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Common Fixed Point for Weakly Compatible Maps in Complete Metric Spaces

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ABSTRACT
In this paper the concept of weakly compatible map in complete metric space has been applied to prove common fixed point theorem for four mappings satisfying implicit relations.

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Keywords
Common fixed point, complete metric space, compatible maps, weakly compatible maps.

1. INTRODUCTION

The study of common fixed point of mappings satisfying contractive type conditions has been a very active field of research activity during the last three decades. In 1922, the Polish mathematician, Banach, proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. His result is called Banach's fixed point theorem or the Banach contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical science and engineering. Many authors have extended, generalized and improved Banach's fixed point theorem in different ways. In [2], Jungck introduced more generalized commuting mappings, called *compatible mappings*, which are more general than commuting and weakly commuting mappings. The concept of the commutativity has generalized in several ways. For this Sessa S [6] has introduced the concept of weakly commuting and Gerald Jungck [2] initiated the concept of compatibility. In 1998, Jungck and Rhoades [4] introduced the notion of weakly compatible and showed that compatible maps are weakly compatible but not conversely. Brian Fisher [1] proved an important Common Fixed Point theorem.
The aim of the present paper is to prove a common fixed point theorem on complete metric spaces. Throughout this paper, let (X,d) be a complete metric space unless mentioned otherwise.

2. PRELIMINARIES
We recall some definitions and known results.

Definition 2.1. A sequence $\{x_n\}$ in a metric space (X,d) is said to be convergent to a point $x \in X$, denoted by $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} d(x_n, x) = 0$.

Definition 2.2. A sequence $\{x_n\}$ in a metric space (X,d) is said to be Cauchy sequence if $\lim_{t\to\infty} d(x_n, x_m) = 0$ for all $n,m > t$.

Definition 2.3. A metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent.

Remark 2.1. In general a convergent sequence in a metric space (X,d) need not be Cauchy but every convergent sequence is a Cauchy sequence whenever metric d is continuous. A metric d on a set X is said to be weakly continuous if every convergent sequence under d is Cauchy.

Definition 2.4. [2] Let S and T be mappings from a metric space (X,d) into itself. The mappings S and T are said to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Definition 2.5. [4] A pair of self mappings S and T of a metric space (X,d) is said to be weakly compatible if $Sx = Tx$ (for some $x \in X$) implies $STx = TSx$.

Definition 2.7. A pair (S,T) of self-mappings of a metric space is said to be semi-compatible if $\lim_{n\to\infty} STx_n = Tx$; whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$.

Proposition 2.1. Let (S,T) be a compatible pair of self maps on a metric space (X,d) and T be continuous. Then the pair (S,T) is weakly compatible.

It is noted that a compatible maps are weakly compatible but weakly compatible maps need
not be compatible.
The converse is not true as seen in following example.

Example 2.1 Let $x = [0,2]$ with usual metric d where $d(x,y) = |x - y|$ for all x and y in X. Let for all x and y in X and $t > 0$,

Define:

$$T(x) = \begin{cases} x; x \in [0,1) \\ 2; x \in [1,2] \end{cases} \quad \text{and} \quad S(x) = \begin{cases} 2 - x; x \in [0,1) \\ 2; x \in [1,2] \end{cases}$$

Let $x_n = 1 - \frac{1}{n}$ then $Tx_n = 1 - \frac{1}{n}$ and $Sx_n = 1 + \frac{1}{n}$

Also $TSx_n = 2$ and $STx_n = 1 + \frac{1}{n}$.

Thus $\lim_{n\to\infty} Tx_n = 1$ and $\lim_{n\to\infty} Sx_n = 1$ and hence $t = 1$.

But $\lim_{n\to\infty} d(STx_n, TSx_n) = \lim_{x\to\infty} |2 - (1 + \frac{1}{n})| = 1 \neq 0$

Hence S and T are not compatible.

Again $\lim_{n\to\infty} d(STx_n, TSx_n) = \lim_{x\to\infty} |2 - (1 + \frac{1}{n})| = 1 \neq 0$

Now we will show that the pair (S,T) is weakly compatible.

Now coincidence points of S and T are in $[1,2]$.

Therefore for any x in $[1,2]$, we have

$Tx = Sx = 2$ and $TSx = 2 = STx$ and $T(2) = 2 = S(2)$

Thus (S,T) is weakly compatible.

3. MAIN RESULT.

3.1 Implicit Relations

Let $\mathbf{F}^*$ be the set of real functions $F(t_1,\ldots t_5) : [0,\infty)^5 \to [0,\infty)$ satisfying the following conditions :

(F_1) F is non increasing in variables t_4 and t_5.

