of lengths ℓ_j of the disjoint intervals is denoted by L . For example, the complement of the Cantor set in the closed unit interval $[0, 1]$ is a Cantor string. Moreover, the topological boundary of a Cantor string is the Cantor set C itself.

Definition 6. The limit of the folding of an n -dimensional manifold M into itself is a manifold N of dimension $n-1$. [22]

3. MAIN RESULTS

In the year (1879-1884), George Cantor coined few problems and consequences in the field of set theory. One of them was Cantor ternary set a classical example of fractals. Under the constructions of Cantor set, we begin with the closed interval $I_0 = [0, 1]$ and divide it into three equal open sub-intervals. And remove the central open interval $I_1 = (\frac{1}{3}, \frac{2}{3})$ such that

$$[0, 1] - I_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

by repeating the process of removing the middle one third from each closed interval, we can define as

$$C = \bigcap_{n=1}^{\infty} I_n$$

where I_{n+1} is constructed as above by trisecting I_n and removing the middle third, I_0 being the closed interval $[0, 1]$.

In this paper, we study the folding of Cantor string. To start the constructions of folding of Cantor string, we take the closed interval $[0,1]$ and divide it into 3 equal subintervals. **Fig.2** below shows the representation of folding of Cantor string by using direct method.

To start the construction, initiator $F_0 = [0, 1]$ is subdivided into three equal subintervals, left(L), right(R) and middle(M).

Drop first and third semi-open intervals $x_1 = \left[0, \frac{1}{3}\right)$ and

$$x_2 = \left(\frac{2}{3}, 1\right]$$
 such that

$$F_1 = [0, 1] - \left[0, \frac{1}{3}\right) \cup \left(\frac{2}{3}, 1\right] = \left[\frac{1}{3}, \frac{2}{3}\right]$$

Again we subdivide the closed interval $F_1 = \left[\frac{1}{3}, \frac{2}{3}\right]$ into three equal subintervals and remove the first and third semi-open intervals, then we get

$$F_2 = \left[\frac{1}{3}, \frac{2}{3}\right] - \left[\frac{1}{3}, \frac{4}{9}\right) \cup \left(\frac{5}{9}, \frac{2}{3}\right] = \left[\frac{4}{9}, \frac{5}{9}\right]$$

Further, repeating the same process again and again, by removing the semi-open subintervals of first and third position at each step from each closed interval, we obtain a sequence

$$\{F_k\}_{k=1}^{\infty}$$

Thus the folding of Cantor string would be the limit F as the intersection of sets F_k i.e. $F = \bigcap F_k$

