International Journal of Computer Applications (0975 – 8887)
Volume 140 – No.4, April 2016

Two Warehouse Inventory Model with Ramp Type Demand and Partial Backordering for Weibull Distribution Deterioration

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ABSTRACT
In the recent years, the effect of deterioration of physical goods has drawn much attention of various researchers. The more the deteriorationis, the more order quantity would be. According to such consideration, taking the deterioration rate into account is necessary. Thus in this paper, we develop the two ware-house inventory model with partial backlogging and two parameter weibull distribution deterioration. In the present market scenario the demand of certain items does not remain constant with time and may increase/decrease for a fixed time interval. The objective of this paper is to derive the optimal replenishment policy considering varying demand rate and deterioration that minimize the present worth of total relevant inventory cost per unit of time. In addition to this single ware-house system is also developed and the results have been compared with the help of numerical example.

Keywords
Weibull deterioration distribution, partialbacklogging,ramp type demand

1. INTRODUCTION
In general, the demand is to be considered either constant or increasing with time. In traditional models many researchers have considered the demand rate as constant, linear time dependent ,stock dependent or exponentially increasing with time and the same trend of considering these type of demand is still continues but it is not always true that the demand occurs in same patterni.e. either constant or increasing with time. The assumption of constant demand rate is usually valid at the mature stage of the life cycle of a product. Several researchers e.g.[1-8] etc. have developed inventory models for items stored in two ware-houses under a variety of modelling assumptions. In these models the demand rate is considered to be constant over a time. However, in practice one would accept demand to vary with time. Donaldson[9] was the first to consider inventory model with time dependent demand and thereafter many researchers such as Goswami and Chaudhuri[10], Bhunia and Maiti[11,12]Banerjee and Agrawal[13] etc. were considered the time dependent demands for two-warehouse inventory systems.

Most of the above mentioned studies consider time varying demand rate either increasing or decreasing with time. However demand cannot increase/decrease continuously over time. e.g in case fashionable goods, demand initially increases with time if customers are attracted by the quality and price and once product accepted in the market, demand stabilises to a constant rate due to new arrivals of fashionable items. This type of stabilization has been considered in the literature since Ritchie[14].This type of demand is considered as “Ramp Type”. Manyresearchers have been considered this type of demand with combination of other assumptions.Wu[15]developed an inventory model withramp type demand rate, Weibulldistribution deterioration and partially backlogged shortages. Giri, Jalan, and Chaudhary[16]used an exponential ramp type function to represent demand. In all above models ,it was assumed that the time point at which demand become constants occurs before the time when the inventory depletes compeletly.Deng,Lin,and Chu[17]considered the case when demand becomes constant after the inventory level becomes zero.Skouri,Konstntras,papachristos,and Ganas[18]further extended the model of Deng et.al. by considering demand as a general type function. All these models considered ramp type demand function for a single ware house model. SwatiAgrawal&Snigdha Banarjee[19,20] developed a two ware-house inventory model with ramp type demand and partially backlogged shortages, one for non-deteriorating items and other with constant deterioration rate in which demand is general ramp type function of time is considered.

Most of the classical inventory models assumed the utility of inventory remains constant during their storage period. But in real life, deterioration does occur during the storage period. The problem of deteriorating inventory has received considerable attention in recent years. Most products such as medicine, blood, fish, alcohol, gasoline,vegetables and radioactive chemicals have finite self-life, and start to deteriorate once they are replenished. Above listed researchers, have taken care of deteriorating items in their models and developed the models accordingly.

In addition with deterioration of inventory, limited storage is also a major practical problem for real situation due to the lack of large storage space at the important market places, forcingretailers to own a small ware-house at important market places, however ,in order to take advantage of attractive price discount offered on bulk purchase, or in anticipation of growth in demand with time, it may be profitable for the retailer to order a quantity that exceeds the capacity of his own ware-house. At this situation retailer need extra space to store the bulk purchased items and hence prefer to rent a house for a limited period. In case deteriorating items,specially equipped storage facility is required to reduce the amount of deterioration. The cost of building such a storage facility for a limited period is usually exorbitant. Hence, it may be difficult for retailer to have such storage facility of his own at the retail outlet. To handle this situation the requirement of another storage space, providing the requisitefacilities become necessity.

In literature, own ware-house is abbreviated as OW and that of rented ware-house is as RW and generally it is assumed that the rented ware house provides better storage facilities as compared to own ware-house and due to this the rate of deterioration is smaller than OW which results more holding cost at RW, therefore retailer prefer to vacant RW by supplying demanded items earlier to the customers and then from OW.Due to arising problem of storage facility the concept of two ware-house inventory modelling come into existence. For two ware-house, time dependent demand was considered by some authors, such as [13-18].

For deteriorating items, Swati Agrawal and Snigdha Banerjeedeveloped inventory model with ramp type demand and partially backlogged shortages and they further extended the their own model by allowing deterioration for stored items and providing the option for using one or two storage facilities. In this paper they have considered constant deterioration rate in both ware-houses with demand rate as general ramp- type function of time.

Motivated by above papers, we developed a two ware-house inventory model for demand rate as general ramp-type function of time and two parameter Weibull distribution deterioration rate. If the shape parameter is equal to 1, then the Weibull distribution is reduced to be constant deterioration rate. Shortages are allowed and partially backordered at constant rate.

2. ASSUMPTIONS AND NOTATION

The mathematical model of the two-warehouse inventory problem is based on the following

Assumption and notations.

2.1 Assumptions

- Demand rate is ramp type.
- The lead time is zero or negligible and initial inventory level is zero.
- The replenishment rate is infinite.
- Shortages are allowed and partially backordered at constant rate.
- Deterioration rate is time dependent and follows a two parameter Weibull distribution where $\alpha, g > 0$denote scale parameter and β,h>1denote the shape parameter.
- The holding cost is constant and higher in RW than OW.
- The deteriorated units cannot be repaired or replaced during the period under review.
- Deterioration occurs as soon as items are received into inventory.

2.2 Notation

The following notation is used throughout the paper:Demand rate (units/unit time) which is ramp type given as

$$f(t_i)=\begin{cases} f(u) & \text{if } t_i > u \\ & for\ i = 1,2,3.. \\ f(t_i) & \text{if } t_i \leq u \end{cases}$$

W Capacity of OW

α Scale parameter of the deterioration rate in OW and 0<α<1

β Shape parameter of the deterioration rate in OW and β>1.

g Scale parameter of the deterioration rate in RW, $\alpha > g$

h Shape parameter of the deterioration rate in RW and h>1.

B Fraction of the demand backordered during the stock out period

C_o Ordering cost per order

d_1 Deterioration cost per unit of deteriorated item in RW

d_2 Deterioration cost per unit of deteriorated item in OW

h_w Holding cost per unit per unit time in OW

h_R Holding cost per unit per unit time in RW such that $h_R > h_o$

s_1 Shortage cost per unit per unit time

s_2 Shortage cost for lost sales per unit

Q_o The order quantity in OW

Q_R The order quantity in RW

Q_M Maximum ordered quantity after a complete time period T

I_k Maximum inventory level in RW k=1,2

T_1 Time with positive inventory in RW

T_1+T_2 Time with positive inventory in OW

T_3 Time when shortage occurs in OW

T Length of the cycle i.e.$T = T_1+T_2+T_3$

$I^t_{ij}(t_i)$ Inventory level for Δ_2-system at time t_i, $0 \leq t_i \leq T_i$, i= 1,2,3& j=1,2,3,4,5.

