Ishikawa Iterates for Logarithmic Function

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ABSTRACT
In this paper the dynamics of the complex logarithmic function is investigated using the Ishikawa iterates. The fractal images generated from the generalized transformation function $z \to \log(z^n + c)$, $n \geq 2$ are analyzed.

Keywords:. : *Complex dynamics, Relative Superior Mandelbrot Set, Relative Superior Julia set, Ishikawa Iteration and Midgets.*

1. INTRODUCTION

The fractals generated from the self-squared function, $z \to z^2 + c$ where z and c are complex quantities, have been studied extensively in the literature[2, 8, 9 & 10]. A multitude of interesting, intriguing and rich families of fractals are generated by changing the complex function $F(z)$. This paper explores the dynamics of a complex logarithmic function.

In 1918, French mathematician Gaston Julia[12] investigated the iteration process of a complex function intensively and attained a Julia set, which is a landmark in the field of fractal theory. The object Mandelbrot set on the other hand was given by Benoit B. Mandelbrot [13] in 1979. Recently, R. L. Devaney [5], [6] and [7] studied widely the behavior of the exponential function and analyzed the Julia sets under different conditions. We briefly recall the well known result for the family of the quadratic polynomial $Q_c(z) = z^2 + c$. Each map Q_c has a single critical point at 0 and so, Q_c has a single critical orbit. The fate of this orbit leads to well known fundamental dichotomy for quadratic polynomials:

(1) If $Q_c^n(0) \to \infty$, then $J(Q_c)$ is a Cantor set.

(2) But if Q_c^n does not tends to ∞ then, $J(Q_c)$ is a connected set.

The set of parameter values of c for which the Julia sets of Q_c is connected forms the well known Mandelbrot set. The Julia set for parameter *c* is defined as the boundary between those values of z_0 that remain bounded after repeated iterations and those that escape to infinity. Julia set is a place where all the chaotic behavior of the complex function occurs. As is well known that Julia sets on the real axis are reflections symmetric while those on the complex plane are rotational symmetric with exception to c(0,0). For a quadratic family, a point at infinity is a super attracting fixed point and so, it is surrounded by an intermediate basin of attraction and if critical orbits tend to infinity, then the critical point must lie in this basin and consequently entire forward orbit lies in this basin.

For the transcendental function, like logarithmic function, Julia set may be defined as closure of the set of the points whose orbits may escape to infinity under the iteration of Q_c . Equivalently, the Julia set is also closure of the set of the repelling periodic points. These two definitions clearly illustrates the chaotic behavior of Julia sets arbitrarily, close to any point in Julia set are the points whose orbits tends to infinity as well as the other points whose orbits are not only bounded but in fact periodic. For a quadratic family, the only singular value is critical value $c = Q_c(0)$, since 0 is the only critical point. Further infinity is the super attracting fixed point for Q_c . The Mandelbrot set on other hand is the set of values of *c* for which the orbit of 0 under Q_c does not tends to infinity. Equivalently, Mandelbrot set takes those values of *c*, for which Julia sets of, Q_c is connected.

2. PRELIMINEARIES

The process of generating fractal images from $z \to \log(z^n + c)$ is similar to the one employed for the self-squared function[21]. Briefly, this process consists of iterating this function up to N times. Starting from a value z_0 we obtain $z_1, z_2, z_3, z_4, \ldots$ by applying the transformation $z \to \log(z^n + c)$.

Definition2.1: Ishikawa Iteration [11]: Let X be a subset of real or complex numbers and $f : X \to X$ for $x_0 \in X$, we have the sequences $\{x_n\}$ and $\{y_n\}$ in X in the following manner:

$$y_n = s'_n f(x_n) + (1 - s'_n)x_n$$
$$x_{n+1} = s_n f(y_n) + (1 - s_n)x_n$$

where $0 \leq s'_n \leq 1$, $0 \leq s_n \leq 1$ and s'_n & s_n are both convergent to non zero number.

Definition 2.2: The sequences x_n and y_n constructed above is called Ishikawa sequences of iterations or Relative Superior sequences of iterates. We denote it by $RSO(x_0, s_n, s'_n, t)$. Notice that $RSO(x_0, s_n, s'_n, t)$ with s'_n=1 is $SO(x_0, s_n, t)$ i.e. Mann's orbit and if we place $s_n = s'_n = 1$ then $RSO(x_0, s_n, s'_n, t)$ reduces to $O(x_0, t)$.

We remark that Ishikawa orbit $RSO(x_0, s_n, s'_n, t)$ with $s'_n = 1/2$ is relative superior orbit.

Now we define Mandelbrot sets for function with respect to Ishikawa iterates. We call them as Relative Superior Mandelbrot sets [23, 27]

Definition 2.3[23, 27]: Relative Superior Mandelbrot set RSM for the function of the form $Q_c(z) = z^n + c$, where *n* = 1, 2, 3, 4… is defined as the collection of $c \in C$ for which the orbit of 0 is bounded *i.e.*

$RSM = \{c \in C : Q_c^k(0) : k = 0, 1, 2...\}$ is bounded.

In functional dynamics, we have existence of two different types of points. Points that leave the interval after a finite number are in stable set of infinity. Points that never leave the interval after any number of iterations have bounded orbits. So, an orbit is bounded if there exists a positive real number, such that the modulus of every point in the orbit is less than this number.

The collection of points that are bounded, *i.e.* there exists M, such that $|Q^n(z)| \le M$, for all *n*, is called as a prisoner set while the collection of points that are in the stable set of infinity is called the escape set. Hence, the boundary of the prisoner set is simultaneously the boundary of escape set and that is Julia set for Q.

Definition 2.4[23, 27]]: The set of points RSK whose orbits are bounded under relative superior iteration of the function Q (z) is called Relative Superior Julia sets. Relative Superior Julia set of Q is boundary of Julia set RSK.

2.1 Generation Process: The basic principle of generating fractals employs the iterative formula: $z_{n+1} \leftarrow f(z_n)$ where z_0 = the initial value of z, and z_i = the value of the complex quantity z at the *i*th iteration. For example, the Mandelbrot's self-squared function for generating fractals is: $f(z) = z^2 + c$, where z and c are both complex quantities.

