Weak Stability Results for Jungck-Ishikawa Iteration

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ABSTRACT

The stability of iterative procedures plays an important role while solving the nonlinear equations obtained out of a physical problem using the advanced computational tools. The main purpose of this paper is to present a weaker stability result for Jungck-Ishikawa iteration process for a map satisfying some general contractive conditions in metric spaces. Some well known recent results are also derived as special cases. An example is given to support the rationality of the used iterative scheme.

General Terms
Computational Mathematics.

Keywords

Weak stability, w^2- stability, equivalent sequence, Jungck Iteration, Jungck-Ishikawa iteration.

1. INTRODUCTION

The first result on stability of iterations on metric spaces was given by Alexander M. Ostrowski [11], although this type of problem was already observed by M Urabe in 1957 for real valued functions. Thereafter, a number of authors have generalized and extended his results in various setting. Czerwik et al. [4] have extended the Ostrowski's result to the setting of generalized metric spaces. Harder and Hicks [5]-[6] have obtained interesting results on stability of a number of iterative procedures with examples. Rhoades [13]-[14] and Rhoades and Saliga [15] have generalized various existing results and established fixed point theorems for various iterative procedures. For a detailed discussion on the role of stability of iterative procedures in analysis, one may refer to Agarwal et al [1], Berinde [2], Czerwik et al. [4], Harder and Hicks [5]-[6], Istrăţescu [8], Osilike [10], Rhoades [13]-[15], Rus et al. [16], Prasad and Sahni [12], Singh et al. [17] Singh and Prasad [18] and references thereof.

Recently Timis and Berinde [20] observed that the concept of stability is not very precise because of the choice of an arbitrary sequence in place of an approximate sequence. With this notion they introduced a weaker concept of stability called weak stability and remarked that every stable iteration is weakly stable but the reverse may not be true (see [20]). Further, Timis [19] defined the concept of weak w^2 - stability of an operator in a metric space. In this paper we obtain weak w^2 - stability result of the Jungck-Ishikawa iteration process for pair of maps satisfying some contractive conditions of general nature.

First we present the basic concepts regarding different iterative procedures used in the sequel.

The most popular iterative procedure called Picard iteration is defined by

$$x_{n+1} = Tx_n, \; n = 0, 1, 2, \ldots \tag{1}$$

When the contractive conditions are slightly weaker, the Picard iteration (1) need not converge to a fixed point of the operator under consideration and some other iterative procedures should be considered. The Mann iteration scheme [2] is defined in the following manner:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \tag{2}$$

where $0 \le \alpha_n \le 1$ and $n = 0, 1, 2, \ldots$.

Notice that (2) reduces to (1) when $\alpha_n = 1$.

For $0 \le \alpha_n, \beta_n \le 1$ and $n = 0, 1, 2, \ldots$, the Ishikawa iteration is defined as

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n Ty_n \\ y_n &= (1-\beta_n)x_n + \beta_n Tx_n, \end{aligned} \tag{3}$$

The Jungck-iteration scheme [7] is defined in the following manner (also see [17]):

Given $S, T : Y \to X$, $T(Y) \subseteq S(Y)$ and $x_0 \in Y$,

$$Sx_{n+1} = f(T, x_n), \; n = 0, 1, \ldots \tag{4}$$

Some of the Jungck iterative schemes are as follows:

In (4) if we put $Sx_{n+1} = Tx_n$ it becomes Jungck-Picard iteration and when $Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Tx_n$, it is called Jungck-

Mann iteration. The Jungck –Ishikawa iteration is defined (see [9]) in the following manner:

$$\left.\begin{aligned} Sx_{n+1} &= (1-\alpha_n)Sx_n + \alpha_n Ty_n \\ Sy_n &= (1-\beta_n)Sx_n + \beta_n Tx_n \end{aligned}\right\} \quad (5)$$

where $n = 0,1,..$and $\{\alpha_n\}$, $\{\beta_n\}$ satisfies

$(i)\,\alpha_0 = 1,\qquad (ii)\,0 \le \alpha_n, \beta_n \le 1,\ n \ge 0,$

$(iii)\sum \alpha_n = \infty,\qquad (iv)\sum_{j=0}^{n}\prod_{i=j+1}^{n}\{1-\alpha_i + a\alpha_i\}$ converges.

For an exhaustive treatment of iterative procedures in nonlinear analysis, Berinde [2] is a good reference.