$I^s(t_i)$ Inventory level in RW at time t_1, $0 \leq t_i \leq T_i$

Δ_1 Identify single ware-house system

Δ_2 Identify two ware-house systems

φ^t Total cost per cycle for Δ_2 system

φ^s Total cost per cycle for Δ_1 system

φ^{ti} The present value of the total relevant inventory cost per unit time for Δ_2 system for case i= 1, 2, 3

φ^{si} The present value of the total relevant inventory cost per unit time for $\Delta_1$systemfor case i= 1, 2

R_s Initial inventory level for Δ_1 system

R_{tj} Initial inventory level for Δ_2 system

The rate of deterioration is given as follows:

t_i Time to deterioration, t_i>0

Instantaneous rate of deterioration in OW

$\theta_1(t_i)=\alpha\beta\, t_i^{(\beta-1)}$where $0 \leq \alpha < 1$ and $\beta > 1$

Instantaneous rate of deterioration in RW

$\theta_2(t_i)=g\ h\ t_i^{(h-1)}$ where$h > 1$,

3. MATHEMATICAL DEVELOPMENT OF MODEL

Descriptionfor Two ware-house model ($\boldsymbol{\Delta_2}$ −system)

In the two ware-house Inventory System, cycle length has been divided into three parts i.e. T_1,T_2 and T_3.For each replenishment, a portion of the replenished quantity is used to clear backlogged shortages, while the rest enters into the system. Wunits of items are stored in the OW and the rest is kept into the RW. To begin with analysis, it is necessary to compare the value of the constant parameter u with the possible values that the decision variables T_1 ,$T_2$and T_3 can take on. This results in the following three cases:

Case-1: $0\le u \le T_1$; Case-2: $0\le u \le T_2$; Case-3: $0\le u \le T_3$

Now we discuss each case separately.

Case-1: $\boldsymbol{0\le u \le T_1}$

In this case, the model parameters are such that the value of T_1 will be at least equal to u.The evolution of stock level in the system is depicted in **Figure-1**.

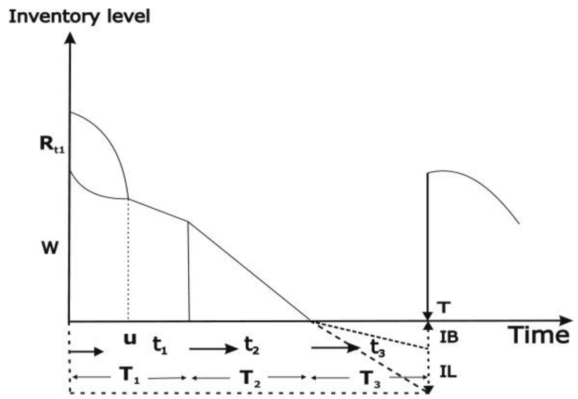


Figure 1:Inventory time graph for case-1 of Two ware house system

The inventory level at RW decreases due to combined effect of increasing demand and deterioration rate in the time interval (0 u) and due to constant demand and deterioration rate in the time interval (u T_1).Hence the inventory level is governed by the following differential equations

$$\frac{dI_{11}^t(t_1)}{dt_1} = -\alpha\beta t_1^{\beta-1} I_{11}^t(t_1) - f(t_1)\ ;\ 0 \le t_1 \le u \quad (1.1)$$

$$\frac{dI_{12}^t(t_1)}{dt_1} = -\alpha\beta t_1^{\beta-1} I_{12}^t(t_1) - f(u);\ u \le t_1 \le T_1 \quad (1.2)$$

With boundary conditions $I_{11}^t(u) = I_{12}^t(u)$ and $I_{12}^t(T_1) = 0$. The solution of 1.1 & 1.2 are resp.

$$I_{11}^t(t_1) = \left(f(u)\left\{(T_1 - u) + \frac{\alpha}{\beta+1}\left(T_1^{\beta+1} - u^{\beta+1}\right)\right\} - \int_{t_1}^{u} f(x) e^{\alpha x^\beta} dx\right) e^{-\alpha t_1^\beta} \quad (1.3)$$

$$I_{12}^t(t_1) = \left(f(u)\left\{(T_1 - u) + \frac{\alpha}{\beta+1}\left(T_1^{\beta+1} - u^{\beta+1}\right)\right\}\right) e^{-\alpha t_1^\beta} \quad (1.4)$$

Since, initially inventory level at RW is $I_1^t(0) = (R_{t1} - w)$, we obtain

$$R_{t1} = w + \left(f(u)\left\{(T_1 - u) + \frac{\alpha}{\beta+1}\left(T_1^{\beta+1} - u^{\beta+1}\right)\right\} - \int_0^u f(x) e^{\alpha x^\beta} dx\right) \quad (1.5)$$

Inventory level at OW during time interval (0 T_1) depletes due to deterioration only till inventory reaches zero at RW. The stock at OW depletes due to thecombined effect of constant

demand and deterioration during time interval (0 T_2). Thus differential equations governing this situation are given as

$$\frac{dI_{13}^t(t_1)}{dt_1} = -ght_1^{h-1} I_{13}^t(t_1);\quad 0 \le t_1 \le T_1 \quad (1.6)$$

$$\frac{dI_{14}^t(t_2)}{dt_2} = -ght_1^{h-1} I_{14}^t(t_2) - f(t_2);\qquad 0 \le t_2 \le T_2 \quad (1.7)$$

With boundary conditions $I_{13}^t(0) = w$and $I_{14}^t(T_2) = 0$. The solution of (1.6) & (1.7) are resp.

$$I_{13}^t(t_1) = we^{-g\, t_1^h} \quad (1.8)$$

$$I_{14}^t(t_2) = \left(f(u)\left\{(T_1 - t_2) + \frac{g}{h+1}\left(T_1^{\beta+1} - t_2^{\beta+1}\right)\right\}\right) e^{-g\, t_2^h} \quad (1.9)$$

During the time interval (0 T_3) at T_3=0 both warehouses are empty, and part of the shortage is backordered at the next replenishment and satisfy the differential equation

$$\frac{dI_{15}^t(t_3)}{dt_3} = -B\, f(u)\ ;\ 0 \le t_3 \le T_3 \quad (1.10)$$

With boundary condition
$I_{15}^t(0) = 0$, the solution of equation (1.10) is given as

$$I_{15}^t(t_3) = -B\, f(u) t_3 \quad (1.11)$$

The amount of inventory lost sales during the shortages period is

LS= (1-B) f (u) T_3

The total demand during time epoch T_1 at RW is

$$\int_0^u f(t_1) dt_1 + \int_u^{t_1} f(u) d\, t_1$$

Hence amount of inventory deteriorated during this period is

$$D_{1R} = R_{t1} - \int_0^u f(t_1) dt_1 - \int_u^{t_1} f(u) d\, t_1$$

The total demand during time epoch $T_1 + T_2$ at OW is $\int_0^{T_2} f(u) d\, t_2$