We propose the use of the transformation function $z \to \log(z^n + c)$ for generating fractal images with respect to Ishikawa iterates, where z and c are the complex quantities and *n* is a real number. Each of these fractal images is constructed as a two-dimensional array of pixels. Each pixel is represented by a pair of (x, y) coordinates. The complex quantities z and c can be represented as:

$$z = z_x + iz_y$$
$$c = c_x + ic_y$$

where $i = \sqrt{(-1)}$ and z_x, c_x are the real parts and z_y & c_y are the imaginary parts of z and c, respectively. The pixel coordinates (x, y) may be associated with (c_x, c_y) or (z_x, z_y).

Based on this concept, the fractal images can be classified as follows:

(a) c-plane fractals, wherein (x, y) is a function of (c_x, c_y)

(b) z-plane fractals, wherein (x, y) is a function of (z_x, z_y).

In the literature, the fractals for *n* = 2 in *z* plane are termed as the Mandelbrot set while the fractals for *n* = 2 in c plane are known as Julia sets

2.2 Generating the fractals: Fractals have been generated from $z \to z^{-n} + c$ using escape-time techniques, for example by Gujar etal.[8, 9] and Glynn [10]. We have used in this paper escape time criteria of Relative Superior Ishikawa iterates for function $z \to \log(z^n + c)$.

Escape Criterion for Quadratics: Suppose that $|z| > \max\{|c|, 2/s, 2/s'\}$, then $|z_n| > (1+\lambda)^n |z|$ and $|z_n| \to \infty$ as $n \to \infty$. So, $|z| \ge |c|$ and $|z| > 2/s$ as well as $|z| > 2/s'$ shows the escape criteria for quadratics.

Escape Criterion for Cubics: Suppose $|z| > \max\{|b|, (|a| + 2/s)^{1/2}, (|a| + 2/s')^{1/2}\}$ then $|z_n| \to \infty$ as $n \to \infty$. This gives an escape criterion for cubic polynomials

General Escape Criterion: Consider $|z| > \max\{|c|, (2/s)^{1/n}, (2/s')^{1/n}\}$ then $|z_n| \to \infty$ as $n \to \infty$ is the escape criterion. (Escape Criterion derived in [23, 27]).

Note that the initial value z_0 should be infinity, since infinity is the critical point of $z \to \log(z^n + c)$. However instead of starting with z_0 = infinity, it is simpler to start with $z_1 = c$, which yields the same result. (*A critical point of* $z \to F(z) + c$ *is a point where* $F'(z) = 0$*).* The role of critical points is explained in [1].

2.3 Midgets of Relative Superior Mandelbrot Set:

The midgets of the RSM set are the small mini Mandelbrot set like images found in the scattered surroundings of the RSM set. The study of midgets in the Mandelbrot set is given by Philip[20] and Romera [24].

We have generated numerous RSM sets for $Q_c(z)$ for various values of *n*. We find fascinating new fractals having several effectively different geometric shapes. However, we have selected a few figures to study midgets in RSM sets for $Q_c(z)$ with n = 2, 4, 6 and 8, wherein s = 0.8 and *s′=0*.9. The Mandelbrot set consists of many small decorations or bulbs attached to the main body. The main body is called "main cardioid" by Devaney [4]-[7] and simply "cardioid" by Philip [20]. Similarly, the biggest bulb attached to the main cardiod is named as "period-2 bulb" by Devaney and "head" by Philip.

Further, the small decorations attached to the main cardiod are called "atom-n" by Philip[20], and "period-n bulb" by Devaney [4], where n is the period of the bulb. Each period-n bulb has a main antenna attached to it. This antenna is named as "spikes" or "tendrils" by Philip [20]. Devaney[4] has shown that the main antenna consists of a number of spokes attached; the number of spokes is the same as on the period of the corresponding bulb. In our study, we largely follow the Devaney's nomenclature and occasionally that of Philip[20].

The RSM set for $Q_c(z)$ with degree n contains n-primary bulbs. We find that a period-2 bulb is connected to each of these (n - 1) primary bulbs. Other period-n bulbs vanish as the value of *s* comes nearer to 0. Further, the main antenna looks disconnected for small values of s and *s′*.

Now we consider the following two cases.

Case I *(s* = 1, s' = 1special case)

We observe that, when *s* = 1 and *s′*=1, the RSM set for *n > 2* is the Mandelbrot set of n^{th} order having n^{th} main cardioids. On zooming, we find only one kind of midgets, *i.e*. mini Mandelbrot sets of n^{th} order, which are near the apex or end of the branch.

Case II (0 < *s* < 1, 0 < s' < 1, general case)

For *n > 2*, we observe that, the RSM set of n^{th} order contains n^{th} distinct main cardioids (see Figs. 2 and 3). On zooming the RSM set for *n > 2*, we get the midgets of mini Mandelbrot set on the main antennas of period-n bulbs. A remarkable feature is observed that, all the midgets are of order 2, i.e., the order of each of the midgets does not depend on the degree of the polynomial $Q_c(z)$. We notice that the order of midgets is not independent of the degree of the polynomial $Q_c(z)$ when *s* = 1, *s′* = 1 *(cf.* Case I).

These observations show that, at least, some of the RSM sets are effectively different from the usual Mandelbrot sets.

3. GEOMETRY OF RELATIVE SUPERIOR MANDELBROT SETS AND RELATIVE SUPERIOR JULIA SETS:

The fractals generated from the equation $z \to \log(z^n + c)$ possesses rotational as well as reflection symmetry. As conjectured by Gujar and Bhavsar in [8, 9], the fractals generated with the exponent *n* are (*n*+1) way rotationally symmetric.

3.1 Relative Superior Mandelbrot sets:

- There are several secondary ovoids or bulbs attached with the main body or the central ovoid. Here, the number of major secondary lobe is (n-1). Besides, this main body of Mandelbrot set is observed to be symmetrical about real axis.