2. PRELIMINARIES

In this section we present the relevant definitions required for our result.

Definition 2.1 [2]. Let (X,d) be a metric space and $\{x_n\}_{n=1}^{\infty} \subset X$ be a given sequence. We shall say that $\{y_n\}_{n=0}^{\infty} \in X$ is an approximate sequence of $\{x_n\}$ if, for any $k \in N$, there exists $\eta = \eta(k)$ such that

$$d(x_n, y_n) \le \eta, \text{ for all } n \ge k$$

Definition 2.2 [2]. Let (X,d) be a metric space and $T: X \to X$ be a map. Let $\{x_n\}$ be an iteration procedure defined by $x_0 \in X$ and

$$x_{n+1} = f(T, x_n),\ n \ge 0$$

Suppose that $\{x_n\}$ converges to a fixed point p of T. If for any approximate sequence $\{y_n\} \subset X$ of $\{x_n\}$,

$\lim_{n\to\infty} d(y_{n+1}, f(T, y_n)) = 0$ implies $\lim_{n\to\infty} y_n = p,$ then we shall say that (5) is weakly *T*-stable or weakly stable in respect to *T*.

Definition 2.4. [3] Two sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are equivalent sequences if $d(x_n, y_n) \to 0$ as $n \to \infty$.

Definition 2.5 [19]. Let (X,d) be a metric space and $T: X \to X$. Let $\{x_n\}$ be an iteration procedure defined by $x_0 \in X$ and

$$x_{n+1} = f(T, x_n),\ n \ge 0 \quad (6)$$

Suppose that $\{x_n\}$ converges to a fixed point p of T. If for any equivalent sequence $\{y_n\} \subset X$ of $\{x_n\}$,

$\lim_{n\to\infty} d(y_{n+1}, f(T, y_n)) = 0$ implies $\lim_{n\to\infty} y_n = p,$ then we shall say that (6) is weak w^2- stable in respect to *T*.

It is remarked that any equivalent sequence is an approximate sequence but there are examples in the literature to show that the converse may not be true (see [19]).

3. MAIN RESULTS

Theorem 3.1. Let (X,d) be a metric space and $S,T: X \to X$ such that $T(X) \subseteq S(X)$ and $S(X)$ or $T(X)$ is a complete subspace of X. Let z be a coincidence point of T and S, i.e., $Sz = Tz = p$. Let $x_0 \in X$ and the sequence $\{Sx_n\}$ generated by (5) converges to p. Let $\{Sx_n\} \subset X$ is an equivalent sequence and define

$$Ss_n = (1-\beta_n)Sy_n + \beta_n Ty_n,\ n \ge 0$$
$$\varepsilon_n = d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ts_n),\quad n \ge 0.$$

If the pair (S,T) satisfies

$$d(Tx, Ty) < \max\{d(Sx, Tx), d(Sy, Ty)\} \quad (7)$$

Then the Jungck-Ishikawa iteration is w^2 -stable.

Proof. Consider $\{Sy_n\}_{n=0}^{\infty}$ to be an equivalent sequence of $\{Sx_n\}$. Then, according to definition, if $\lim_{n\to\infty} d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ts_n) = 0$ implies that $\lim_{n\to\infty} Sy_n = p$, then the J-Ishikawa iteration is w^2 -stable. In order to prove this, we suppose that $\lim_{n\to\infty} d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ts_n) = 0.$

Therefore, $\forall \varepsilon > 0,\ \exists\ n_0 = n(\varepsilon)$ such that
$d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ts_n) < \varepsilon,\ \forall n \ge n_0$.

So,

$$\begin{aligned} d(p, Sy_{n+1},) &\le d(p, Sx_{n+1}) + d(Sx_{n+1}, Sy_{n+1}) \\ &\le d(p, Sx_{n+1}) + d[(1-\alpha_n)Sx_n + \alpha_n Ty_n, Sy_{n+1}] \\ &\le d(p, Sx_{n+1}) + d[(1-\alpha_n)Sx_n + \alpha_n Ty_n, \\ &\quad (1-\alpha_n)Sy_n + \alpha_n Ts_n] + d[(1-\alpha_n)Sy_n + \alpha_n Ts_n, Sy_{n+1}] \\ &\le d(p, Sx_{n+1}) + (1-\alpha_n)d(Sx_n, Sy_n) + \\ &\quad \alpha_n d(Ty_n, Ts_n) + \varepsilon_n \\ &\le d(p, Sx_{n+1}) + (1-\alpha_n)d(Sx_n, Sy_n) + \\ &\quad \alpha_n[\max\{d(Sy_n, Ty_n), d(Ss_n, Ts_n)\}] + \varepsilon_n \end{aligned}$$