Hence amount of inventory deteriorated during this period is

$$D_{1w} = w - \int_0^{T_2} f(u) d\, t_2$$

Thus the total present worth inventory cost during the cycle length consist of the following cost elements

- Ordering cost C_o
- Inventory holding cost in RW
- Inventory holding cost in OW

- Shortages cost
- Lost sales cost
- Deterioration cost in RW
- Deterioration cost in OW

Hence the total relevant inventory cost per unit of time during cycle is given by

$$\varphi^{t1}(T_1, T_2, T_3) = \frac{1}{T}\begin{bmatrix} C_o + h_R\left(\int_0^{u} I_{11}^{t}(t_1)dt_1 + \int_0^{T_1} I_{12}^{t}(t_1)dt_1\right) + \\ h_w\left(\int_0^{T_1} I_{13}^{t}(t_2)dt_2 + \int_0^{T_2} I_{14}^{t}(t_2)dt_2\right) + \\ s_1\left(\int_0^{T_3}\left(-I_5^{t}(t_3)\right)dt_3\right) + s_2 LS + d_1 D_{1R} + d_2 D_{1w} \end{bmatrix} \quad (1.12)$$

Case-2: $0 \leq u \leq T_2$

In this case evolution of inventory level in the system is depicted by **Figure-2**.

The inventory level at RW decreases due to combined effect of increasing demand and deterioration rate in the time interval (0 T_1).Hence the inventory level is governed by differential equation

$$\frac{dI_{21}^{t}(t_1)}{dt_1} = -\alpha\beta t_1^{\beta-1} I_{21}^{t}(t_1) - f(t_1); \ 0 \leq t_1 \leq T_1 \quad (2.1)$$

With boundary condition $I_{21}^{t}(T_1) = 0$. The solution of equation(2.1) is

$$I_{21}^{t}(t_1) = \left(\int_0^{T_1} f(x)e^{\alpha x^{\beta}}dx - \int_0^{t_1} f(x)e^{\alpha x^{\beta}}dx\right)e^{-\alpha t_1^{\beta}} \quad (2.2)$$

Further, since initially inventory level in RW is $I_{21}^{t}(0) = (R_{t2} - W)andT_1 > 0$,we obtain

$$R_{t2} = \left(w + \int_0^{T_1} f(x)e^{\alpha x^{\beta}}dx)\right) > W$$

Remark-1:If $T_1 = 0$ the above inequality does not satisfy and the inventory level at RW will be zero. This situation arises when the initial inventory level is less than or equal to W and corresponds to Case-1 of the **Δ_1- system**, discussed later in section (4.0).

Applying similar arguments to OW over the time interval (0 T_1) and (0 T_2) the following differential equations are obtained

$$\frac{dI_{22}^{t}(t_1)}{dt_1} = -ght_1^{h-1} I_{22}^{t}(t_1); \qquad 0 \leq t_1 \leq T_1 \quad (2.3)$$

$$\frac{dI_{23}^{t}(t_2)}{dt_2} = -ght_2^{h-1} I_{23}^{t}(t_2) - f(t_2); \qquad 0 \leq t_2 \leq u \quad (2.4)$$

$$\frac{dI_{24}^{t}(t_2)}{dt_2} = -ght_2^{h-1} I_{24}^{t}(t_2) - f(u); \qquad u \leq t_2 \leq T_2 \quad (2.5)$$

With boundary conditions$I_{22}^{t}(0) = W$,$I_{22}^{t}(T_1) = I_{23}^{t}(0)$ and $I_{24}^{t}(T_2) = 0$ The solution of (2.3), (2.4) & (2.5 are respectively

$$I_{22}^{t}(t_1) = we^{-g\,t_1^{h}} \quad (2.6)$$

$$I_{23}^{t}(t_2) = \left(We^{-g\,T_1^{h}} - \int_0^{t_2} f(x)e^{g\,x^{h}}dx\right)e^{-g\,t_2^{h}} \quad (2.7)$$

$$I_{24}^{t}(t_2) = \left(f(u)\left\{(T_2 - t_2) + \frac{g}{h+1}\left(T_2^{\beta+1} - t_2^{\beta+1}\right)\right\}\right)e^{-g\,t_2^{h}} \quad (2.8)$$

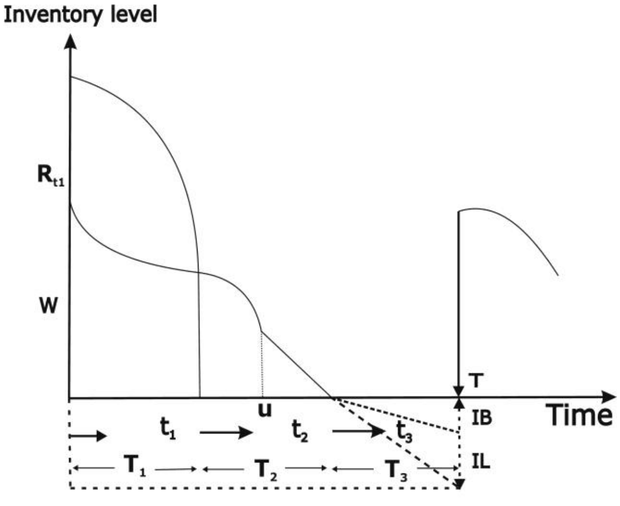


Figure- 2:Inventory time graph for case-2 of Two ware house system

Further, during the time interval (0 T_3) at T_3=0 both warehouses are empty, and part of the shortage is backordered at the next replenishment and is same as equation (1.10) of case-1, i.e.

$$I_{15}^{t}(t_3) = -B\,f(u)t_3 \quad (2.9)$$

The amount of inventory lost sales during the shortages period is

LS= (1-B) f (u) T_3

The total demand during time epoch T_1 at RW is$\int_0^{T_1} f(t_1)dt_1$ and therefore the amount of deteriorated inventory at RW is

$$D_{2R} = R_{t2} - \int_0^{T_1} f(t_1)dt_1$$

The total demand during time epoch $T_1 + T_2$ at OW is $\int_0^{u} f(t_2)dt_2 - \int_u^{T_2} f(u)d\,t_2$

And amount of inventory deteriorated during the period$T_1 + T_2$ at OW is

$$D_{2w} = w - \int_0^{u} f(t_2)dt_2 - \int_u^{T_2} f(u)d\,t_2$$

Hence in this case the total cost per unit of time during cycle is given by

$$\varphi^{t2}(T_1, T_2, T_3) = \frac{1}{T}\Big[C_o + h_R\left(\int_0^{T_1} I_{21}^{t}(t_1)dt_1\right) + h_w\left(\int_0^{T_1} I_{22}^{t}(t_2)dt_2 + \int_0^{u} I_{23}^{t}(t_2)dt_2 + \int_u^{T_2} I_{24}^{t}(t_2)dt_2\right) + s_1\left(\int_0^{T_3} -I_5^{t}(t_3)dt_3\right) + s_2 LS + d_1 D_{2R} + d_2 D_{2w}\Big] \quad (2.10)$$

Case-3: $0 \leq u \leq T_3$

In this case evolution of inventory level in the system is depicted by **Figure-3**.