- As the value of *s* tend to 1 and *s′* tends to 1, the Relative Superior Mandelbrot sets of logarithmic function converts to the general Mandelbrot sets of logarithmic function, hence we can say that the Relative Superior Mandelbrot sets of logarithmic function is the general case of the usual Mandelbrot sets of logarithmic function.

- In case of quadratic polynomial, among all the secondary ovoids, only one that is the major secondary ovoid happens to be quite larger than rest of the other ovoids. Moreover, as the value of *s* and *s′* changes, the rest of the ovoids vanishes except the major secondary ovoid.

- In case of the cubic polynomial, the central body is bifurcated into two lobes, where each primary ovoid contains a major secondary ovoid along with other ovoids. As, the value of *s* and *s′* varies, then the major secondary lobes also shows the bifurcations.

- In case of biquadractic function, the central body is bifurcated into three lobes, where two major lobes contains bigger secondary major ovoid while the third one which remains comparatively a smaller lobe, has a very small secondary major lobe.

- We also observe that the Relative Superior Mandelbrot sets of logarithmic function had their Midgets for quadratic, Biquadratic and other even valued function.

4. FIXED POINTS

4.1 Fixed points of quadratic polynomial

Table 1: Orbit of F(z) at **s=1 and s'=1** for **(z_0=1.115279339+0.004573602931i)**

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
205	2.2608	215	2.2609
206	2.2605	216	2.2607
207	2.2609	217	2.2608
208	2.2611	218	2.261
209	2.2607	219	2.2608
210	2.2606	220	2.2608
211	2.261	221	2.2608
212	2.261	222	2.2608
213	2.2607	223	2.2608
214	2.2607	224	2.2608

We skipped 204 iterations and after 219 iterations value converges

Figure 1. **:** Orbit of F(z) at **s=1 and s'=1** for **(z_0=1.115279339+0.004573602931i)**

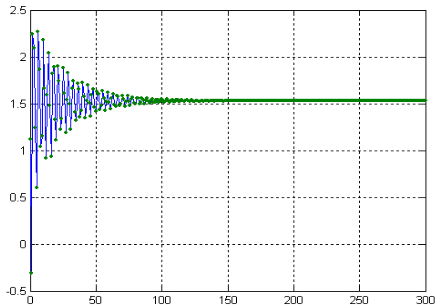


Table 2: Orbit of F(z) at **s=0.5 and s'=0.2** for **(z_0= -0.2719720527+0.006189740197i)**

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1	0.27204	8	0.77381
2	1.0321	9	0.77413
3	0.92033	10	0.77418
4	0.7621	11	0.77419
5	0.75008	12	0.77418
6	0.78331	13	0.77418
7	0.77306	14	0.77418

Here we observe that the value converges to a fixed point after 10 iterations

Figure 2. Orbit of F(z) at **s=0.5 and s'=0.2** for **(z_0= -0.2719720527+0.006189740197i)**

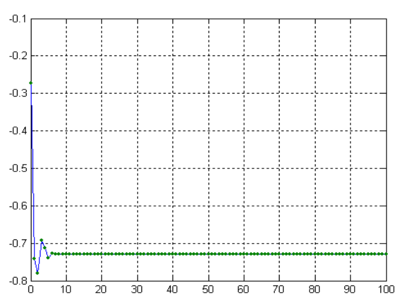


Table 3: Orbit of F(z) at **s=0.5 and s'=0.4** for **(z_0= 0.1221501844-0.001193880049i)**

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1	0.12216	9	1.7162
2	1.3071	10	1.716
3	1.6111	11	1.7159
4	1.6914	12	1.7158
5	1.7096	13	1.7157
6	1.7151	14	1.7157
7	1.7166	15	1.7157
8	1.7167	16	1.7157

Here the value converges to a fixed point after 13 iterations

Figure 3. Orbit of F(z) at **s=0.5 and s'=0.4** for **(z_0= 0.1221501844-0.001193880049i)**

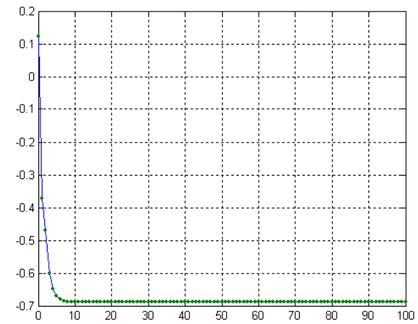
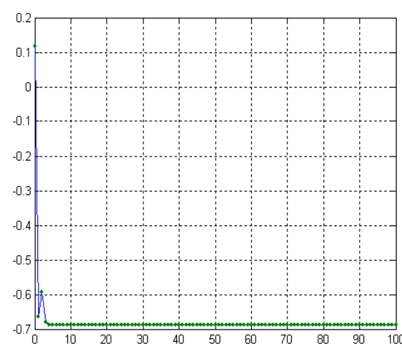


Table 4 Orbit of F(z) at **s=0.8 and s'=0.4** for **(z_0= 0.116875247+0.008872315035i)**

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1	0.11721	7	1.7157
2	2.1293	8	1.7157
3	1.7603	9	1.7157
4	1.7258	10	1.7157
5	1.7188	11	1.7157
6	1.7164	12	1.7157

Here the value converges to a fixed point after 07 iterations

Figure 4. Orbit of F(z) at **s=0.8 and s'=0.4** for **(z_0= 0.116875247+0.008872315035i)**



4.2 Fixed points of Cubic polynomial

Table 1: Orbit of F(z) at **s=1 and s'=1** for **(z_0= 1.159503886+0.09059823882i)**

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1	1.163	10	4.5376
2	1.878	11	4.5371
3	3.1287	12	4.5368
4	4.4407	13	4.5367
5	4.637	14	4.5367
6	4.6023	15	4.5367
7	4.5678	16	4.5367

8	4.5502	17	4.5367
9	4.5423	18	4.5367

Here we observe that the value converges to a fixed point after 13 iterations

Figure 1. Orbit of F(z) at **s=1 and s'=1** for **(z_0= 1.159503886+0.09059823882i)**