If $\max\{d(Sy_n, Ty_n), d(Ss_n, Ts_n)\} = d(Sy_n, Ty_n)$

Then,

$$\begin{aligned} d(p, Sy_{n+1},) &\le d(p, Sx_{n+1}) + (1-\alpha_n)d(Sx_n, Sy_n) + \\ &\quad \alpha_n d(Sy_n, Ty_n) + \varepsilon_n \end{aligned}$$

But,

$$d(Sy_n, Ty_n) \le d(Sy_n, Sx_n) + d(Sx_n, p) + d(Tp, Ty_n)$$

and $d(Tp, Ty_n) < \max\{d(Sp, Tp), d(Sy_n, Ty_n)\}$

If $\max\{d(Sp, Tp), d(Sy_n, Ty_n)\} = d(Sp, Tp)$, then $\lim_{n\to\infty} d(Sy_n, Ty_n) = 0$

and if $\max\{d(Sp, Tp), d(Sy_n, Ty_n)\} = d(Sy_n, Ty_n)$

Then,

$$d(Sy_n, Ty_n) \le d(Sy_n, Sx_n) + d(Sx_n, p) + d(Sy_n, Ty_n)$$

or,

$$0 \le d(Sy_n, Sx_n) + d(Sx_n, p)$$

Now, since Sx_n and Sy_n are equivalent sequences, therefore $\lim_{n\to\infty} d(Sy_n, Sx_n) = 0$.

Thus, $\lim_{n\to\infty} d(Sy_n, Ty_n) = 0$ (8)

If $\max\{d(Sy_n, Ty_n), d(Ss_n, Ts_n)\} = d(Ss_n, Ts_n)$

Then,

$$d(p, Sy_{n+1},) \le d(p, Sx_{n+1}) + (1-\alpha_n) d(Sx_n, Sy_n) + \alpha_n d(Ss_n, Ts_n) + \varepsilon_n$$

Now,

$$\begin{aligned} d(Ss_n, Ts_n) &\le d(Ss_n, Sy_n) + d(Sy_n, p) + d(Tp, Ts_n) \\ &= d((1-\beta_n)Sy_n + \beta_n Ty_n, (1-\beta_n)Sx_n + \beta_n Tx_n) \\ &\quad + d(Sy_n, p) + d(Tp, Ts_n) \\ &\le (1-\beta_n) d(Sy_n, Sx_n) + \beta_n d(Ty_n, Tx_n) + d(Sy_n, p) \\ &\quad + d(Tp, Ts_n) \\ &\le (1-\beta_n) d(Sy_n, Sx_n) + \beta_n [d(Ty_n, Sy_n) + d(Sy_n, Sx_n) \\ &\quad + d(Sx_n, Tx_n)] + d(Sy_n, p) + d(Tp, Ts_n) \end{aligned}$$

Again,

$d(Sx_n, Tx_n) \le d(Sx_n, Sy_n) + d(Sy_n, p) + d(Tp, Tx_n)$ and

$$d(Tp, Tx_n) < \max\{d(Sp, Tp), d(Sx_n, Tx_n)\}.$$

If $\max\{d(Sp, Tp), d(Sx_n, Tx_n)\} = d(Sp, Tp)$, then $\lim_{n\to\infty} d(Sx_n, Tx_n) = 0$ (9)

and if $\max\{d(Sp, Tp), d(Sx_n, Tx_n)\} = d(Sx_n, Tx_n)$, then $0 \le d(Sx_n, Sy_n) + d(Sy_n, p)$

and $\lim_{n\to\infty} d(Sx_n, Tx_n) = 0$ (10)

Now,

$$d(Tp, Ts_n) < \max\{d(Sp, Tp), d(Ss_n, Ts_n)\}$$

If $\max\{d(Sp, Tp), d(Ss_n, Ts_n)\} = d(Sp, Tp)$, then by (8), (9) and (10), $\lim_{n\to\infty} d(Ss_n, Ts_n) = 0$

and if $\max\{d(Sp, Tp), d(Ss_n, Ts_n)\} = d(Ss_n, Ts_n)$, then

$$\begin{aligned} 0 &\le d(Sy_n, Sx_n) + \beta_n [d(Ty_n, Sy_n) + d(Sx_n, Tx_n)] \\ &\quad + d(Sy_n, p) \end{aligned}$$

and by (8), (9) and (10), $\lim_{n\to\infty} d(Ss_n, Ts_n) = 0$.