The inventory level both at RW and OW become zero before the demand is stabilizes, Thus inventory level at both the ware-houses decreases due to combined effect of increasing demand and deterioration rate in the time interval (0 T_1).Hence the inventory level governed by differential equation is given as

$$\frac{dI_{31}^{t}(t_1)}{dt_1} = -\alpha\beta t_1^{\beta-1} I_{31}^{t}(t_1) - f(t_1); \quad 0 \le t_1 \le T_1 \qquad (3.1)$$

With boundary condition $I_{21}^{t}(T_1) = 0$. The solution of (3.1) is

$$I_{31}^{t}(t_1) = \left(\int_0^{T_1} f(x)e^{\alpha x^{\beta}}dx - \int_0^{t_1} f(x)e^{\alpha x^{\beta}}dx\right)e^{-\alpha t_1^{\beta}}$$

Further, since initially, inventory level in RW is $I_{31}^{t}(0) = (R_{t3} - W)$and $T_1 > 0$, therefore we obtain

$$R_{t3} = \left(w + \int_0^{T_1} f(x)e^{\alpha x^{\beta}}dx)\right) > W$$

Remark-2: If $T_1 = 0$ the above inequality does not satisfy and the inventory level at RW will be zero. This situation arises when the initial inventory level is less than or equal to W and corresponds to Case-2 of the **Δ_1- system**, discussed later in section (4.0).

At OW, inventory level decreases due to deterioration over time interval (0 T_1) and over (0 T_2) due to the increasing demand rate and deterioration. Thus the following differential equations are obtained related to this situation

$$\frac{dI_{32}^{t}(t_1)}{dt_1} = -ght_1^{h-1} I_{32}^{t}(t_1); \quad 0 \le t_1 \le T_1 \qquad (3.2)$$

$$\frac{dI_{33}^{t}(t_2)}{dt_2} = -ght_2^{h-1} I_{33}^{t}(t_2) - f(t_2)\,; 0 \le t_2 \le T_2 \qquad (3.3)$$

With boundary conditions$I_{32}^{t}(0) = W$, and $I_{33}^{t}(T_2) = 0$,the solution of (3.2) & (3.3) are respectively

$$I_{32}^{t}(t_1) = we^{-g\,t_1^{h}} \qquad (3.4)$$

$$I_{33}^{t}(t_2) = \left(\int_0^{T_1} f(x)e^{gx^{h}}dx - \int_0^{t_1} f(x)e^{gx^{h}}dx\right)e^{-gt_1^{h}} \qquad (3.5)$$

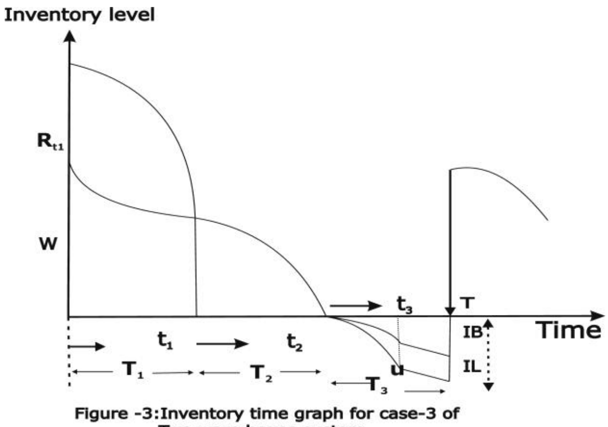


Figure -3:Inventory time graph for case-3 of Two ware house system

Further, during the time interval (0 T_3) at T_3=0 both warehouses are empty, and part of the shortage is backordered at the next replenishment and differential equations governing this situation are given as.

$$\frac{dI_{34}^{t}(t_3)}{dt_3} = -B\,f(t_3); \qquad 0 \le t_3 \le u \qquad (3.6)$$

$$\frac{dI_{35}^{t}(t_3)}{dt_3} = -B\,f(u); \qquad 0 \le t_3 \le T_2 \qquad (3.7)$$

With boundary conditions$I_{34}^{t}(0) = 0$, and $I_{34}^{t}(u) = I_{35}^{t}(u)$,the solution of (3.6) & (3.7) are respectively

$$I_{34}^{t}(t_3) = -B\int_0^{t_3} f(x)dx \qquad (3.8)$$

$$I_{35}^{t}(t_3) = -B\left(f(u)(t_3 - u) + \int_0^{u} f(x)dx\right) \qquad (3.9)$$

The amount of inventory lost sales during the shortages period is

$$LS= (1\text{-}B)\left(\int_0^{u} f(t_3)dt_3 + \int_0^{T_3} f(t_3)dt_3\right)$$

The total demand during time epoch T_1 at RW is$\int_0^{T_1} f(t_1)dt_1$ and therefore

The amount of deteriorated inventory at RW is

$$D_{3R} = R_{t3} - \int_0^{T_1} f(t_1)dt_1$$

The total demand during time epoch $T_1 + T_2$ at OW is $\int_u^{T_2} f(t_2)\ d\,t_2$ and therefore

Amount of inventory deteriorated during the period $T_1 + T_2$ at OW is

$$D_{3w} = W - \int_u^{T_2} f(t_2)\ d\,t_2$$

Hence in this case the total cost per unit of time during cycle is given by

$$\varphi^{t3}(T_1,T_2,T_3) = \frac{1}{T}\Big[C_o + h_R\left(\int_0^{T_1} I_{31}^{t}(t_1)dt_1\right) + h_w\left(\int_0^{T_1} I_{32}^{t}(t_2)dt_2 + \int_0^{T_2} I_{33}^{t}(t_2)dt_2\right) + s_1\left(\int_0^{u} -I_{34}^{t}(t_3)dt_3 + \int_u^{T_3} -I_{35}^{t}(t_3)dt_3\right) + s_2 LS + d_1 D_{3R} + d_2 D_{3w}\Big] \qquad (3.10)$$

Combining (1.12), (2.10) & (3.10) ,the cost function $\varphi^{t}(T_1,T_2,T_3)$ of the problem results in the following three-branch function corresponding to the three cases.

$$\varphi^{t}(T_1,T_2,T_3) = \begin{cases} \varphi^{t1}(T_1,T_2,T_3) & 0 \le u \le T_1 \\ \varphi^{t2}(T_1,T_2,T_3) & 0 \le u \le T_2 \\ \varphi^{t3}(T_1,T_2,T_3) & 0 \le u \le T_3 \end{cases} \qquad (3.11)$$

Optimality condition forΔ_2 −system

The optimal problem can be formulated as

Minimize:$\varphi^{t}(T_1,T_2,T_3)$
Subject to: $(T_1 > 0, T_2 > 0, T_3 > 0)$

To find the optimal solution of the equation the following condition must be satisfied

$\frac{\partial\varphi^t(T_1,T_2,T_3))}{\partial T_1}$=0; $\frac{\partial\varphi^t(T_1,T_2,T_3)}{\partial T_2}=0$; $\frac{\partial\varphi^t(T_1,T_2,T_3)}{\partial T_3}=0$
(3.12)

Solving equation (3.12) respectively forT_1 ,T_2 ,T_3 , we can obtain T_1^*, T_2^*, T_3^*, and T and with these optimal values we can find the total minimum inventory cost from equation (3.11) for three branches separately.