(F_2) There is an $h_1 > 0$ and $h_2 > 0$ such that $h = h_1 h_2 < 1$ and if $u \geq 0$, $v \geq 0$ satisfy

(F_a) $u \leq F(v, v, u, u+v, 0)$ or
$u \leq F(v, u, v, u+v, 0)$
then we have $u \leq h_1 v$.

and if $u \geq 0$, $v \geq 0$ satisfy

(F_b) $u \leq F(v, v, u, 0, u+v)$ or
$u \leq F(v, u, v, 0, u+v)$
then we have $u \leq h_2 v$.

(F_3) If $u \geq 0$ is such that $u \leq F(u, 0, 0, u, u)$ or
$u \leq F(0, u, 0, 0, u)$ or
$u \leq F(0, 0, u, u, 0)$
then $u = 0$.

3.2 Fixed Point Theorem

Let S, T, I and J be self-mappings of a complete metric space (X,d) satisfying the following conditions:

(a) $S(X) \subset J(X)$, $T(X) \subset I(X)$.

(b) $d(Sx, Ty) \leq F(d(Ix, Jy); d(Ix, Sx); d(Jy, Ty); d(Ix, Ty); d(Sx, Jy))$

For all x and y in X where $F \in \mathbf{F^*}$.

Then S, T, I and J have unique common fixed point z in X. Further z is the unique common fixed point of S and I and of T and J.

Proof

let $x_0 \in X$

Since $S(X) \subset J(X)$, $T(X) \subset I(X)$,
we can choose x_{2n}, x_{2n+1} and x_{2n+2} such that

$Sx_{2n} = Jx_{2n+1}$ and $Tx_{2n+1} = I_{2n+2}$, n = 0, 1,2….

Using (b) we have

$d(Sx_{2n},Tx_{2n+1})$
$\leq F(d(Ix_{2n}, Jx_{2n+1}); d(Ix_{2n}, Sx_{2n}); d(Jx_{2n+1}, Tx_{2n+1}); d(Ix_{2n}, Tx_{2n+1}); d(Sx_{2n}, Jx_{2n+1}))$
$= F(d(Tx_{2n-1}, Sx_{2n}); d(Tx_{2n-1}, Sx_{2n}); d(Sx_{2n}, Tx_{2n+1}); d(Tx_{2n-1}, Tx_{2n+1}); 0)$
$\leq F(d(Sx_{2n}, Tx_{2n-1}); d(Sx_{2n}, Tx_{2n-1}); d(Sx_{2n}, Tx_{2n+1}); d(Tx_{2n-1},Tx_{2n+1}); 0)$
$\leq F(d(Sx_{2n}, Tx_{2n-1}); d(Sx_{2n}, Tx_{2n-1}); d(Sx_{2n},Tx_{2n+1}); d(Sx_{2n}, Tx_{2n-1})+d(Sx_{2n},Tx_{2n+1}); 0)$

Thus by property (F_a), $d(Sx_{2n}, Tx_{2n+1}) \leq h_1 d(Sx_{2n}, Tx_{2n-1})$.
Similarly, $d(Tx_{2n-1},Sx_{2n}) \leq h_2 d(Sx_{2n-2}, Tx_{2n-1})$.
Therefore $d(Sx_{2n}, Tx_{2n+1}) \leq h\, d(Sx_{2n-2}, Tx_{2n-1})$.
From this we can deduce that $d(Sx_{2n}, Tx_{2n+1}) \leq h^n d(Sx_0, T_1)$.
$d(Tx_{2n+1}, Sx_{2n+2}) \leq h_2 h^n d(Sx_0, T_1)$
for n = 1, 2, ……

Since h < 1, the sequence

$\{Sx_0, Tx_1, Sx_2, \ldots Tx_{2n-1}, Sx_{2n}, Tx_{2n+1}, \ldots\}$

is a Cauchy sequence.

Since (X, d) is complete metric space, this sequence has a limit z in X and the subsequences $\{Sx_{2n}\} = \{Jx_{2n+1}\}$ and $\{Tx_{2n+1}\} = \{I_{2n+2}\}$ converge to the point z.

We suppose that the mapping I is continuous, so that the sequences $\{I^2x_{2n}\}$ and $\{ISx_{2n}\}$ converge to the point Iz. Since S and I are weakly commute, we have

$$d(ISx_{2n}, SIx_{2n}) \leq d(Ix_{2n}, Sx_{2n})$$

So that the sequence $\{SIx_{2n}\}$ converges to the point Iz.

