Q-Vague Groups and Vague Normal Sub Groups with Respect to (T, S) Norms

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ABSTRACT

In this Paper, Q-Vague sets and Q-Vague normal subgroups are studied. The study of Vague groups initiated by Ranjit Biswas [2006] is continued and Q-Vague homologous groups characterized as normal groups which admit a particular type of Q-Vague groups with respect to (mini, max) norms.

Keywords: Q-Vague set, Q-Vague group, Q-Vague-cut group, Q-Vague normal group, Q-Vague centralizer, Homologous group.

1. INTRODUCTION AND PRELIMINARIES

The theory of fuzzy groups defined by Rosenfeld [1971] is the first application of fuzzy theory in Algebra. Since then a number of works has been done in the area of fuzzy algebra, Gau.W.L. and Bueher. D.J. [1993] has initiated the study of Vague sets as an improvement over the theory of fuzzy sets to interpret and solve real life problems which are in general Vague. Recently, Biswas [2006] defined the notion of Vague groups analogous to the idea of Rosenfeld [1971]. The notion of Q-fuzzy groups is defined by [2009]. The objective of this paper is to contribute further to the study Q-Vague groups and introducing concepts of Q-Vague normalizer, Q-Vague centralizer and Q-Vague homologous group by imposing fitness condition that can be removed. In this paper, we characterized the Q-Vague normal groups and homologous Q-Vague group which admit a particular type of Q-fuzzy groups.

Definition 1.1: A Q-Vague set (or in-short QVS) in the universe of discourse X is characterized by two membership functions given by

1. a truth membership function $t_A : X \times Q \rightarrow [0,1]$
2. a false membership function $f_A : X \times Q \rightarrow [0,1]$ such that $t_A(x,q) + f_A(x,q) \leq 1$, for all $x \in X$ and $q \in Q$.

Definition 1.2: The interval $[t_A(x,q), 1 - f_A(x,q)]$ is called the Q-Vague Value of X in A, and it is denoted by $V_A(x,q)$. So $V_A(x, q) = [t_A(x, q), 1- f_A(x, q)]$.

Definition 1.3: A Q-Vague set 'A' of X with $t_A(x, q) = 0$ and $f_A(x, q) =1$ for all $x \in X$ and $q \in Q$ is called Zero Q-Vague set of X. A Q-Vague set 'A' of X with $t_A(x,q) = 1$ and $f_A(x,q) =0$ for all $x \in X$ and $q \in Q$ is called Unit Q-Vague set of X.

Definition 1.4: A Q-Vague set 'A' of a set 'X' with $t_A(x, q) = \alpha$ and $f_A(x, q) =(1- \alpha)$ for all $x \in X$ is called α-Q-Vague set of X where $\alpha \in [0, 1]$.

Definition 1.5: Let Q and G be a set and group respectively. A Q-Vague set 'A' of G is called a Q-Vague group of G if for all x,y in G and $q \in Q$.

(QVG1) $V_A(xy, q) \geq T\{V_A(x, q), V_A(y, q)\}$ and

(QVG2) $V_A(x^{-1}, q) \geq V_A(x, q)$

Thus $t_A(xy, q) \geq T\{t_A(x, q), t_A(y, q)\}$

$f_A(xy, q) \leq S\{f_A(x, q), f_A(y, q)\}$ and

$t_A(x^{-1}, q) \geq t_A(x, q)$, $f_A(x^{-1}, q) \leq f_A(x, q)$.

Here the element xy stands for x • y.

Definition 1.6: The α- cut A_α of the Q-Vague set 'A' is the (α, α) cut of A and hence given by $A_\alpha = \{x / x \in G, t_A(x, q) \geq \alpha\}$.

Definition 1.7: Let 'A' be a Q-Vague group (QVG) of G. Then 'A' is called Q-Vague normal subgroup (QVNG) is $V_A(xy, q) = V_A(yx, q)$ for $x, y \in G$, $q \in Q$.

Definition 1.8: Let 'A' be a Q-Vague group of G. The set N(A) ={a ∈ G / $V_A(axa^{-1}, q)=V_A(x, q)$} for x ∈ G is called Q-Vague normalizer of A.

Definition 1.9: Let 'A' be a Q-Vague group of G. Then C(A) = {a∈G / $V_A([a,x]_q) = V_A(e,q)$} for all x ∈ G, q ∈ Q is called Q-Vague Centralizer of A where $[a, x]_q = (a^{-1}x^{-1}ax, q)$.

2. CHARACTERIZATIONS OF Q-VAGUE NORMAL GROUPS

The following theorem is first started.