4. DESCRIPTION FOR SINGLE WARE-HOUSE ($\boldsymbol{\Delta_1}$ −SYSTEM)

In this system, we consider the two cases one with unlimited capacity of a ware-house i.e. a rented ware-house (RW) system and other one with limited capacity i.e. own ware-house (OW) system, separately and shall briefly present the analysis. As stated in remarks 1 and 2, this may be treated as a particular case of $\boldsymbol{\Delta_2}$ **−system** by relaxing the condition $T_1 > 0$

4.1RW Case

At $t_1 = 0$, the amount of inventory ordered enters into the system. Part of it used to fulfil the backlogged shortages of the previous period and R_{s1} units are stored in RW.The following two subcases arises according to increasing demand period and must be examined:

Subcase-1: $0 \le u \le T_1$; Subcase-2: $0 \le u \le T_3$

Now each case will be considered separately.

Subcase-1: $0 \le u \le T_1$

In this case inventory level depleted due to combined effect of both increasing demand and deterioration over time period (0 u) and due to constant demand over ($0T_1$).At the point of time T_1, inventory vanishes and shortages partially backlogged accumulate till timeT_3.The situation is depicted by **Figure-4**.

The total cost function per unit of time is found to be

$$\varphi^{s1}(T_1,T_3) = \frac{1}{T}\Big[C_o + h_R\left(\int_0^u I_{11}^t(t_1)dt_1 + \int_u^{T_1} I_{12}^t(t_1)dt_1\right) + s_1\left(\int_0^{T_3} -I_3^t(t_3)dt_3\right) + s_2LS + d_1D_{1R}\Big] \quad (4.1)$$

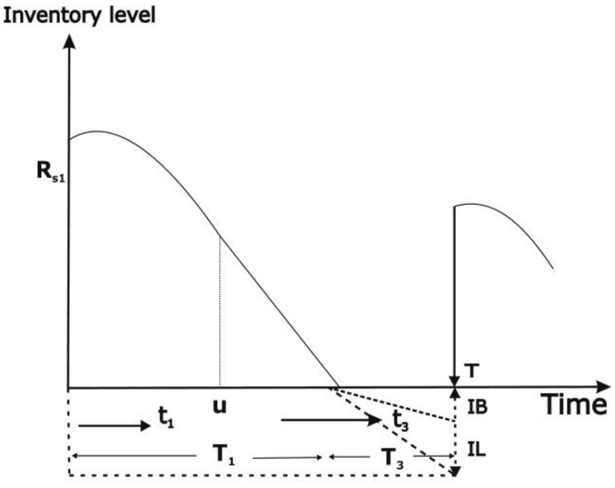


Figure 4:Inventory- time graph for case-1 of single warehouse system

The level R_{s1} found to be

$$R_{s1} = \left(f(u)\left\{(T_1-u) + \frac{\alpha}{\beta+1}\left(T_1^{\beta+1} - u^{\beta+1}\right)\right\} - \int_0^u f(x)e^{\alpha x^\beta}dx\right)$$

Case2: $0 \le u \le T_3$

In this case inventory level depleted due to increasing demand and deterioration over time period ($0T_1$). At the point of time T_1 , inventory vanishes and shortages occur due to increasing demand in the period (0 u) and due to constant demand over time period ($0T_3$) and partially backlogged accumulate till time T_3. The situation is presented by **Figure-5**.

The total cost function per unit of time is found to be

$$\varphi^{s2}(T_1,T_3) = \frac{1}{T}\Big[C_o + h_R\left(\int_0^{T_1} I_{21}^t(t_1)dt_1\right) + s_1\left(\int_0^u -I_{22}^t(t_3)dt_3 + \int_u^{T_3} -I_{23}^t(t_3)dt_3\right) + s_2LS + d_1D_{2R}\Big] \quad (4.2)$$

The level R_{s2} found to be

$$R_{s2} = \left(\int_0^{T_1} f(x)e^{\alpha x^\beta}dx\right)$$

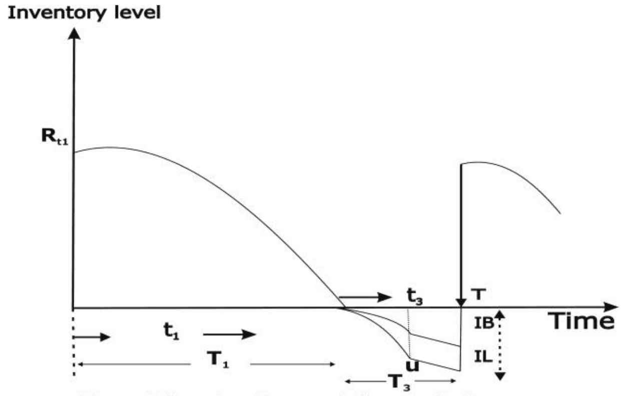


Figure -5:Inventory time graph for case-2 of single ware house system

Thus cost function per unit of time for the cycle for these two subcases shall be given by two branch function combining equations (4.1) & (4.2) as follows

$$\varphi^s(T_1,T_3) = \begin{cases} \varphi^{s1}(T_1,T_3) & 0 \le u \le T_1 \\ \varphi^{s2}(T_1,T_3) & 0 \le u \le T_3 \end{cases} \quad (4.3)$$

Optimality condition forRW Case ($\boldsymbol{\Delta_1}$-system)

The optimal problem for this system can be formulated as

Minimize:$\varphi^s(T_1,T_3)$

Subject to: $(T_1 > 0, T_3 > 0)$

To find the optimal solution of the equation the following condition must be satisfied

$\frac{\partial\varphi^s(T_1,T_3)}{\partial T_1}$=0;$\frac{\partial\varphi^s(T_1,T_3)}{\partial T_3}=0$;

4.2 OW Case

At $t_1 = 0$, the amount of inventory ordered, enters into the system. Part of it used to fulfil the backlogged shortages of the previous period and a maximum of W units are stored in OW. The following two sub cases arises according to increasing demand period and must be examined:

Subcase-1: $0 \le u \le T_1$; Subcase-2: $0 \le u \le T_3$

Now each case will be considered separately.

Subcase-1: $0 \le u \le T_1$

In this case inventory level depleted due to combined effect of both increasing demand and deterioration over time period $(0\ \ u)$ and due to constant demand over $(0T_1)$.At the point of time T_1, inventory vanishes and shortages partially backlogged accumulate till time T_3. The situation is presented by **Figure-6**.