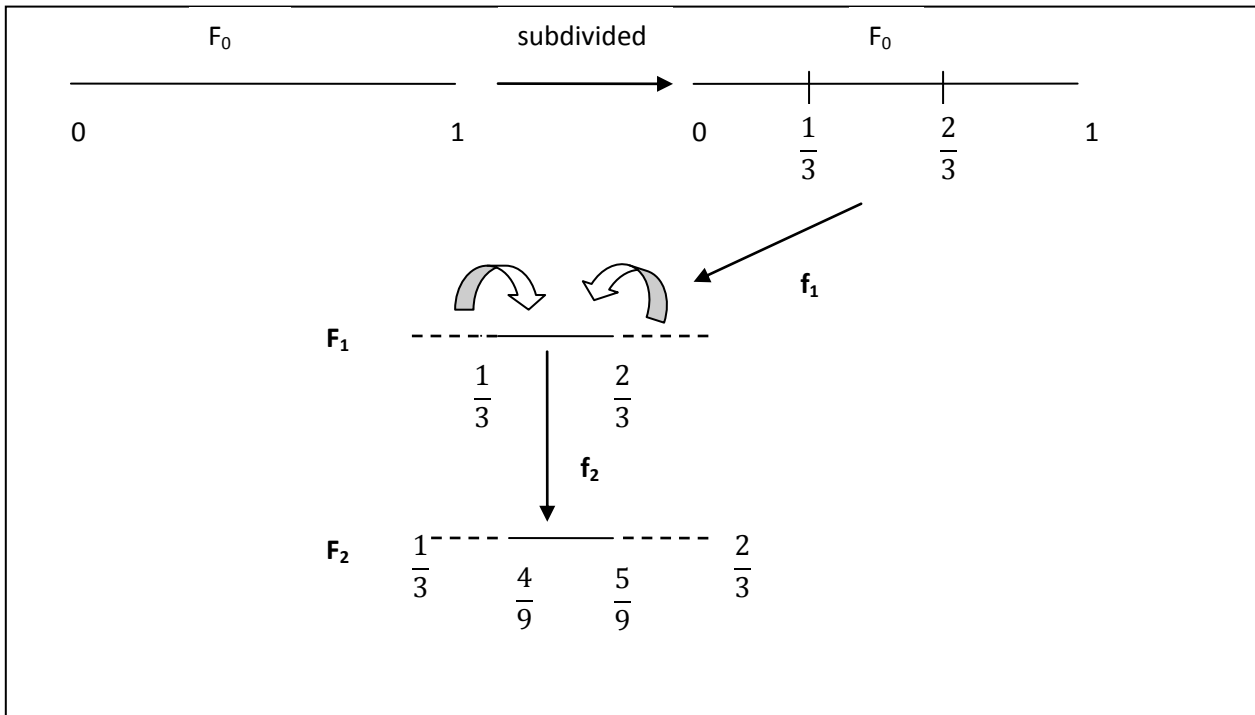


Fig. 2 Folding of Cantor string by direct method

Theorem 3.1 Let f be the folding map on $[0, 1]$ defined by $f(x) = \frac{x+1}{3}$. Then the Cantor string F defined above satisfies the inclusion $F_k \supseteq F_{k+1}$ for all $k \in \{0, 1, 2, \dots\}$.

Proof. In the starting of this section, we study the folding of Cantor set by simply removing the one-third semi-open intervals of first and third step.

Now using the map $f(x) = \frac{x+1}{3}$, we generate the folding which is quite different from the method mentioned above.

In Fig.3, by using the mapping $f(x) = \frac{x+1}{3}$ on initiator $[0, 1]$, we study the folding of Cantor string in the following way:

First, let $x \in [0, 1]$ and then using the map f , we get

$$f_1(F_0) = f_1([0, 1]) = \left[\frac{1}{3}, \frac{2}{3}\right] = F_1$$

$$\Rightarrow F_0 \supseteq F_1$$

Now, take $x \in \left[\frac{1}{3}, \frac{2}{3}\right]$ and then using the mapping f , we get

$$f_2(F_1) = f_2\left(\left[\frac{1}{3}, \frac{2}{3}\right]\right) = \left[\frac{4}{9}, \frac{5}{9}\right] = F_2$$

$$\Rightarrow F_1 \supseteq F_2$$

Again, taking $x \in \left[\frac{4}{9}, \frac{5}{9}\right]$ and then using the mapping f , we get

$$f_3(F_2) = f_2\left(\left[\frac{4}{9}, \frac{5}{9}\right]\right) = \left[\frac{13}{27}, \frac{14}{27}\right] = F_3$$

$$\Rightarrow F_2 \supseteq F_3$$

Fig.3 shows the geometrical representation of the folding by using a folding map.