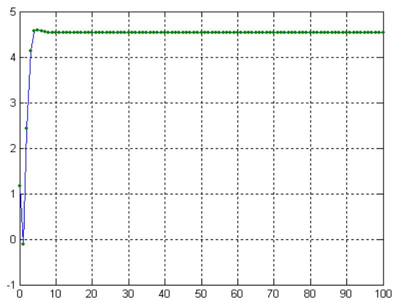


Table 2: Orbit of F(z) at **s=0.5 and s'=0.1** for **(z_0 = 0.739164036+0.03249605129i)**

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1	0.73988	12	0.80468
2	0.6624	13	0.80488
3	2.106	14	0.80534
4	0.84776	15	0.80495
5	1.0982	16	0.80517
6	0.33581	17	0.80507
7	1.019	18	0.80511
8	0.69424	19	0.80509
9	0.84457	20	0.8051
10	0.79229	21	0.8051
11	0.80849	22	0.8051

Here we observe that the value converges to a fixed point after 20 iterations

Figure 2 Orbit of F(z) at **s=0.5 and s'=0.1** for **(z_0 = 0.739164036+0.03249605129i)**

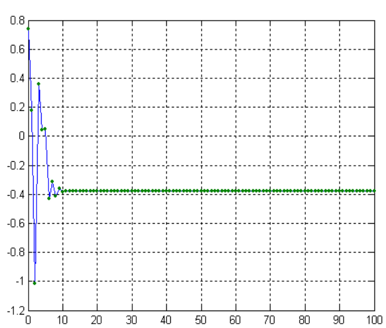


Table 3: Orbit of F(z) at **s=0.5 and s'=0.3** for **(z_0 = 0.4806782686+0.02610157227i)**

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
20	0.80485	27	0.8051
21	0.80502	28	0.80509
22	0.80512	29	0.80509
23	0.80516	30	0.80509
24	0.80516	31	0.80509
25	0.80514	32	0.8051
26	0.80512	33	0.8051

We skipped 19 iterations and after 32 iterations value converges

Figure 3. Orbit of F(z) at **s=0.5 and s'=0.3** for **(z_0 = 0.4806782686+0.02610157227i)**

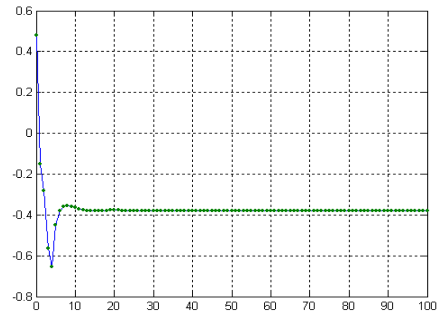
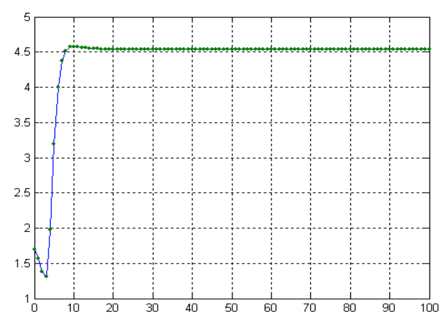


Table 4 Orbit of F(z) at **s=0.8 and s'=0.3** for **(z_0= 1.694305119-0.002625316039i)**

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
15	4.5539	24	4.5374
16	4.5489	25	4.5372
17	4.5452	26	4.537
18	4.5427	27	4.5369
19	4.5409	28	4.5368
20	4.5396	29	4.5368
21	4.5387	30	4.5368
22	4.5381	31	4.5367
23	4.5377	32	4.5367

We skipped 14 iterations and after 31 iterations value converges

Figure 4. Orbit of F(z) at **s=0.8 and s'=0.3** for **(z_0= 1.694305119-0.002625316039i)**



4.3 Fixed points of Bi-quadratic polynomial

Table 1: Orbit of F(z) at **s=1 and s'=1** for **(z_0= 1.36233755-0.02045744357i)**

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1	1.3625	8	8.6066
2	1.048	9	8.6118
3	2.6441	10	8.6129
4	6.0664	11	8.6131
5	7.9792	12	8.6132
6	8.4729	13	8.6132

7	8.5828	14	8.6132

Here we observe that the value converges to a fixed point after 12 iterations

Figure 1 Orbit of F(z) at **s=1 and s'=1** for **(z_0= 1.36233755-0.02045744357i)**

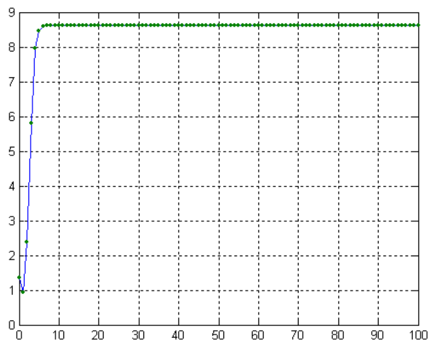


Table 2: Orbit of F(z) at **s=0.5 and s'=0.1** for **(z_0= 0.4283854228-0.03991188002i)**

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
30	0.88893	37	0.88866
31	0.88848	38	0.88878
32	0.88883	39	0.88874
33	0.88881	40	0.8887
34	0.8886	41	0.88876
35	0.8888	42	0.88873
36	0.88876	43	0.88873

We skipped 29 iterations and after 42 iterations value converges

Figure 2. : Orbit of F(z) at **s=0.5 and s'=0.1** for **(z_0= 0.4283854228-0.03991188002i)**

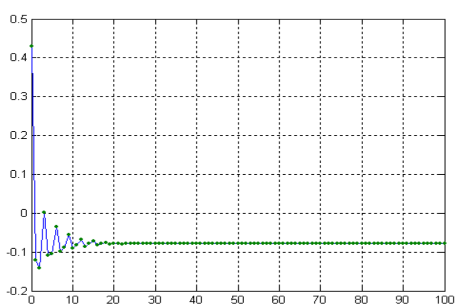


Table 3: Orbit of F(z) at **s=0.5 and s'=0.3** for **(z_0= 0.3088081894-0.01017917836i)**

