Anti L-Fuzzy M-Cosets of M-Groups

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ABSTRACT
This paper contains some definitions and results of anti L-fuzzy M-cosets of a M-group and generalized characteristic anti L-fuzzy M-subgroup of a M-group. Using homomorphism and anti-homomorphism in anti L-fuzzy M-cosets of a M-group is studied. Some properties of anti L-fuzzy M-cosets of a M-group are also established.

Keywords:
L-fuzzy subset, homomorphism, anti-homomorphism, anti L-fuzzy M-subgroup, anti L-fuzzy M-coset, pseudo anti L-fuzzy M-coset, anti L-fuzzy M-Middle coset, generalized characteristic anti L-fuzzy M-subgroup.

INTRODUCTION
The notion of fuzzy sets was introduced by **L.A. Zadeh** [10]. Fuzzy set theory has been developed in many directions by many researchers and has evoked great interest among mathematicians working in different fields of mathematics, such as topological spaces, functional analysis, loop, group, ring, near ring, vector spaces, automation. In 1971, **Rosenfield** [1] introduced the concept of fuzzy subgroup. Motivated by this, many mathematicians started to review various concepts and theorems of abstract algebra in the broader frame work of fuzzy settings. In [2], **Biswas** introduced the concept of anti- fuzzy subgroups of groups. **Palaniappan. N** and **Muthuraj,** [6] defined the homomorphism, anti-homomorphism of a fuzzy and an anti-fuzzy subgroups. In this paper we define a new algebraic structure of anti L-fuzzy M-cosets of a M-group and study some their related properties.

1. PRELIMINARIES

1.1 Definition: Let X be a non-empty set and L = (L, $\leq$) be a lattice with least element 0 and greatest element 1. A **L-fuzzy subset** A of X is a function A : X $\rightarrow$ L.

1.2 Definition: Let G be a M-group. A L-fuzzy subset A of G is said to be **anti L-fuzzy M-subgroup** (ALFMSG) of G if its satisfies the following axioms:

(i) $\mu_A(mxy) \leq \mu_A(x) \vee \mu_A(y)$,
(ii) $\mu_A(x^{-1}) \leq \mu_A(x)$, for all x & y in G.

1.3 Definition: Let A and B be two anti L-fuzzy M-subgroups of a M-group G. Then A and B are said to be **conjugate anti L-fuzzy M-subgroups** of G if for some g in G, $\mu_A(x) = \mu_B(g^{-1}xg)$, for every x in G.

1.4 Definition: Let A be an anti L-fuzzy M-subgroup of a M-group G. For any a$\in$`G, aA defined by $(a\mu_A)(x) = \mu_A(a^{-1}x)$, for every x in G is called an **anti L-fuzzy M-coset** of the M-group G.

1.5 Definition: Let A be an anti L-fuzzy M-subgroup of a M-group G and H = { $x\in G /\mu_A(x) = \mu_A(e)$ }, then O(A), **order of A** is defined as O(A) = O(H).

1.6 Definition: Let A be an anti L-fuzzy M-subgroup of a M-group G. Then for any a, b$\in$G, an **anti L- fuzzy M-middle coset** aAb of G is defined by $(a\mu_A b)(x) = \mu_A(a^{-1}x b^{-1})$, for every x$\in$G.

1.7 Definition: Let A be an anti L-fuzzy M-subgroup of a M-group G and a$\in$G. Then the **pseudo anti L-fuzzy M-coset** $(aA)^p$ is defined by $((a\mu_A)^p)(x) = p(a)\mu_A(x)$, for every x$\in$G and for some p$\in$P.

1.8 Definition: Let A be a L-fuzzy subset of X. For t$\in$L, the lower level subset of A is the set, A_t = { $x \in X : \mu_A(x) \leq t$ }.This is called a **L-fuzzy lower level subset** of A.

1.9 Definition : Let G be a M-group. An anti L-fuzzy M-subgroup A of G is said to be **anti L-fuzzy normal M-subgroup** (ALFNMSG) of G if $\mu_A(xy) = \mu_A(yx)$, for all x and y in G.

1.10 Definition: Let (G, **.**) and (G^{l}, **.**) be two M-groups. A map f : G $\rightarrow$ G^{l} is called **M-group isomorphism** if the following conditions are satisfied:

(i) f is a bijection,
(ii) f(xy)=f(x)f(y) for all x and y in G.

1.11 Definition: Let (G, **.**) and (G^{l}, **.**) be two M-groups. A map f : G $\rightarrow$ G^{l} is called a **M-group anti-isomorphism** if the following conditions are satisfied:

(i) f is a bijection,
(ii) f(xy)=f(y)f(x) for all x and y in G.