Thus we have $\lim_{n\to\infty} d(p, Sy_{n+1}) = 0$.

Corollary 3.2. [19] Let (X, d) be a complete metric space and $T : X \to X$ a mapping satisfying

$$d(Tx, Ty) < \max\{d(x, Tx), d(y, Ty)\}$$

Let $x_0 \in X$ and the sequence $\{x_n\}$ generated by $x_{n+1} = Tx_n$, for all $n \ge 0$ and the sequence $\{x_n\}$ converges to *p*, the unique fixed point of T. Then the Picard iteration is w^2-stable.

Proof. If $S = id$, the identity map and $\alpha_n = \beta_n = 1$, then result follows from Theorem 3.1.

Theorem 3.3. Let (X, d) be a metric space and $S, T : X \to X$ such that $T(X) \subseteq S(X)$ and $S(X)$ or $T(X)$ is a complete subspace of X. Let z be a coincidence point of T and S, i.e., $Sz = Tz = p$. Let $x_0 \in X$ and the sequence $\{Sx_n\}$ generated by (5) converges to *p*. Let $\{Sx_n\} \subset X$ is an equivalent sequence and define

$$\begin{aligned} Ss_n &= (1-\beta_n)Sy_n + \beta_n Ty_n, \ n \ge 0 \\ \varepsilon_n &= d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ts_n), \quad n \ge 0. \end{aligned}$$

If the pair (S, T) satisfies

$$d(Tx, Ty) < \max\{d(Sx, Tx), d(Sy, Ty), d(Sx, Sy)\} \tag{11}$$

Then the Jungck-Ishikawa iteration is w^2-stable.

Proof. Here we have,

$$\begin{aligned} d(p, Sy_{n+1},) &\le d(p, Sx_{n+1}) + d(Sx_{n+1}, Sy_{n+1}) \\ &\le d(p, Sx_{n+1}) + d[(1-\alpha_n)Sx_n + \alpha_n Ty_n, Sy_{n+1}] \\ &\le d(p, Sx_{n+1}) + d[(1-\alpha_n)Sx_n + \alpha_n Ty_n, \\ &\quad (1-\alpha_n)Sy_n + \alpha_n Ts_n] + d[(1-\alpha_n)Sy_n + \alpha_n Ts_n, Sy_{n+1}] \end{aligned}$$

$$\begin{aligned} &\le d(p, Sx_{n+1}) + (1-\alpha_n) d(Sx_n, Sy_n) + \\ &\quad \alpha_n d(Ty_n, Ts_n) + \varepsilon_n \\ &\le d(p, Sx_{n+1}) + (1-\alpha_n) d(Sx_n, Sy_n) + \\ &\quad \alpha_n [\max\{d(Sy_n, Ty_n), d(Ss_n, Ts_n), d(Sy_n, Ss_n)] + \varepsilon_n \end{aligned}$$

Now we have to prove result only when

$\max\{d(Sy_n,Ty_n), d(Ss_n,Ts_n), d(Sy_n,Ss_n)\} = d(Sy_n,Ss_n)$

We have,

$$d(Sy_n,Ss_n) = d((1-\beta_n)Sx_n + \beta_n Tx_n, (1-\beta_n)Sy_n + \beta_n Ty_n)$$
$$\le (1-\beta_n)d(Sx_n,Sy_n) + \beta_n d(Tx_n,Ty_n)$$

But,

$$d(Tx_n,Ty_n) \le d(Tx_n,Sx_n) + d(Sx_n,Sy_n) + d(Sy_n,Ty_n)$$

So,

$$d(Sy_n,Ss_n) \le d(Sx_n,Sy_n) + \beta_n d(Tx_n,Sx_n) + \beta_n d(Sy_n,Ty_n)$$

By, (8) $\lim_{n\to\infty} d(Sy_n,Ty_n) = 0$ and by (9) and (10) $\lim_{n\to\infty} d(Sx_n,Tx_n) = 0$ and since Sx_n and Sy_n are equivalent sequences, therefore $\lim_{n\to\infty} d(Sy_n,Sx_n) = 0$ and