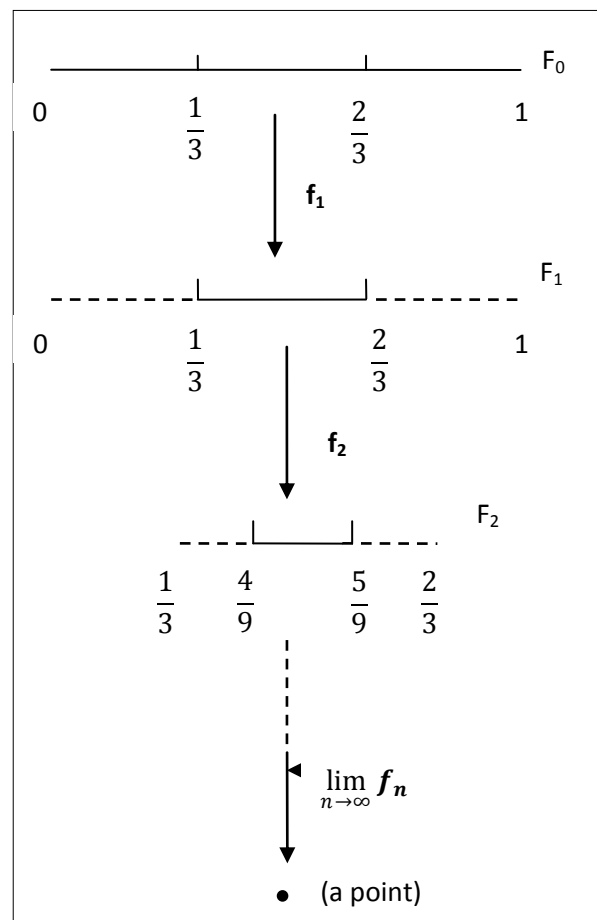


Fig. 3

Further repeating the same process and substituting the value of previous steps in the mapping $f(x) = \frac{x+1}{3}$, we get the following table:

Table-1

$f(x) = \frac{x+1}{3}$		
$f_1([0, 1])$	$\left[\frac{1}{3}, \frac{2}{3}\right]$	F_1
$f_2\left(\left[\frac{1}{3}, \frac{2}{3}\right]\right)$	$\left[\frac{4}{9}, \frac{5}{9}\right]$	F_2
$f_3\left(\left[\frac{4}{9}, \frac{5}{9}\right]\right)$	$\left[\frac{13}{27}, \frac{14}{27}\right]$	F_3
$f_4\left(\left[\frac{13}{27}, \frac{14}{27}\right]\right)$	$\left[\frac{40}{81}, \frac{41}{81}\right]$	F_4
$f_5\left(\left[\frac{40}{81}, \frac{41}{81}\right]\right)$	$\left[\frac{121}{243}, \frac{122}{243}\right]$	F_5
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$f_n(F_{n-1})$	$\left[\frac{\sum_{k=1}^n 3^{k-1}}{3^n}, \frac{1 + \sum_{k=1}^n 3^{k-1}}{3^n}\right]$	F_n
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This implies that the inclusion $F_k \supseteq F_{k+1}$ holds for all $k \in \{0, 1, 2, \dots\}$. This completes the proof of the theorem.

Theorem 3.2 The limit of folding by cut of Classical Cantor set $[0, 1]$ is a 0-dimensional manifold.

Proof. Here we take closed interval $F_0 = [0, 1]$ and divided it into 3 equal subintervals.

We remove $p_1 = \left[0, \frac{1}{3}\right]$ and $p_2 = \left[\frac{2}{3}, 1\right]$ such that

$$[0, 1] - p_1 \cup p_2 = \left[\frac{1}{3}, \frac{2}{3}\right] = F_1.$$

Next subdivide F_1 into 3 equal subintervals and then remove two semi open intervals $p_3 = \left[\frac{1}{3}, \frac{4}{9}\right]$ and $p_4 = \left[\frac{5}{9}, \frac{2}{3}\right]$,

we get $F_1 - p_3 \cup p_4 = \left[\frac{4}{9}, \frac{5}{9}\right] = F_2$

In this way we divide the middle closed interval in three equal sub-intervals and then remove the outer two semi-closed intervals. We then find a set of closed intervals, i.e.,

$$F = [0, 1] - \cup p_n = \cap_{n=1}^{\infty} F_n$$

In each step, we removed $[a, a+w)$ and $(b-w, b]$ intervals from $[a, b]$, where $w = \frac{b-a}{3}$.

Theorem 3.3 The folding of Cantor string is nonempty.