1.12 Definition: An anti L-fuzzy M-subgroup A of a M-group G is called a **generalized characteristic anti L-**

fuzzy M-subgroup (GCALFMSG) if for all x and y in G, O(x) = O(y) implies $\mu_A(x) = \mu_A(y)$.

2 PROPERTIES OF ANTI L-FUZZY M-COSETS

2.1 Theorem: Let A be an anti L-fuzzy M-subgroup of a finite M-group G, then O(A) / O(G).

Proof: Let A be an anti L-fuzzy M-subgroup of a finite M-group G with e as its identity element.
Clearly H ={x∈G / $\mu_A(x) = \mu_A(e)$ } is a M-subgroup of the M-group G for H is a lower t-level subset of a M-group G where t = $\mu_A(e)$.
By Lagranges theorem, O(H) / O(G).
Hence by the definition of the order of the anti L-fuzzy M-subgroup of the M-group G, we have O(A) / O(G).

2.2 Theorem: Let A and B be two L-fuzzy subsets of a M-abelian group G. Then A and B are conjugate L-fuzzy subsets of the M-group G if and only if A = B.

Proof: Let A and B be conjugate L-fuzzy subsets of M-group G, then for some y∈G, we have, $\mu_A(x) = \mu_B(y^{-1}xy)$, for every x∈G
$= \mu_B(yy^{-1}x)$,
since G is a M-abelian group
$= \mu_B(ex) = \mu_B(x)$.
Therefore, $\mu_A(x) = \mu_B(x)$.
Hence A = B.
Conversely, if A = B, then for the identity element 'e' of M-group G
we have, $\mu_A(x) = \mu_B(e^{-1}xe)$, for every x∈G.
Hence A and B are conjugate L-fuzzy subsets of the M-group G.

2.3 Theorem: If A and B are conjugate L-fuzzy M-subgroups of the M-group G, then O(A) = O(B).

Proof: Let A and B be conjugate L-fuzzy M-subgroups of the M-group G.
Now, O(A)= order of{x∈G / $\mu_A(x) = \mu_A(e)$ }
= order of { x∈G / $\mu_B(y^{-1}xy) = \mu_B(y^{-1}ey)$ }
= order of{x∈G / $\mu_B(x) = \mu_B(e)$ }
= O(B).
Hence O(A) = O(B).

2.4 Theorem: Let A be an anti L-fuzzy M-subgroup of a M-group G, then the pseudo anti L-fuzzy M-coset $(aA)^p$ is an anti L-fuzzy M-subgroup of a M-group G, for every a∈G.

Proof: Let A be an anti L-fuzzy M-subgroup of a M-group G.
For every x and y in G, we have,
$((a\mu_A)^p)(mxy^{-1}) = p(a)\mu_A(mxy^{-1})$
$\leq p(a) \{ \mu_A(x) \vee \mu_A(y)\}$
$=\{p(a)\mu_A(x) \vee p(a)\mu_A(y)\}$
$=((a\mu_A)^p)(x)\vee((a\mu_A)^p)(y)$.
Therefore, $((a\mu_A)^p)(mxy^{-1}) \leq ((a\mu_A)^p)(x) \vee ((a\mu_A)^p)(y)$.
Hence $(aA)^p$ is an anti L-fuzzy M-subgroup of a M-group G.

2.5 Theorem: If A is an anti L-fuzzy M-subgroup of a M-group G, then for any a∈G, the anti L-fuzzy M-middle coset aAa^{-1} of G is also an anti L-fuzzy M-subgroup of G.

Proof: Let A be an anti L-fuzzy M-subgroup of a M-group G and a∈G.
To prove $aAa^{-1} = (x, a\mu_A a^{-1}, a\nu_A a^{-1})$ is an anti L-fuzzy M-subgroup of a M-group G .
Let x and y in G. Then,
$(a\mu_A a^{-1})(mxy^{-1}) = \mu_A(ma^{-1}xy^{-1}a)$,
by the definition
$= \mu_A(ma^{-1}xaa^{-1}y^{-1}a)$
$= \mu_A(m(a^{-1}xa)(a^{-1}ya)^{-1})$
$\leq \mu_A(a^{-1}xa)\vee\mu_A((a^{-1}ya)^{-1})$
$\leq \mu_A(a^{-1}xa) \vee \mu_A(a^{-1}ya)$,
since A is an ALFMSG of G
$=(a\mu_A a^{-1})(x)\vee(a\mu_A a^{-1}) (y)$.
Therefore, $(a\mu_A a^{-1})(mxy^{-1}) \leq (a\mu_A a^{-1})(x) \vee (a\mu_A a^{-1})(y)$.
Hence aAa^{-1} is an anti L-fuzzy M-subgroup of a M-group G.