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
11	0.89619	20	0.88834
12	0.8852	21	0.88897
13	0.88884	22	0.88867
14	0.89048	23	0.88868
15	0.88673	24	0.88882
16	0.8901	25	0.88866
17	0.88822	26	0.88877
18	0.88861	27	0.88873
19	0.88914	28	0.88873

We skipped 10 iterations and after 27 iterations value converges

Figure 3. Orbit of F(z) at **s=0.5 and s'=0.3** for **(z_0= 0.3088081894-0.01017917836i)**

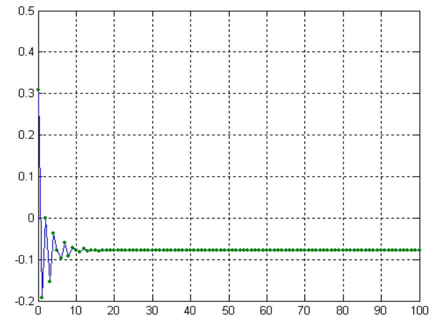
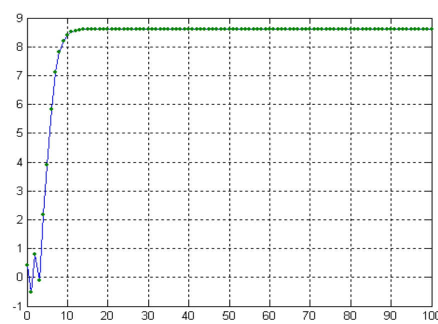


Table 4 Orbit of F(z) at **s=0.8 and s'=0.3** for **(z_0= 0.4057819286-0.0129093121i)**

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
11	8.4042	20	8.6122
12	8.5055	21	8.6127
13	8.5579	22	8.6129
14	8.5848	23	8.613
15	8.5986	24	8.6131
16	8.6057	25	8.6131
17	8.6094	26	8.6132
18	8.6112	27	8.6132

We skipped 10 iterations and after 26 iterations value converges

Figure 4. Orbit of F(z) at **s=0.8 and s'=0.3** for **(z_0= 0.4057819286-0.0129093121i)**



5. GENERATION OF RELATIVE SUPERIOR MANDELBROT SETS:

We generate Relative Superior Mandelbrot Sets. We present here some Relative Superior Mandelbrot sets for cubic and biquadratic function.

5.1 Relative Superior Mandelbrot Sets for Quadratic function:
Figure 1: Relative Superior Mandelbrot Set for s= s'=1

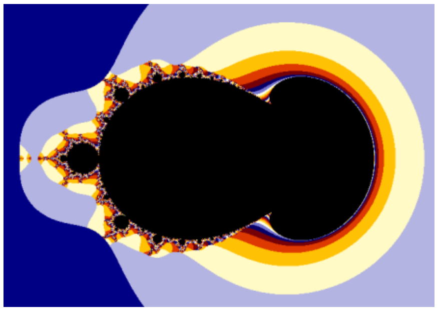


Figure 2: Relative Superior Mandelbrot Set for s=0.4, s'=0.8

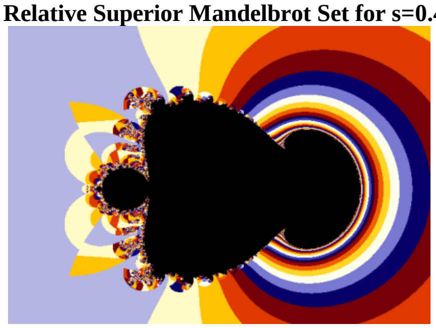


Figure 3: Relative Superior Mandelbrot Set for s=0.5, s'=0.7

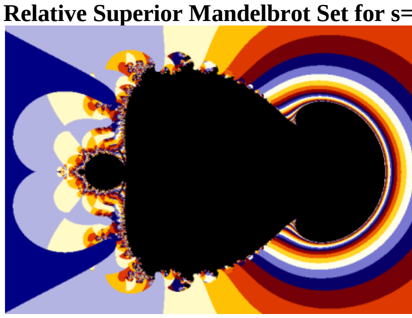
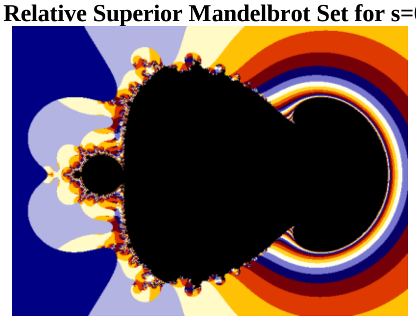


Figure 4: Relative Superior Mandelbrot Set for s=0.6, s'=0.8



5.2 Relative Superior Mandelbrot Sets for Cubic function:

Figure 1: Relative Superior Mandelbrot Set for s=s'=1

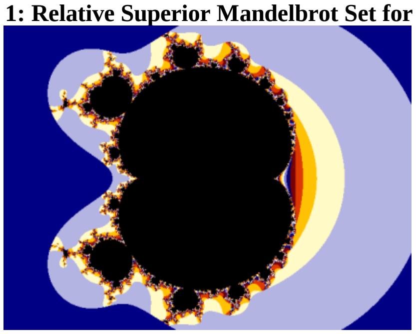


Figure 2: Relative Superior Mandelbrot Set for s=0.3, s'=0.8

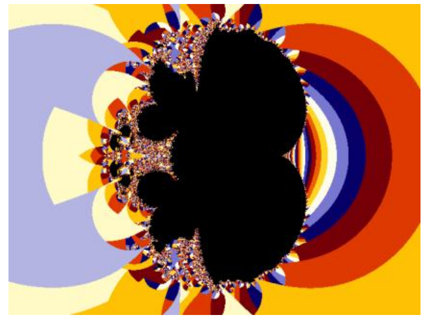
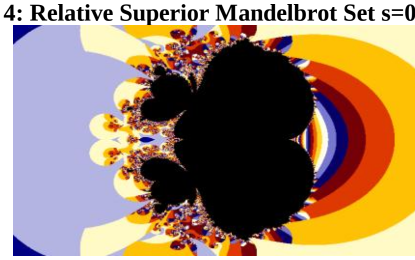