The total cost function per unit of time is found to be

$\varphi^{s1}(T_1,T_3) =$
$\frac{1}{T}\Big[C_o + h_W\left(\int_0^u I_{11}^t(t_1)dt_1 + \int_u^{T_1} I_{12}^t(t_1)dt_1\right) +$
s10T3−I3tt*3d*t*3+s2LS+*d *1*D*1W*
(4.4)

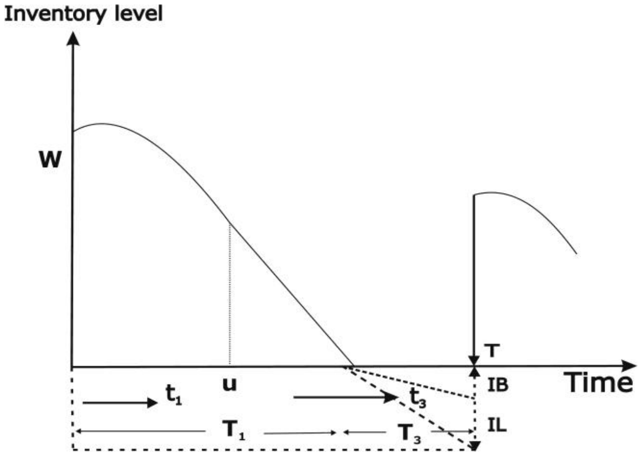


Figure 6:Inventory- time graph for case-1 of single warehouse system

Subcase2: $0 \le u \le T_3$
In this case inventory level depleted due to increasing demand and deterioration over time period $(0T_1)$. At the point of time T_1, inventory vanishes and shortages occur due to increasing demand in the period $(0\ \ u)$ and due to constant demand over time period $(0T_3)$ and partially backlogged accumulate till timeT_3. The situation is presented by **Figure-7**.

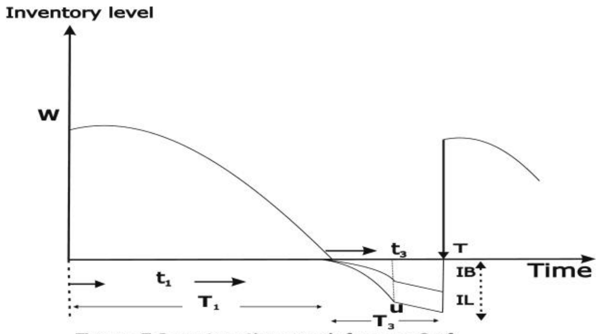


Figure -7:Inventory time graph for case-2 of single ware house system

The total cost function per unit of time is found to be

$\varphi^{s2}(T_1,T_3) = \quad \frac{1}{T}\Big[C_o + h_W\left(\int_0^{T_1} I_{21}^t(t_1)dt_1\right) +$
s10u−I22tt*3d*t*3+uT3*−I23tt*3d*t*3+*
*s2LS+*d *1*D*2W* (4 Thus cost function per unit of time for the cycle for these two cases shall be given by two branch function combining equations (4.1) & (4.2) as follows

$$\varphi^{s}(T_1,T_3) = \begin{cases} \varphi^{s1}(T_1,T_3) & 0 \le u \le T_1 \\ \varphi^{s2}(T_1,T_3) & 0 \le u \le T_3 \end{cases}$$

(4.6)

Optimality condition forOW case ($\boldsymbol{\Delta_1}$system)

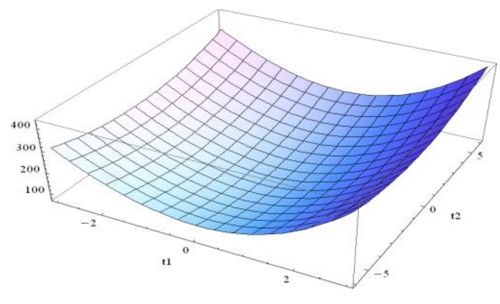
The optimal problem for this system can be formulated as

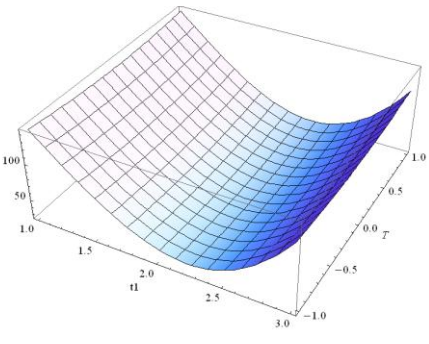
Minimize:$\varphi^{s}(T_1,T_3)$

Subject to: $(T_1 > 0, T_3 > 0)$

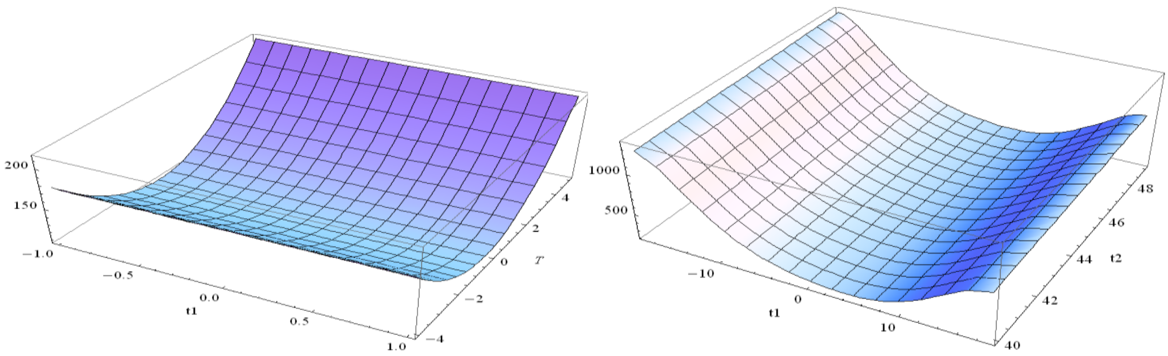
To find the optimal solution of the equation the following condition must be satisfied

$$\frac{\partial\varphi^{s}(T_1,T_3)}{\partial T_1}=0;\frac{\partial\varphi^{s}(T_1,T_3)}{\partial T_3} = 0$$





RW-1 **OW-2**



RW-12 **OW-1**

Figure-8: Graphical representation of convexity of Δ_1- system

Table-1

	Δ_1-system							
	RW				OW			
Subcase	T_1^*	T_3^*	T	Total cost	T_1^*	T_3^*	T	Total cost
1	0.5605	5.6578	6.218	172.34	18.216	48.364	66.58	864.39
2	0.5671	4.4464	5.013	192.99	2.2598	0.689	2.949	29.740

Table-2

	Δ_1-system									
	RW					OW				
Subcase	T_1^*	T_3^*	T	R_{sr}	Total cost	T_1^*	T_3^*	T	R_{sw}	Total cost
1	17.0459	2.3329	19.38	2448	4328.5	17.6866	1.3418	19.028	50	1957.94
2	1.0064	0.2953	1.302	99	525.89	2.2215	0.0688	2.2903	50	4.0813