2.6 Theorem: Let A be an anti L-fuzzy M-subgroup of a M-group G and aAa^{-1} be an anti L-fuzzy M-middle coset of G, then $O(aAa^{-1}) = O(A)$, for any a∈G.

Proof: Let A be an anti L-fuzzy M-subgroup of a M-group G and a∈G.
By Theorem 2.5, the anti L-fuzzy M-middle coset aAa^{-1} is an anti L-fuzzy M-subgroup of G.
Further by the definition of an anti L-fuzzy M-middle coset of a M-group G, we have,
$(a\mu_A a^{-1})(x) = \mu_A(a^{-1}x a)$, for every x in G.
Hence for any a∈G, A and aAa^{-1} are conjugate anti L-fuzzy M-subgroups of a M-group G as there exists a∈G such that $(a\mu_A a^{-1})(x) = \mu_A(a^{-1}xa)$, for every x∈G.
By Theorem 2.3, $O(aAa^{-1}) = O(A)$, for any a∈G.

2.7 Theorem: Let A be an anti L-fuzzy M-subgroup of a M-group G and B be a L-fuzzy subset of a M-group G. If A and B are conjugate L-fuzzy subsets of the M-group G, then B is an anti L-fuzzy M-subgroup of a M-group G.

Proof: Let A be an anti L-fuzzy M-subgroup of a M-group G and B be a L-fuzzy subset of a M-group G. And, let A and B be conjugate L-fuzzy subsets of the M-group G.
To prove B is an anti L-fuzzy M-subgroup of the M-group G.
Let x and y in G and m in M. Then mxy^{-1}∈G.
Now, $\mu_B(mxy^{-1}) = \mu_A(mg^{-1}xy^{-1}g)$, for some g∈G
$= \mu_A(mg^{-1}xgg^{-1}y^{-1}g)$
$= \mu_A((mg^{-1}xg) (g^{-1}yg)^{-1})$
$\leq \mu_A(g^{-1}xg) \vee \mu_A((g^{-1}yg)^{-1})$
$\leq \mu_A(g^{-1}xg) \vee \mu_A(g^{-1}yg)$,
since A is an ALFMSG of G
$\leq \mu_B(x) \vee \mu_B(y)$, since A and B are conjugate L-fuzzy subsets of the M-group G.
Therefore, $\mu_B(mxy^{-1}) \leq \mu_B(x) \vee \mu_B(y)$.

Hence B is an anti L-fuzzy M-subgroup of the M-group G.

2.8 Theorem: Let A be an anti L-fuzzy M-subgroup of a M-group G .Then $aA_t = (aA)_t$, for every $a \in G$ and $t \in L$.

Proof: Let A be an anti L-fuzzy M-subgroup of a M-group G and let $x \in G$.

Now, $x \in (aA)_t \Leftrightarrow (a\mu_A)(x) \le t$
$\Leftrightarrow \mu_A(a^{-1}x) \le t$
$\Leftrightarrow a^{-1}x \in A_t$
$\Leftrightarrow x \in aA_t$.

Therefore, $aA_t = (aA)_t$, for every $x \in G$.

2.9 Theorem: Let A be an anti L-fuzzy M-subgroup of a M-group G. Then xA = yA, for x and y in G if and only if $\mu_A(x^{-1}y) = \mu_A(y^{-1}x) = \mu_A(e)$.

Proof: Let A be an anti L-fuzzy M-subgroup of a M-group G.

Let xA = yA , for x and y in G.

Then, $x\mu_A(x) = y\mu_A(x)$ and $x\mu_A(y) = y\mu_A(y)$,
which implies that, $\mu_A(x^{-1}x) = \mu_A(y^{-1}x)$ and $\mu_A(x^{-1}y) = \mu_A(y^{-1}y)$.

Hence $\mu_A(e) = \mu_A(y^{-1}x)$ and $\mu_A(x^{-1}y) = \mu_A(e)$

Therefore, $\mu_A(x^{-1}y) = \mu_A(y^{-1}x) = \mu_A(e)$.

Conversely, let $\mu_A(x^{-1}y) = \mu_A(y^{-1}x) = \mu_A(e)$, for x and y in G.