Proposition 2.1: If 'A' is a Q-Vague normal group of a group G, then K = { x∈G / $V_A(x,q) = V_A(e,q)$} is a crisp normal subgroup of G.

Proof: 'A' is a Q-Vague normal group of G.

Let x, y ∈ K and q ∈ Q implies $V_A(x,q) = V_A(e,q)$ and $V_A(y,q) = V_A(e,q)$.

Consider $V_A(x^{-1}y,q) \geq T \{V_A(x,q), V_A(y,q)\}$

$= T \{ V_A(e,q), V_A(e,q) \}$

$= V_A(e,q) \geq V_A(x^{-1}y,q)$

implies $V_A(x^{-1}y,q) = V_A(e,q)$, and so $x^{-1}y$ ∈ H. Therefore 'K' is a crisp subgroup of G.

Let x ∈ G, y ∈ K. Consider $V_A(xyx^{-1},q) = V_A(y,q) = V_A(e,q)$ implies xyx^{-1} ∈ K implies that K is a crisp normal subgroup of G.

Proposition 2.2: Let 'A' be a Q-Vague normal subgroup of G. Then α-cut. A_α is a crisp normal subgroup of G.

Proof: $A_\alpha = \{ x \in G / t_A(x, q) \geq \alpha \}$. Let x, y ∈$A_\alpha$ implies $t_A(x, q) \geq \alpha$ and $t_A(y, q) \geq \alpha$.

Consider $t_A(xy^{-1},q) \geq \text{Min} \{ t_A(x,q), t_A(y,q) \}$

$\geq \text{Min} \{ \alpha, \alpha\} = \alpha$

implies $t_A(xy^{-1},q) \geq \alpha$ and so $xy^{-1} \in A_\alpha$. Therefore A_α is a crisp subgroup of G.

Now, for all x∈G, y ∈A_α. Consider $t_A(xyx^{-1},q) = t_A(y,q) \geq \alpha$ implies $xyx^{-1} \in A_\alpha$.

Proposition 2.3: If A and B are two Q-Vague normal groups of G, then A ∩ B is also Q-Vague normal subgroups of G.

Proof: If A and B are two Vague groups of G, then A ∩ B is also Vague group of G. [Proposition 4.4 [8]).

Now, $t_{A \cap B}(xy,q) = T \{ t_A(xy, q), t_B(xy, q)\}$

$= T \{ t_A(yx, q), t_B(yx, q)\}$

$= t_{A\cap B}(yx, q)$

Also $f_{A\cap B}(xy,q) = S \{f_A(xy,q), f_B(xy,q)\}$

$= S \{ f_A(yx,q), f_B(yx,q)\}$

$= f_{A\cap B}(yx,q)$

implies that $V_{A\cap B}(xy,q) = V_{A\cap B}(yx,q)$ thus A∩B is a Q-Vague normal subgroup in G.

Proposition 2.4: Let 'A' be a Q-Vague group of G and B be a Q-vague normal group of G. Then (A ∩ B) is a Q-vague normal group of the group K = {x∈ G / $V_A(x,q) = V_A(e,q)$}.

Proof: Since 'A' is a Q-vague group of G then K = {x∈ G / $V_A(x,q) = V_A(e,q)$} is a crisp subgroup of G. Also (A ∩ B) is a Q-vague group of G. Now we wish to show that (A ∩ B) is a Q-vague normal group of K. Let x, y ∈ K then xy ∈ K and yx ∈ K implies

$V_A(xy,q) = V_A(e,q)$ and $V_A(yx,q) = V_A(e, q)$ implies$V_A(xy,q) = V_A(yx,q)$.

Since 'B' is a Q-vague normal group of G, then $V_B(xy,q) = V_B(yx,q)$.

Consider $t_{A\cap B}(xy, q) = T\, t_A(xy, q), t_B(xy, q)\}$

$=T\{t_A(yx, q), t_B(yx, q)\}$

$= t_{A\cap B}(yx, q)$.

Also, $f_{A\cap B}(xy,q) = S \{f_A(xy,q), f_B(xy,q)\}$

$= S \{ f_A(yx,q), f_B(yx,q)\}$

$= f_{A\cap B}(yx,q)$.

Therefore $V_{A\cap B}(xy,q) = V_{A\cap B}(yx,q)$ thus A∩B is a Q-Vague normal group of K.

Proposition 2.5: Let 'A' be a Q-vague group of G. Then 'A' is Q-vague normal group of G if and only if $V_A([x, y]_q) \geq V_A(x, q)$ for all x,y in G, where $[x, y]_q = (x^{-1}y^{-1}xy, q)$.