Figure 3: Relative Superior Mandelbrot Set for s=0.4 s'=0.6

Figure 4: Relative Superior Mandelbrot Set s=0.4, s'=0.8



5.3 Relative Superior Mandelbrot Sets for Bi-quadratic function:

Figure 1: Relative Superior Mandelbrot Set for s=s'=1

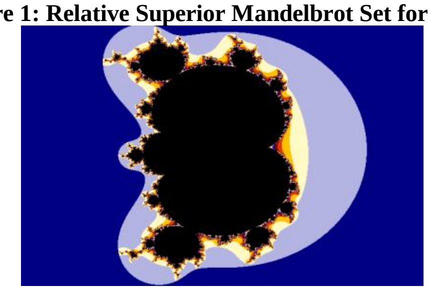


Figure 2: Relative Superior Mandelbrot Set for s=0.3, s'=0.8

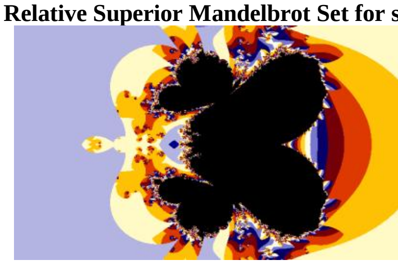


Figure 3: Relative Superior Mandelbrot Set for s=0.4, s'=0.7

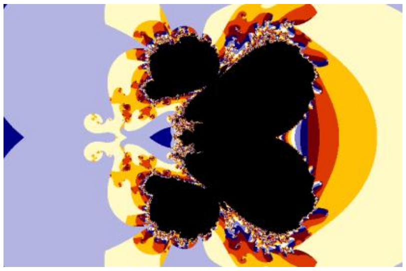


Figure 4: Relative Superior Mandelbrot Set for s=0.5, s'=0.5

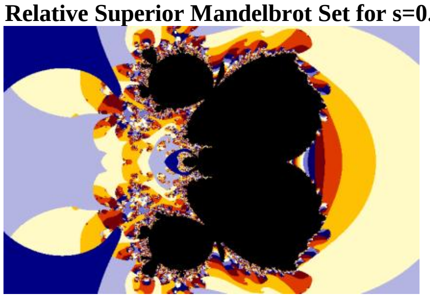


Figure 5:Relative Superior Mandelbrot Set for n=7, s=1, s'=1

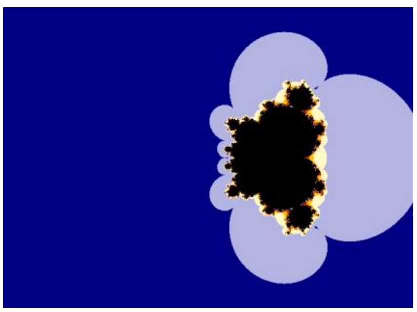


Figure 6:Relative Superior Mandelbroot Set for n=7, s=0.4, s'=0.6

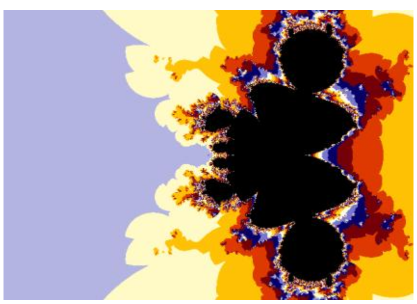


Figure 7: Relative Superior Mandelbrot Set for n=12, s=0.5, s'=0.8

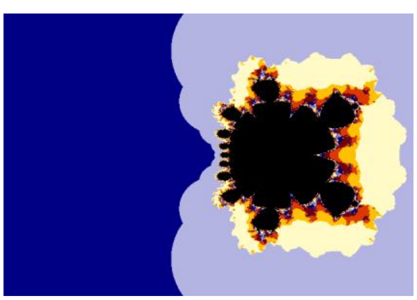
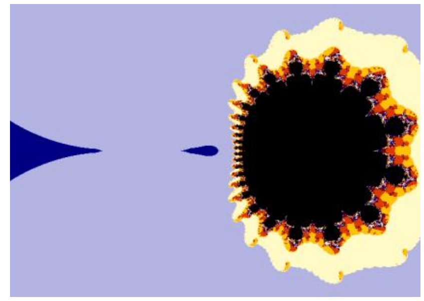


Figure 8:Relative Superior Mandelbrot Set for n= 25, s=0.6, s'=0.8

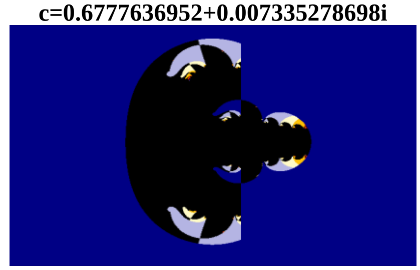


6. GENERATION OF RELATIVE SUPERIOR JULIA SETS:

We present here some filled Relative Superior Julia sets for quadratic, cubic and biquadratic function.

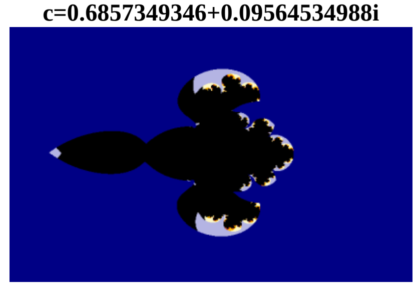
6.1 Relative Superior Julia sets for Quadratic function:

Figure 1: Relative Superior Julia Set for s=0.4, s'=0.8, c=0.6777636952+0.007335278698i



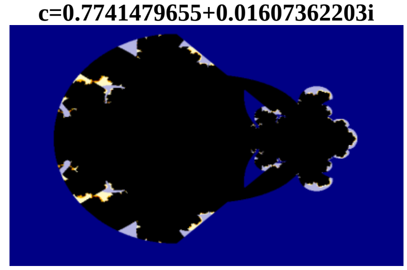
6.2: Relative Superior Julia sets for Cubic function:

Figure 1: Relative Superior Julia Set for s=0.4, s'=0.8, c=0.6857349346+0.09564534988i