	Δ_2-system				
Case	T_1^*	T_2^*	T_3^*	T^*	Total cost
1	1.0176	7.0212	0.1206	8.1594	407.44
2	0.8332	7.1320	0.2392	8.2044	462.94
3	0.8561	1.4240	0.2775	2.5576	484.95

Table-3

Sensitivity analysis for the case-1 of two ware-house model with change in the value of one parameter keeping rest unchanged

Initial value of parameter	change value of parameter	T_1^*	T_2^*	T_3^*	Change in cycle length T^*	% change in cycle length	Change in Total cost	% change in total cost
30	15	1.0364	6.8895	0.3355	8.2614	-1.25009	254.00	-37.66
A	45	1.0108	7.0616	0.0484	8.1208	0.473074	560.50	37.57
4.5	2.25	1.0332	6.9514	0.2382	8.2228	-0.77702	302.40	-25.78
b	6.75	1.0098	7.0535	0.0632	8.1265	0.403216	512.31	25.74
500	250	0.9194	7.0582	0.0561	8.0337	1.540554	376.54	-7.58
c	750	1.1133	6.9833	0.1856	8.2822	-1.50501	437.85	7.46
4.5	2.25	1.0333	6.9392	0.2579	8.2304	-0.87016	308.42	-24.30
u	6.75	1.0098	7.0707	0.3190	8.3995	-2.94262	495.70	21.66
3	1.5	1.8368	7.0365	0.0937	8.967	-2.94262	394.85	-3.09
h_1	4.5	0.6948	7.0182	0.1259	7.8389	3.927985	409.91	0.61
2	1.9	0.9225	7.0114	0.0464	7.9803	2.195014	372.72	-8.52
h_2	2.1	1.1999	7.0400	0.2649	8.5048	-4.23315	486.42	19.38
8	4	1.0163	7.0217	0.2394	8.2774	-1.44618	407.03	-0.09

S_c	12	1.0181	7.0211	0.0806	8.1198	0.48533	407.58	0.04
9	8.1	1.0155	7.0208	0.0464	8.0827	0.94002	406.76	-0.16
L_c	9.9	1.0188	7.0221	0.1942	8.2351	-0.92776	407.78	0.08
W	25	1.1023	7.0455	0.0777	8.2255	0.81075	387.76	-4.83
50	75	0.9231	6.9968	0.0627	7.9826	-0.81011	427.18	4.85
0.6	0.54	1.0189	7.0207	0.0099	8.0495	1.346913	407.85	0.10
B	0.9	0.9917	7.0312	0.4437	8.4666	-3.76498	399.24	-2.01
4	2	1.1667	7.0292	0.2108	8.4067	-3.03086	449.66	10.36
d_1	6	0.9927	7.0261	0.1120	8.1308	0.350516	403.42	-0.98
4.5	4.05	1.1264	7.0323	0.1946	8.3533	-2.3764	442.05	8.49
d_2	4.95	0.9050	7.0108	0.0450	7.9608	2.434003	37.04	-90.90
0.02	0.01	1.0417	7.0215	0.1200	8.1832	-0.29169	407.18	-0.06
α	0.03	0.9959	7.0209	0.1211	8.1379	0.2635	407.68	0.06
2	1	1.0237	7.0213	0.1207	8.1657	-0.07721	407.48	0.01
β	3	1.0146	7.0213	0.1205	8.1564	0.036767	407.37	-0.01
0.05	0.04	1.2431	7.8814	0.0278	9.1523	-12.1689	481.22	18.11
G	0.06	0.8471	6.3852	0.0049	7.2372	11.3023	353.30	-13.28
2	1	4.7889	34.392	3.7761	42.957	-426.473	2118.2	419.88
H	2.2	1.1560	6.3660	0.2157	7.7377	5.168272	183.83	-54.88

Note: Because B,,L_c,h_2,h_1 and g are found to be unreasonable when $\mp 50\%$ changes are made therefore sensitivity analysis is performed by changing $\mp 5\%$ to $\mp 20\%$ at a time.

5. NUMERICAL EXAMPLES

To analyse the model, we consider the demand rate function to be f(t)= Ae^{bt}, where A is the initial demand rate at t=0 and b is the shape parameter .If b=o, demand remain constant with time, while b>0 implies that demand increases exponentially with time. In view of Tailor series expansion, for finite values of t, small value of b would be good representations of the linear, quadratic etc. polynomials of degree depending upon the magnitudes of b and u.We consider two different parameter sets corresponding to the situations where (i) $\boldsymbol{\Delta_1}$ –**system** is optimal and (ii) $\boldsymbol{\Delta_2}$ –**system** is optimal.

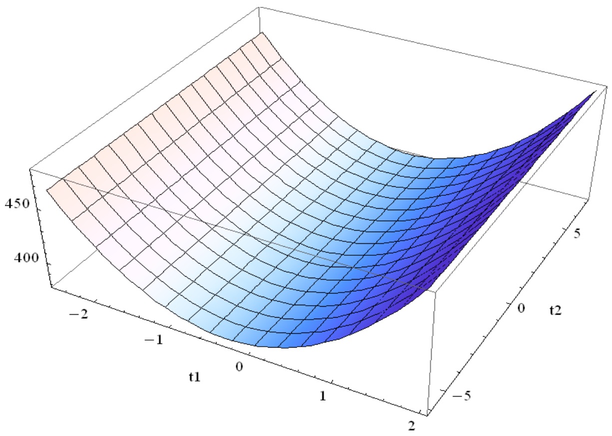
Example-1: Consider the following set of values of parameters:$A = 30. b = 4.5, C_o = 500, u = 0.5, h_1 = 3.0, h_2 = 2.0, W = 50, S_c = 0.15, L_c = 0.20, d_1 = 4.0, d_2 = 4.5, \alpha = 0.02, \ \beta = 2, g = 0.05, h = 2, B = 0.6$,.The optimal results obtained from the $\boldsymbol{\Delta_1}$ –system has shown in **Table-1**. For these parameters of value the RW has optimal solution in subcase-1 and OW has optimal value in subcase-2.

Example2:
Next we increase the value ofS_c and L_c much higher than .Let us take $S_c = 8$ and $L_c = 9$,Keeping the other set of values of parameters same $as\ in\ example - 1$.For the $\boldsymbol{\Delta_1}$ –system the optimal values obtained are shown in **Table -2**. In subcase-1($R_{s1} = 2448 > W$) is much higher than W.We must check for $\boldsymbol{\Delta_2}$ –**system** and found that the total cost in three cases of $\boldsymbol{\Delta_2}$ – system are less expensive than in subcase-1of RW and OW of$\boldsymbol{\Delta_1}$ – **system**.