Proof: Suppose 'A' is Q-vague normal group of G.

For all x,y ∈ G, $V_A([x,y]_q) = V_A(x^{-1}y^{-1}xy, q)$

$= V_A(x^{-1}(y^{-1}xy), q)$

$\geq T \{ V_A(x^{-1},q), V_A(y^{-1}xy, q)\}$

$= T\{ V_A(x,q), V_A(x,q)\} = V_A(x,q)$.

Therefore $V_A([x,y]_q) \geq V_A(x,q)$.

Conversly, suppose $V_A([x,y]_q) \geq V_A(x,q)$ for x,z in G.

It follows that $V_A(x^{-1}zx, q) = V_A(ex^{-1}zx, q) = V_A(zz^{-1}x^{-1}zx, q) = V_A(z[z,x]_q)$

$\geq T\{ V_A(z, q), V_A([z,x]_q)\} = V_A(z,q)$

Therefore $V_A(x^{-1}zx, q) \geq V_A(z,q)$ for all x,z in G and $q \in Q$.

It follows that $V_A(z,q) = V_A(xx^{-1}zxx^{-1}, q) \geq T\{ V_A(x, q), V_A(x^{-1}zx, q)\}$.

Now if $T\{V_A(x, q), V_A(x^{-1}zx, q)\} = V_A(x,q)$, then $V_A(z, q) \geq V_A(x,q)$ for all x,z in G.

Implying the constant set and in this case the result is holds trivially.

If $T\{V_A(x, q), V_A(x^{-1}zx, q)\} = V_A(x^{-1}zx, q)$, then $V_A(z, q) \geq V_A(x^{-1}zx, q)$ for all x, z in G.

This implies$V_A(z,q) = V_A(x^{-1}zx, q)$. Thus 'A' is a Q-vague normal group of G.

Proposition 2.6: Let ' A' be a Q-Vague normal group of G. Then (i) Q-normalizer N(A) is a crisp subgroup of G (ii) 'A' is Q-vague normal group of N(A).

Proof: (i) 'A' is a Q-vague group of G and $N(A) = \{a \in G / V_A(ax^{-1}a, q)\} = V_A(x,q)$ for all $x \in G\}$.

Now let x, y $\in$ N(A) implies $V_A(xax^{-1}, q) = V_A(a, q)$ and $V_A(yby^{-1}, q) = V_A(b, q)$.

So $V_A(xy^{-1}a(xy^{-1})^{-1},q) = V_A(xy^{-1}axyx^{-1},q) = V_A(y^{-1}ay,q) = V_A(y,q) = xy^{-1} \in N(A)$

Therefore, N(A) is a crisp subgroup of G.

(ii) Suppose 'A' is a Q-vague normal group of G. Let $a \in G$, $x \in A$, $V_A(xax^{-1},q) = V_A(a,q)$

implies $a \in N(A)$and so $G \subseteq N(A) \subseteq G$. Thus N(A) = G. Convexly, N(A) = G giving $V_A(axa^{-1},q) = V_A(x,q)$ for all $x \in G$. So 'A' is Q-vague normal group of G.

Let $x \in A$. Therefore $\alpha \in N(A) \subseteq G$, and then $V_A(x\alpha x^{-1},q) = V_A(\alpha,q)$ gives 'A' is a Q-vague normal group of N(A).

Proposition 2.7: Let 'A' be a Q-vague normal group of G and $K = \{ x \in G / V_A(x, q) = V_A(e,q) \}$. Then $K \subseteq$ **C(**A).

Proof: Let $x \in K$ implies $V_A(x,q) = V_A(e,q)$ for all $y \in G$. Consider

$V_A([x, y] q) = V_A(x^{-1}y^{-1}xy, q)$

$= V_A(x^{-1}(y^{-1}xy), q)$

$\geq T\{V_A(x, q), V_A(y, q)\}$

$= V_A(x, q) = V_A(e, q)$ implies

$V_A([x, y] q) \geq V_A(e, q)$.

But $V_A([x,y]_q) \leq V_A(e,q)$ gives $V_A([x,y]_q) = V_A(e,q)$ and so $x \in$ **C(**A), thus $K \subseteq$ **C(**A).

Proposition 2.8: Let 'A' be a Q-Vague group of a group G. Then $K = \{ x \in G / V_A(x, q) = V_A(e, q)\}$ is a normal subgroup of N(A).

Proof: Let $x \in K$, $y \in G$ implies

$V_A(x, q) = V_A(e, q)$.