6.3 Relative Superior Julia sets for Bi-quadratic function:

Figure 1: Relative Superior Julia Set for s=0.4, s'=0.8, c=0.7741479655+0.01607362203i



7. RELATIVE SUPERIOR MIDGET OF THE LOGARTHMIC FUNCTION:

7.1 Relative Superior Midget of the quadratic function:

Figure 1: Relative Superior Midget for s=1, s'=1

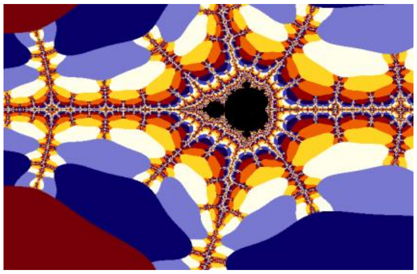


Figure 2: Relative Superior Midget for s=0.8, s'= 0.8

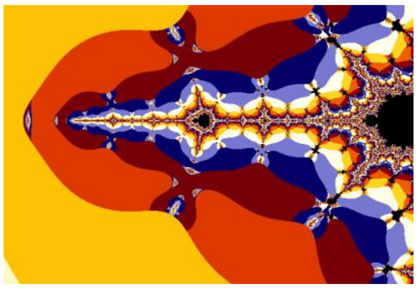
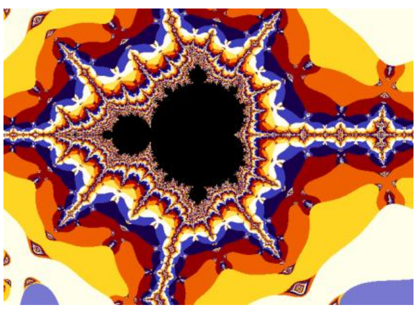


Figure 3: Zoom of Relative Superior Midget for s=0.8, s'= 0.8



7.2 Relative Superior Midget of the bi-quadratic function:

Figure 1: Relative Superior Midget for s=1, s'=1

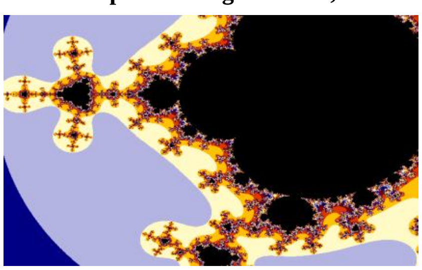


Figure 2: Relative Superior Midget for s=0.8, s'= 0.9

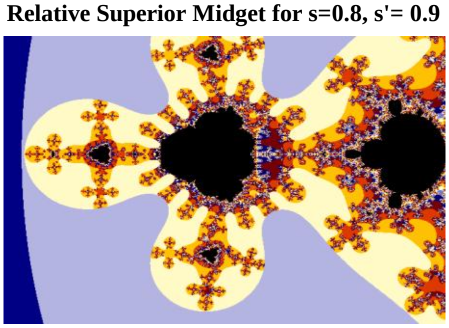
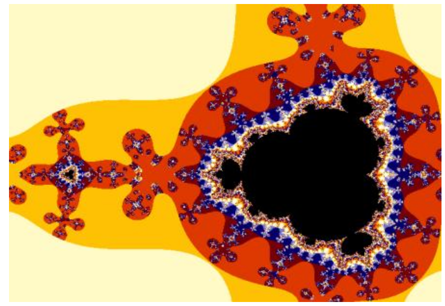


Figure 3: Zoom of Relative Superior Midget for s=0.8, s'= 0.9



7.3 Relative Superior Midget of the function for n = 6

Figure 1: Relative Superior Midget for s=1, s'=1

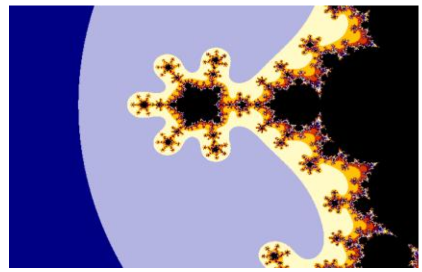


Figure 2: Relative Superior Midget for s=0.8, s'= 0.9

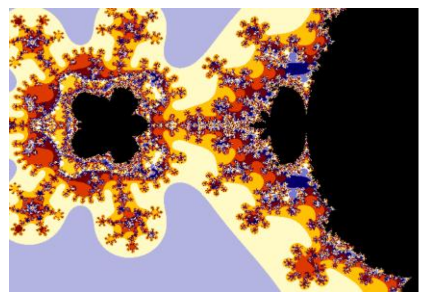


Figure 3: Disconnected bulb of period-3 bulb for s=0.8, s'= 0.9

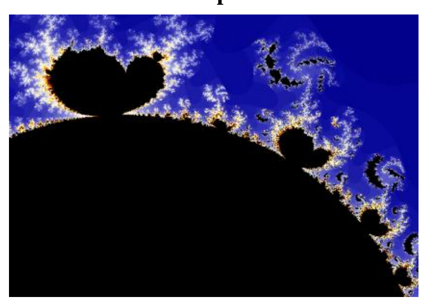
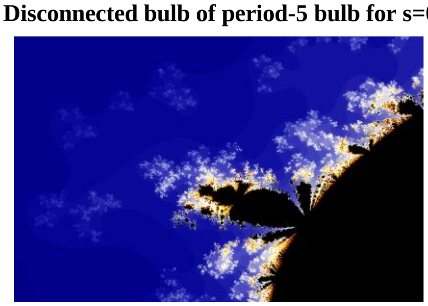
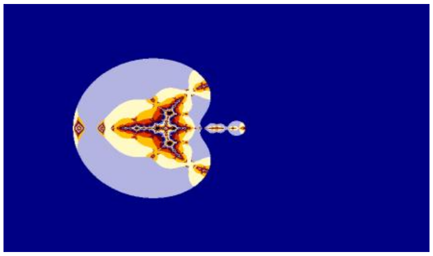