For every g in G and we have,

$x\mu_A(g) = \mu_A(x^{-1}g)$
$= \mu_A(x^{-1}yy^{-1}g)$
$\le \mu_A(x^{-1}y) \vee \mu_A(y^{-1}g)$
$\le \mu_A(e) \vee \mu_A(y^{-1}g)$
$= \mu_A(y^{-1}g)$
$= y\mu_A(g)$.

Therefore, $x\mu_A(g) \le y\mu_A(g)$ --------------(1).

And, $y\mu_A(g) = \mu_A(y^{-1}g)$
$= \mu_A(y^{-1}xx^{-1}g)$
$\le \mu_A(y^{-1}x) \vee \mu_A(x^{-1}g)$
$\le \mu_A(e) \vee \mu_A(x^{-1}g)$
$= \mu_A(x^{-1}g)$
$= x\mu_A(g)$.

Therefore, $y\mu_A(g) \le x\mu_A(g)$ --------------(2).

From (1) and (2) we get,
$x\mu_A(g) = y\mu_A(g)$ ----------------(3).
we get , xA = Ya.

2.10 Theorem: Let A be an anti L-fuzzy M-subgroup of a M-group G and xA = yA, for x and y in G. Then $\mu_A(x) = \mu_A(y)$.

Proof: Let A be an anti L-fuzzy M-subgroup of a M-group G and xA = yA, for x and y in G.

Now, $\mu_A(x) = \mu_A(yy^{-1}x)$
$\le \mu_A(y) \vee \mu_A(y^{-1}x)$
$= \mu_A(y) \vee \mu_A(e)$, by Theorem 2.9
$= \mu_A(y)$.

Therefore, $\mu_A(x) \le \mu_A(y)$ ---------------- (1).

And, $\mu_A(y) = \mu_A(xx^{-1}y)$
$\le \mu_A(x) \vee \mu_A(x^{-1}y)$
$= \mu_A(x) \vee \mu_A(e)$,
by Theorem 2.9
$= \mu_A(x)$.

Therefore, $\mu_A(y) \le \mu_A(x)$ ---------------- (2).

From (1) and (2) we get, $\mu_A(x) = \mu_A(y)$.

2.11 Theorem: Let A be an anti L-fuzzy M-subgroup of a M-group G and $xA_t = yA_t$, for x and $y \in G - A_t$, $t \in L$. Then $\mu_A(x) = \mu_A(y)$.

Proof: Let A be an anti L-fuzzy M-subgroup of a M-group G and $xA_t = yA_t$, for x and $y \in G - A_t$, $t \in L$.

But $y^{-1}x$ and $x^{-1}y \in A_t$.

Now, $\mu_A(x) = \mu_A(yy^{-1}x)$
$\le \mu_A(y) \vee \mu_A(y^{-1}x)$
$= \mu_A(y)$.

Therefore, $\mu_A(x) \le \mu_A(y)$ ---------------- (1).

And, $\mu_A(y) = \mu_A(xx^{-1}y)$
$\le \mu_A(x) \vee \mu_A(x^{-1}y)$
$= \mu_A(x)$.

Therefore, $\mu_A(y) \le \mu_A(x)$ --------------- (2).

From (1) and (2) we get, $\mu_A(x) = \mu_A(y)$.

2.12 Theorem: If A is an anti L-fuzzy normal M-subgroup of a M-group G, then the set G / A = { xA : $x \in G$ } is a M-group with the operation (xA)(yA) = (xy)A.

Proof: Let x and $y \in G$, xA and yA $\in G / A$.

Clearly, $y^{-1} \in G$.

Therefore, $y^{-1}A \in G / A$.

Now, $(xA)(y^{-1}A) = (xy^{-1})A \in G / A$.

Hence G / A is a M-group.

2.13 Theorem: Let f: G → H be a homomorphism of M-groups and let B be an anti L-fuzzy normal M-subgroup of H and $A = f^{-1}(B)$. Then φ : G / A → H / B such that φ(xA) = f(x)B, for every $x \in G$, is an isomorphism of M-group.

Proof: Clearly φ is onto.

Let xA and yA $\in G / A$.

Now , φ(xA) = φ(yA),
which implies that, f(x)B = f(y)B,
By Theorem 2.9,
$\mu_B(f(x)^{-1}f(y)) = \mu_B(f(y)^{-1}f(x)) = \mu_B(f(e))$,
which implies that
$\mu_B(f(x^{-1})f(y)) = \mu_B(f(y^{-1})f(x)) = \mu_B(f(e))$,
which implies that
$\mu_B(f(x^{-1}y)) = \mu_B(f(y^{-1}x)) = \mu_B(f(e))$, since f is a homomorphism.

Using Theorem 2.9, we get, xA = yA.