Consider $V_A(xyx^{-1},q) \geq T\{V_A(x,q), V_A(y,q)\}$

$= V_A(y, q) = V_A(eye, q)$

$= V_A(x^{-1}(xyx^{-1}x), q)$

$\geq T\{ V_A(x^{-1}, q), V_A(xyx^{-1}x, q)\}$

$= T\{ V_A(e, q), V_A(xyx^{-1}, q)\}$

$= V_A(xyx^{-1}, q)$ for all $y \in G$.

$V_A(xyx^{-1},q) = V_A(x,q)$, for all $y \in G$ gives $x \in N(A)$ and so $K \subseteq N(A)$.

Now, for all $a \in N(A)$, $V_A(axa^{-1},q) = V_A(x,q)$ for all $x \in G$. Thus $V_A(axa^{-1},q) = V_A(y,q) = V_A(e,q)$ for all $y \in K$, giving that $aya^{-1} \in K$, and so K is a normal subgroup of N(A).

Proposition 2.9: Let 'A' be a Q-vague group of G then C(A) is a normal subgroup of G.

Proof: $C(A) = \{a \in G / V_A([a, x]_q) = V_A(e,q)$, for all $x \in G\}$.

Let $a \in C(A)$ gives $V_A([a, x]_q) = V_A(e, q)$. So $V_A(a^{-1}x^{-1}x, q) = V_A(e, q)$.

Thus $V_A((xa)^{-1}ax, q) = V_A(e,q)$ implies $V_A(xa, q) = V_A(ax, q)$.

Let a, b $\in$ C(A) . Then $V_A([a, x]_q) = V_A(e,q)$ and $V_A([b, x]_q) = V_A(e,q)$

Consider

$V_A([ab^{-1}, x]_q) = V_A((ab^{-1})^{-1}x^{-1}ab^{-1}x, q) = V_A(b(a^{-1}x^{-1}ab^{-1}x, q))$

$= V_A((a^{-1}x^{-1}ax, x^{-1}b^{-1}xb), q) = V_A([a, x]_q, [x, b]_q)$

$\geq T\{V_A([a, x]_q), V_A([x,b]_q)\}$

$= T\{V_A(e,q), V_A(e,q)\} = V_A(e,q)$

Therefore, $V_A([ab^{-1}, x]_q) \geq V_A(e,q)$

$\geq V_A([ab^{-1}, x]_q)$ gives

$V_A([ab^{-1}, x]_q) = V_A(e,q)$ and so $ab^{-1} \in C(A)$. Therefore C(A) is a subgroup of G.

Now, for all a $\in$ C(A) for all g $\in$ G, it follows that

$V_A([g^{-1}ag, x]_q) = V_A((g^{-1}ag)^{-1}x^{-1}g^{-1}agx, q) = V_A([g, a]_q, (a^{-1}(gx)^{-1}a(gx)\}$

$= V_A([g, a]_q, V_A[a, gx]_q) = T\{V_A[g, a]_q, [a, gx]\}$

$= T\{V_A(e,q), V_A(e,q)\}$

$= V_A(e,q)$

Then $V_A([g^{-1}ag, x]_q) \geq V_A(e,q)$

$\geq V_A([g^{-1}ag, x]_q)$ gives

$V_A([g^{-1}ag, x]_q) = V_A(e, x)$.

Thus $V_A([g^{-1}ag, x]_q) = V_A(e, q)$ gives $g^{-1}ag \in C(A)$, and so C(A) is a crisp normal subgroup of G.

Proposition 2.10: If 'A' is a Q-vague group of G and θ is a homomorphism of G, then Q-vague set A^θ is also Q-vague group of G.

Proof: Let x,y $\in$ G and q $\in$ Q. Then

$t_A^\theta(xy,q) = t_A(\theta(xy, q)) = t_A(\theta(x,q), \theta(y,q)) \geq T\{t_A(\theta(x,q), t_A(\theta(y,q)\}$

$= T\{t_A^\theta(x,q), t_A^\theta(y,q)\}$

Also

$f_A^\theta(xy,q) = f_A(\theta(xy, q)) = f_A(\theta(x,q), \theta(y,q)) \leq S\{f_A(\theta(x,q), f_A(\theta(y,q)\}$

$= S\{f_A^\theta(x,q), f_A^\theta(y,q)\}$

Again

$t_A^\theta(x^{-1},q) = t_A(\theta(x^{-1}, q)) = t_A(\theta(x, q)^{-1})$

$= t_A(\theta(x, q)) = t_A(\theta(x, q)$ for all $x \in G$

Similarly, $f_A^\theta(x^{-1},q) = f_A(\theta(x, q)$ for all $x \in G$, Thus $_A^\theta$ is a Q-vague group of G.