Using (b) we have,

$d(SIx_{2n},Tx_{2n+1})$
$\leq F(d(I^2x_{2n}, Jx_{2n+1}); d(I^2x_{2n}, SIx_{2n}); d(Jx_{2n+1}, Tx_{2n+1}); d(I^2x_{2n}, Tx_{2n+1}); d(SIx_{2n},Jx_{2n+1}))$

By letting $n \to \infty$, we get

$$d(Iz, z) \leq F(d(Iz, z); 0; 0; d(Iz, z); d(Iz, z))$$

Therefore by property (F_3), we get $d(Iz, z) = 0$, i.e. $Iz = z$.

Again by using (b) we have,

$d(Sz,Tx_{2n+1}) \leq F(d(Iz, Jx_{2n+1}); d(Iz, Sz); d(Jx_{2n+1}, Tx_{2n+1}); d(Iz, Tx_{2n+1}); d(Sz,Jx_{2n+1}))$

By letting $n \to \infty$, we get

$$d(Sz, z) \leq F(0; d(z, Sz); 0; 0; d(Sz, z))$$

Therefore by property (F_3), we get $d(Sz, z) = 0$, i.e. $Sz = z$.

Since $S(X) \subset J(X)$, there is a point y in X such that $Jy = z$.

Therefore by (b), we have

$d(z, Ty) = d(Sz, Ty) \leq F(d(Iz, Jy); d(Iz, Sz); d(Jy, Ty); d(Iz, Ty); d(Sz,Jy))$

so that $d(z, Ty) \leq F(0; 0; d(z, Ty); d(z, Ty); 0)$

Therefore by property (F_3), we get $d(z, Ty) = 0$, i.e. $Ty = z$.

Since T and J are weakly commute, we have

$$d(Tz, Jz) = d(TJy, JTy) \leq d(Jy, Ty) = 0$$

Thus Tz = Jz and so that by (b), we have

$$d(z, Tz) = d(Sz, Tz) \leq F(d(Iz, Jz); d(Iz, Sz); d(Jz, Tz); d(Iz, Tz); d(Sz, Jz))$$
$$= F(d(z, Tz); d(z, z); d(Tz, Tz); d(z, Tz); d(z, Tz))$$
$$= F(d(z, Tz); 0; 0; d(z, Tz); d(z, Tz))$$

Therefore by property (F_3), we get $d(z, Tz) = 0$,
i.e. Tz =z i.e. z = Tz = Jz.
Since Iz = Sz = z, we get z = Tz = Jz = Iz = Sz
Thus z is a common fixed point of S, T, I and J.
On the other way the proof is similar if mapping J is continuous.
Now if we consider that the mapping S or T is continuous, in the similar way we can prove that z is a common fixed point of S, T, I and J.

Uniqueness : For uniqueness let us suppose that there is another fixed point u of S and I.
Therefore by (b), we have

$$d(Su, Tz) = d(u, z) \leq F(d(Iu, Jz); d(Iu, Su); d(Jz, Tz); d(Iu, Tz); d(Su,Jz))$$
$$= F(d(u, z); 0; 0; d(u, z); d(u,z))$$

Therefore by property (F_3), we get $d(u, z) = 0$, i.e. u =z.
Similarly it can be proved that z is the unique common fixed point of T and J.
Hence the theorem.

Remark

Let $\mathbf{G}^*$ be the set of real functions $G(t_1, t_2, t_3) : [0,\infty)^3 \rightarrow [0,\infty)$ satisfying the following conditions :

(G_1) $G(1, 1, 1) = h < 1$.

(G_2) If $u \geq 0, v \geq 0$ be such that $u \leq G(u, u, u)$ or $u \leq G(v, v, u)$ or $u \leq G(v, u, v)$ then we have $u \leq hv$.

It should be noted that $G^* \subset F^*$ but $G^* \neq F^*$.

Corollary Let S, T, I and J be self-mappings of a complete metric space (X,d) satisfying the following conditions:

(a) $S(X) \subset J(X)$, $T(X) \subset I(X)$.

(b) $d(Sx, Ty) \leq G(d(Ix, Jy); d(Ix, Sx); d(Jy, Ty))$
For all x and y in X where $G \in \mathbf{G^*}$.

Then S, T, I and J have unique common fixed point z in X . Further z is the unique common fixed point of S and I and of T and J.

Proof : Proof is follows form the theorem because $G^* \subset F^*$.

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