Hence φ is one-one .

Now , φ((xA)(yA)) = φ((xy)A))
= f(xy)B
= (f(x)f(y))B,
since f is a homomorphism
= (f(x)B) (f(y)B)
= φ (xA) φ(yA).

Therefore, φ((xA)(yA)) = φ (xA) φ(yA).

Hence φ is an isomorphism.

2.14 Theorem: Let f : G → H be an anti-homomorphism of M-groups and let B be an anti L-fuzzy normal M-subgroup of H and $A = f^{-1}(B)$. Then φ : G / A → H / B such that φ(xa) = f(x)B, for every x∈G, is an anti-isomorphism of M-group.

Proof: Clearly φ is onto. Let xA and yA ∈G / A.
Now , φ(xA) = φ(yA),
which implies that f(x)B = f(y)B,
By Theorem 2.9
$\mu_B(f(x)^{-1}f(y)) = \mu_B(f(y)^{-1}f(x)) = \mu_B(f(e))$,
which implies that
$\mu_B(f(x^{-1})f(y)) = \mu_B(f(y^{-1})f(x)) = \mu_B(f(e))$,
which implies that
$\mu_B(f(yx^{-1})) = \mu_B(f(xy^{-1})) = \mu_B(f(e))$,
since f is an anti-homomorphism.
Using Theorem 2.9, we get, xA = yA.
Hence φ is one-one.
Now, φ((xA)(yA)) = φ((xy)A))
= f(xy)B
= (f(y)f(x))B, since f is an anti-homomorphism
= (f(y)B)(f(x)B)
= φ(yA) φ(xA).
Therefore, φ((xA)(yA)) = φ(yA) φ(xa).
Hence φ is an anti-isomorphism.

In the following proposition ◦ is the composition operation of functions :

2.15 Theorem: Let A be an anti L-fuzzy M-subgroup of a M-group H and f is an isomorphism from a M-group G onto H. Then we have the following:

(i) If A is a generalized characteristic anti L-fuzzy M-subgroup (GCALFMSG) of H, then A◦f is a generalized characteristic anti L-fuzzy M-subgroup of G.

(ii) If A is a generalized characteristic anti L-fuzzy M-subgroup (GCALFMSG) and f is an automorphism on G, then A◦f = A.

Proof: Let A be a generalized characteristic anti L-fuzzy M-subgroup (GCFALMSG) of H.
We know that,"Let A be an anti L-fuzzy M-subgroup of a M-group H and f is an isomorphism from a M-group G onto H. Then A◦f is an anti L-fuzzy M-subgroup of a M-group G."
Hence A◦f is an anti L-fuzzy M-subgroup of G.
Let x and y∈G and O(x) = O(y).
Then we have, $(\mu_A\circ f) (x) = \mu_A(f(x))$
$= \mu_A(f(y))$,
as O(f(x)) = O(f(y))
$= (\mu_A\circ f)(y)$, as f is an isomorphism,
which implies that $(\mu_A\circ f)(x) = (\mu_A\circ f)(y)$.
Therefore, A◦f is a generalized characteristic anti L-fuzzy M-subgroup of G.
Thus (i) is proved.
(ii)Clear.

2.16 Theorem: Let A be an anti L-fuzzy M-subgroup of a M-group H and f is an anti-isomorphism from a M-group G onto H. Then we have the following:

(i) If A is a generalized characteristic anti L-fuzzy M-subgroup (GCALFMSG) of H, then A◦f is a generalized characteristic anti L-fuzzy M-subgroup of G.

(ii) If A is a generalized characteristic anti L-fuzzy M-subgroup (GCALFMSG) and f is an anti-automorphism on G, then A◦f = A.

Proof: Let A be a generalized characteristic anti L-fuzzy M-subgroup (GCALFMSG) of H.
We know that, "Let A be an anti L-fuzzy M-subgroup of a M-group H and f is an anti-isomorphism from a M-group G onto H. Then A◦f is an anti L-fuzzy M-subgroup of a M-group G."
Hence A◦f is an anti L-fuzzy M-subgroup of G.
Let x and y∈G and O(x) = O(y).
Then we have, $(\mu_A\circ f)(x) = \mu_A(f(x))$
$= \mu_A(f(y))$, as O(f(x)) = O(f(y))
$= (\mu_A\circ f)(y)$,
as f is an anti-isomorphism,
which implies that $(\mu_A\circ f)(x) = (\mu_A\circ f)(y)$.
Therefore, A◦f is a generalized characteristic anti L-fuzzy M-subgroup of G.
Thus (i) is proved.
(ii)Clear.

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