Proposition 2.11: If A is a Q-vague characteristic group of G, it is a Q-vague normal group of G.

Proof: Let x,y $\in$ G and q $\in$ Q. Consider the map $\theta : G \rightarrow G'$ given by $\theta(g,q) = (x^{-1}gx, q)$ for all $g \in G$. Clearly, θ is an automorphism of G. Now $t_A(xy,q) = t_A^\theta(xy, q) = t_A(\theta(xy, q)) = t_A(x^{-1}xyx, q) = t_A(yx, q)$.

Similarly, $f_A(xy,q) = f_A(yx,q)$ for all x, y $\in$ G. Therefore 'A' is a Q-Vague normal group of G.

Definition 2.12: (Q-Homologous Vague groups) : Let A and B be two Q-vague groups of a group G. If there exists Φ $\in$ Aut(G) such that $V_A(x, q) = V_B(\Phi(x, q))$ for all x $\in$ G. Thus $t_A(x, q) = t_B(\Phi(x, q))$ and $f_A(x, q) = f_B(\Phi(x, q))$. Then A and B are homologous Vague group of G.

Proposition 2.13: Let 'B' be a Q-vague group of G. Φ $\in$ Aut (G). Let 'A' be a Q-vague set of G such that $V_A(x,q) = V_B(\Phi(x,q))$ for all x $\in$ G. Then A and B are Q-homologous Vague group of G.

Proof: $V_A(xy, q) = V_B(\Phi(xy, q)) = V_B(\Phi(x, q), \Phi(y, q))$

$\geq T\{V_B(\Phi(x,q), V_B(\Phi(y,q)\}$

$= T\{V_A(x,q), V_A(y,q)\}$

Also $V_A(x^{-1}, q) = V_B(\Phi(x^{-1}, q)$

$= V_B((\Phi(x,q))^{-1}) \geq V_B((\Phi(x,q)) = V_A(x, q)$.

Therefore 'A' is a Q-vague group of G. Hence A and B are Q-homologous Vague groups of G.

Proposition 2.14: Let A and B be Q-homologous Vague group of G. Then C(A) and N(A) are Q-homologous subgroups of G.

Proof: Let A and B are Q-homologous vague groups of G. Then C(A), C(B) are subgroups of G. Now C(A), C(B) are to be proved Q-homologous subgroups of G. For this, it is enough to check that, there exits an automorphism Φ of G such that Φ(C(A)) = C(B).

Since A and B are -homologous Vague groups, there exsits Φ $\in$ Aut(G) such that

$V_A(x, q) = V_B(\Phi(x, q))$; $V_B(x, q) = V_A(\Phi^{-1}(x, q))$. For $x \in G$, so for $a \in C(A)$, it follows that

$V_B([\Phi(a), x]_q) = V_B((\Phi(a))^{-1} x^{-1} \Phi(a)x, q) = V_B(\Phi(a^{-1}) x^{-1} \Phi(a)x, q)$

$= V_B(\Phi(a^{-1} \Phi^{-1}(x^{-1}) a \Phi^{-1}(x), q)$

$= V_A(a^{-1} \Phi^{-1}(x^{-1}) a \Phi^{-1}(x), q)$

$= V_A([(a, \Phi^{-1}(x)]_q) = V_A(e, q)$

$= V_A(\Phi^{-1}(x,q) = V_B(e,q)$ for all $x \in G$.

Therefore $\Phi(C(A)) \subseteq C(B)$ -----> **(1).**

On the other hand, for all $a \in C(B)$.

$V_B([\Phi^{-1}(a), x]_q) = V_A(\Phi^{-1}(a^{-1}) x^{-1} \Phi^{-1}(a)x, q) = V_A(\Phi^{-1}(a^{-1} \Phi(x^{-1}) a \Phi(x), q)$

$= V_A(\Phi^{-1}[a, \Phi(x)] q)$

$= V_B([(a, \Phi(x)]_q) = V_B(e, q)$

$= V_B(\Phi(e, q) = V_A(e, q)$.

Therefore $\Phi^{-1}(a, q) \in C^{-1}(A)$ gives

$C(B) \subseteq \Phi(C(A))$ ------------> **(2).**

From (1) and (2), $\Phi(C(A)) = C(B)$. Hence C(A) and C(B) are Q-homologous subgroup of G.

3. CONCLUSION

Ranjit Biswas [6] introduced the concept of Vague groups and others [1], [5] were discussed. [8] investigated the concept of Q-Fuzzy groups. In this paper we investigate a new kind of Q-Vague groups and its characterizations.

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