Figure 4: Disconnected bulb of period-5 bulb for s=0.8, s'= 0.9



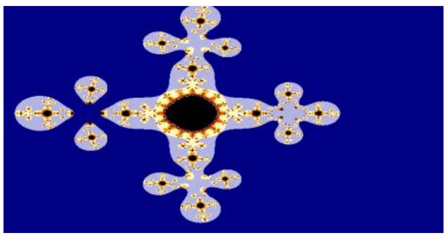
7.3 Relative Superior Julia of Midget of the quadratic function:

Figure 1: Relative Superior Julia Set for s=0.8, s'=0.8, c=0.2090087+0.0000122i



7.4 Relative Superior Julia of Midget of the bi- quadratic function:

Figure 1: Relative Superior Julia Set for s=0.8, s'=0.9, c=0.2090087+0.0000122i



8. CONCLUSION

In the dynamics of complex logarithmic polynomial $z \to \log(z^n + c)$, where $n \geq 2$, the fractals generated with exponent *n* are found as (n + 1) way rotationally symmetric. There are several ovoids or bulbs attached with the main body. The number of major secondary lobe is (n – 1). Besides this, for the polynomial of degree greater than two, the central body is bifurcated into (n – 1) lobes.

The midgets observed for the logarithmic function are derived for even polynomials while for the odd function, bulbs gets disconnected.

9. REFERENCES

[1] B. Branner, "*The Mandelbrot Set*", Proceedings of Symposia in Applied Mathematics39 (1989), 75-105.

[2] Robert L. Devaney, "*A First Course in Chaotic Dynamical Systems: Theory and Experiment*", Addison-Wesley, 1992. MR1202237.

[3] R. L. Devaney, "*The fractal geometry of the Mandelbrot set. 2. How to count and how to add. Symposium in Honor of Benoit Mandelbrot",* Fractals 3 (1995), no. 4, 629-640. MR1410283(99d:58095)

[4] R. L. Devaney, "*The fractal geometry of the Mandelbrot set:I, Period of Bulbs*", In Fractals, Graphics and mathematics education, MAA Notes,58(2002), 61-68.

[5] R. L. Devaney and M. Krych, "Dynamics of Exp(z)", Ergodic Theory and Dynamical Systems 4(1984), 35- 52.

[6] R. L. Devaney, D.M. Look and D. Umnisky, "*The Escape Trichotomy for Singular perturbed Rational maps*", Indiana Univ. Mathematics Journal 54(2005), 267-285.

[7] R. L. Devaney and X. Jarque, "Indecomposable Continua in Exponential Dynamics", Conformal Geom. Dynamical 6(2002), 1-12.

[8] U. G. Gujar and V. C. Bhavsar, Fractals from $z \leftarrow z^{\alpha} + c$ in the Complex c-Plane, Computers and Graphics 15, 3 (1991), 441-449.

[9] U. G. Gujar, V. C. Bhavsar and N. Vangala, Fractals from $z \leftarrow z^{\alpha} + c$ in the Complex z-Plane, Computers and Graphics 16, 1 (1992), 45-49.

[10] E. F. Glynn, The Evolution of the Gingerbread Man, Computers and Graphics 15,4 (1991), 579-582.

[11] S. Ishikawa, "*Fixed points by a new iteration method*", Proc. Amer. Math. Soc.44 (1974), 147-150.

[12] G. Julia, "*Sur 1' iteration des functions rationnelles*", J Math Pure Appli. 8 (1918), 737-747

[13] B. B. Mandelbrot, *The Fractal Geometry of Nature*, W. H. Freeman, New York, 1983.

[14] Eike Lau and Dierk Schleicher, "*Symmetries of fractals revisited.*", Math. Intelligencer (18)(1)(1996), 45-51. MR1381579 Zbl 0847.30018.

[15] J. Milnor, "*Dynamics in one complex variable; Introductory lectures*", Vieweg (1999).

[16] Shizuo Nakane, and Dierk Schleicher, "*Non-local connectivity of the tricorn and multicorns*", Dynamical systems and chaos (1) (Hachioji, 1994), 200-203, World Sci. Publ., River Edge, NJ, 1995. MR1479931.

[17] Shizuo Nakane, and Dierk Schleicher, "*On multicorns and unicorns: I. Antiholomorphic dynamics. hyperbolic components and real cubic polynomials"*, Internat. J. Bifur. Chaos Appl. Sci. Engrg, (13)(10)(2003), 2825-2844. MR2020986.

[18] Ashish Negi, "*Generation of Fractals and Applications*", Thesis, Gurukul Kangri Vishwvidyalaya, (2005).

[19] M.O.Osilike, "*Stability results for Ishikawa fixed point iteration procedure*", Indian Journal of Pure and Appl. Math., 26(1995), 937-945.

[20] A. G. D. Philip: "*Wrapped midgets in the Mandelbrot set"*, Computer and Graphics 18 (1994), no. 2, 239-248.

[21] H. Peitgen and P. H. Richter, "*The Beauty of Fractals*", Springer-Verlag, Berlin, 1986.

[22] B. E. Rhoades, "*Fixed point iterations for certain nonlinear mappings",* J. Math. Anal. 183 (1994), 118-120.

[23] Rajeshri Rana, Yashwant S Chauhan and Ashish Negi. Article: *Non Linear Dynamics of Ishikawa Iteration*. International Journal of Computer Applications 7(13):43–49, October 2010. Published By Foundation of Computer Science.ISBN: 978-93-80746-97-5

[24] M.Romera, G. Pastor and F. Montoya, "*On the cusp and the tip of a midget in the Mandelbrot set antenna",* Phys. Lett. A 221 (1996), no 3-4, 158-162. MR1409563 (97d: 58073).

[25] K. W. Shirriff, "*An investigation of fractals generated by* $z \to z^{-n} + c$", Computers and Graphics 13, 4(1993), 603-607.

[26] R. Winters, "*Bifurcations in families of Antiholomorphic and biquadratic maps*", Thesis, Boston Univ. (1990).

[27] Yashwant S Chauhan, Rajeshri Rana and Ashish Negi. Article: *New Julia Sets of Ishikawa Iterates*. International Journal of Computer Applications 7(13):34–42, October 2010. Published By Foundation of Computer Science. ISBN: 978-93-80746-97-5