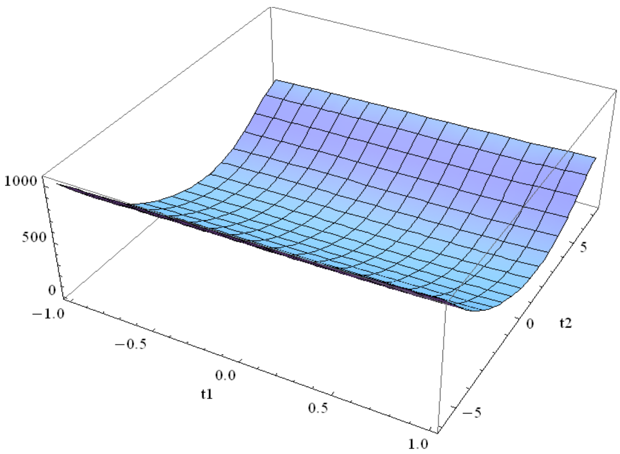
Numerical results
We conclude from the above numerical result as follows:

1. From Table-1, when all the given conditions and constraints are satisfied, the optimal solution is obtained. In this example the minimal present value of total relevant inventory cost per unit time in an appropriate unit for subcase -1 is 172.34 and 192.99 for subcase -2, for RW-Case and 864.39 for subcase-1 and 29.74 for subcase-2 for OW-Case. From the above result we see that in subcase- 2,OW- system is less expensive while in case -1 ,RW system is less expensive.
2. From Table-2, when all the given conditions and constraints are satisfied, the optimal solution is obtained. In this example the minimal present value of total relevant inventory cost per unit time in an appropriate unit is less expensive for each case for $\boldsymbol{\Delta_2}$- system when there is bulk amount of inventory purchased as compared to subcase-1 of RW and OW of $\boldsymbol{\Delta_1}$ – **system**. In subcase-2 of RW $\boldsymbol{\Delta_1}$ – **system** is less expensive when capacity of ware house is taken to be unlimited and in case limited capacity i.e. below 50 units of inventory subcase-2 of OW is very less expensive as compared to $\boldsymbol{\Delta_2}$ – **system.**
3. The convexity of graphs shown in **Figure-8** for each case of $\boldsymbol{\Delta_1}$ – **system** and in **Figure- 9** for each cases of $\boldsymbol{\Delta_2}$-**system** shows that there are points where inventory system has minimal cost depending upon the capacity of ware-houses and cycle length and that point is unique.
4. The parabolic graphs shown in **Figure-10** for $\boldsymbol{\Delta_1}$-system and in **Figure-11** for $\boldsymbol{\Delta_2}$- system shows that inventory cost is directly proportional to the cycle length and has a minimal point where the cost function is minimal.

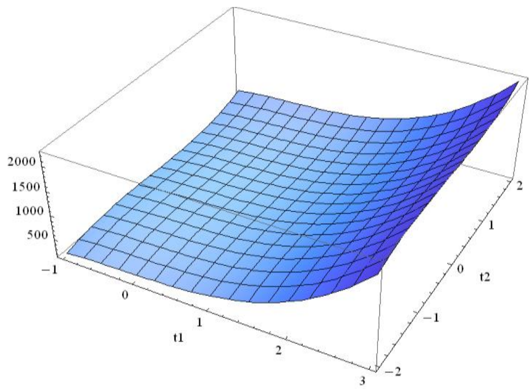
2



Csae-1

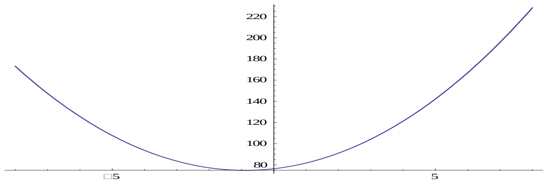


Case-2

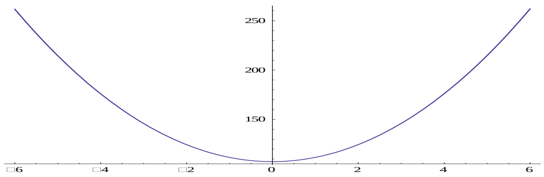


Case-3

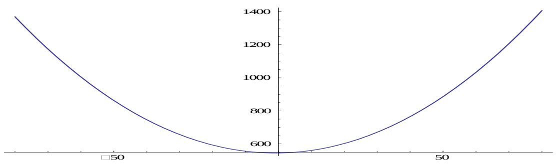
Figure-9: Graphical representation of convexity for Δ_2– system



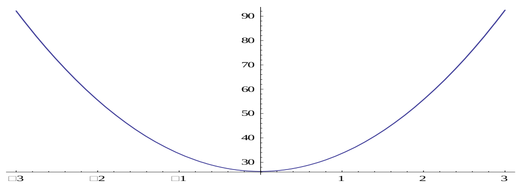
RW-1



RW-1

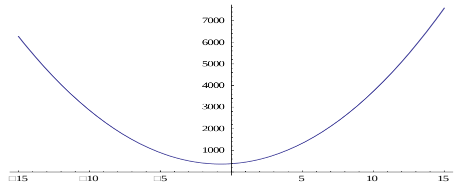


OW -1

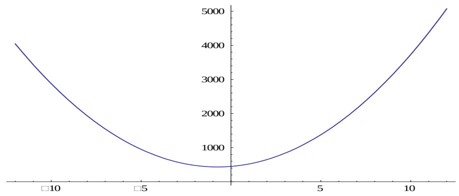


OW -2

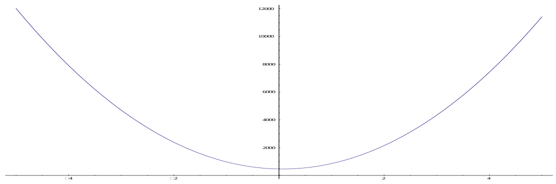
Figure-10: Graph presenting Cycle length verses Inventory cost for Δ_1 -system



Case-1



Csae-2



Case-3

Figure-11: Graph presenting Cycle length verses Inventory cost for Δ_2–system

5.1 Sensitivity Analysis

It is clear from table -3 that the following phenomenon can be obtained.

- As the value of A,b,c and u and holding cost in OW increases, the total present worth relevant inventory cost per unit of time increases and is highly sensitive to these parameters.
- Total present worth relevant inventory cost per unit of time increases as h_2,s_c,L_c, W increases and is moderately sensitive to these parameters change.
- Total present worth relevant inventory cost per unit of time decreases asd_1,d_2,B,g increases and slightly sensitive to B , moderately sensitive to d_1 and highly sensitive to g and d_2 but consistent with change of parameters α and β.
- Total present worth relevant inventory cost per unit of time is very highly sensitive to the shape parameter h and increases as h increases.

6. CONCLUSION

In this paper, a deterministic inventory model is presented to determine the optimal replacement cycle for two warehouse inventory problem under varying rate of deterioration and partial backlogging .The model assumes that the capacity of distributors' warehouse is limited. The optimization technique is used to derive the optimum replenishment policy i.e. to minimize the total relevant cost of the inventory system. A numerical example is presented to illustrate the model. When there is single ware house is assumed in the inventory system then the total relevant cost per unit time of the system are higher than the two warehouse model. This model is most useful for the instant deteriorating items under Weibull distribution deterioration rate as inventory cost depending on demand is indirectly proportional to demand. Further this paper can be enriched by incorporating other types of time dependent demand and another extension of this model may be for a bulk release pattern with combination of other realistic factors. In practice, now days pricing and advertising also have effect on the demand rate and must be taken into consideration.

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