Proof. In the folding of F_n to form F_{n+1} leaves a closed interval. For example removing $\left[0, \frac{1}{3}\right)$ and $\left(\frac{2}{3}, 1\right]$ from $[0,1]$ leaves $\left[\frac{1}{3}, \frac{2}{3}\right]$. In fact, since the folding of Cantor string is the infinite intersection of each F_n , F contains at least one sub-interval and is clearly nonempty and infinite.

Theorem 3.4 The folding of Cantor string is uncountable.

Proof. Georg Cantor demonstrated that real numbers cannot be put into one-to-one correspondence with the natural numbers and therefore that the set of real numbers has a greater cardinality than the set of natural numbers. By folding method, we get a nonempty closed subinterval of real numbers which is uncountable.

Theorem 3.5 The limit of folding of Cantor string $[0, 1]$ is equal to the limit of retraction.

Proof. Let $r_1 : I_1 = (a, b) \rightarrow [a+w, b-w]$, be the retraction and the limits of folding for $[0,1]$ are given by $f_i : I_i = [a, b] \rightarrow [a+w, b-w]$, where $w = \frac{b-a}{3}$.

$$\begin{aligned} \text{Let } r_1 : C_1 &\rightarrow C_2, C_2 \subset C_1 \\ r_2 : C_2 &\rightarrow C_3 \\ &\dots\dots\dots \\ r_n : C_n &\rightarrow C_{n+1} \end{aligned}$$

Then $r_i \equiv f_i$, fig.4 represents that there are homeomorphisms h_i such that

$$h_{n+1} \circ \lim_{n \rightarrow \infty} r_n = \lim_{n \rightarrow \infty} f_n \circ h_{n+1}.$$

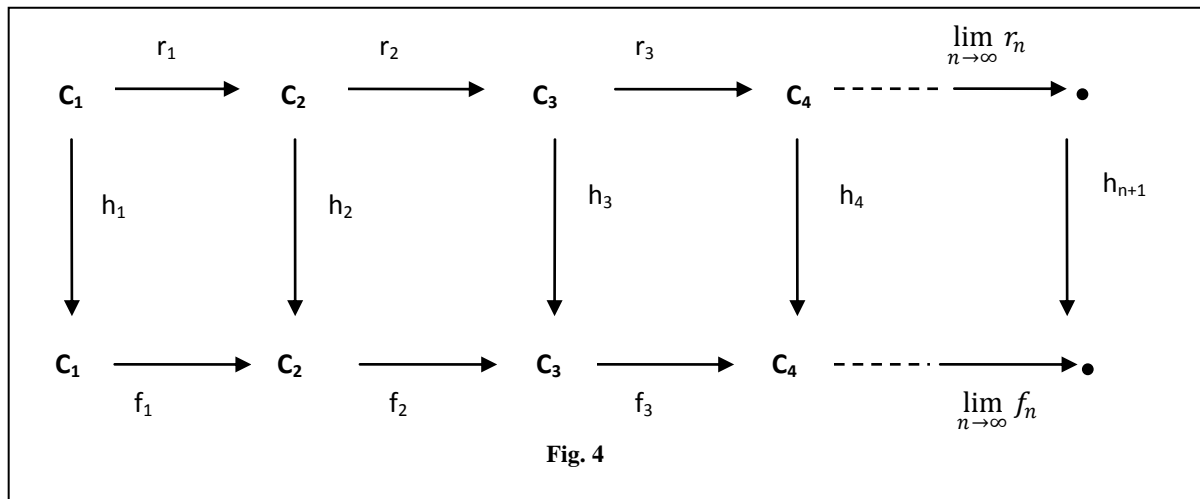


Fig. 4

4. CONCLUSION

In this paper, different Folding methods have been introduced in the study of Cantor string. Also, the limits of folding and retraction are identical. We have drawn different diagrams to give the description of our approach. Thus, our work is the application of folding in the field of Cantor set theory.

5. ACKNOWLEDGMENTS

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