$$\lim_{n\to\infty} d(Sy_n,Ss_n) = 0 \qquad (12)$$

Thus $\lim_{n\to\infty} d(p,Sy_{n+1}) = 0$

Corollary 3.4. [19] Let (X,d) be a complete metric space and $T: X \to X$ a mapping satisfying

$$d(Tx,Ty) < \max\{d(x,Tx), d(y,Ty), d(x,y)\}$$

Let $x_0 \in X$ and the sequence $\{x_n\}$ generated by $x_{n+1} = Tx_n$, for all $n \ge 0$ and the sequence $\{x_n\}$ converges to *p*, the unique fixed point of T. Then the Picard iteration is w^2-stable.

Proof. If $S = id$, the identity map, and $\alpha_n = \beta_n = 1$, then result follows from Theorem 3.3.

Theorem 3.5. Let (X,d) be a metric space and $S,T: X \to X$ such that $T(X) \subseteq S(X)$ and $S(X)$ or $T(X)$ is a complete subspace of X. Let z be a coincidence point of T and S, i.e., $Sz = Tz = p$. Let $x_0 \in X$ and the sequence $\{Sx_n\}$ generated by (5) converges to *p*. Let $\{Sx_n\} \subset X$ is an equivalent sequence and define

$$Ss_n = (1-\beta_n)Sy_n + \beta_n Ty_n,\ n \ge 0$$
$$\varepsilon_n = d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ts_n), \quad n \ge 0.$$

If the pair (S,T) satisfies

$$d(Tx,Ty) < \max\{d(Sx,Tx), d(Sy,Ty), d(Sx,Sy), d(Sx,Ty), d(Sy,Tx)\} \qquad (13)$$

Then the Jungck-Ishikawa iteration is w^2-stable.

Proof.

Here we have,

$$d(p,Sy_{n+1}) \le d(p,Sx_{n+1}) + d(Sx_{n+1},Sy_{n+1})$$
$$\le d(p,Sx_{n+1}) + d[(1-\alpha_n)Sx_n + \alpha_n Ty_n, Sy_{n+1}]$$
$$\le d(p,Sx_{n+1}) + d[(1-\alpha_n)Sx_n + \alpha_n Ty_n,$$
$$(1-\alpha_n)Sy_n + \alpha_n Ts_n] + d[(1-\alpha_n)Sy_n + \alpha_n Ts_n, Sy_{n+1}]$$

$$\le d(p,Sx_{n+1}) + (1-\alpha_n)d(Sx_n,Sy_n) +$$
$$\alpha_n d(Ty_n,Ts_n) + \varepsilon_n$$

$$\le d(p,Sx_{n+1}) + (1-\alpha_n)d(Sx_n,Sy_n) +$$
$$\alpha_n[\max\{d(Sy_n,Ty_n), d(Ss_n,Ts_n), d(Sy_n,Ss_n),$$
$$d(Sy_n,Ts_n), d(Ss_n,Ty_n)] + \varepsilon_n$$

Now if

$$\max\{d(Sy_n,Ty_n), d(Ss_n,Ts_n), d(Sy_n,Ss_n),$$
$$d(Sy_n,Ts_n), d(Ss_n,Ty_n) = d(Sy_n,Ts_n)$$

Then, $d(Sy_n,Ts_n) \le d(Sy_n,Ss_n) + d(Ss_n,Ts_n)$ and by (12) $\lim_{n\to\infty} d(Sy_n,Ss_n) = 0$ and by (8), (9) and (10) $\lim_{n\to\infty} d(Ss_n,Ts_n) = 0$.

If

$$\max\{d(Sy_n,Ty_n), d(Ss_n,Ts_n), d(Sy_n,Ss_n),$$
$$d(Sy_n,Ts_n), d(Ss_n,Ty_n) = d(Ss_n,Ty_n)$$

Then, $d(Ss_n,Ty_n) \le d(Ss_n,Sy_n) + d(Sy_n,Ty_n)$

And by (12), $\lim_{n\to\infty} d(Sy_n,Ss_n) = 0$ and by (8) $\lim_{n\to\infty} d(Sy_n,Ty_n) = 0$.

Thus $\lim_{n\to\infty} d(p,Sy_{n+1}) = 0$.

Corollary 3.6. [19] Let (X,d) be a complete metric space and $T: X \to X$ a mapping satisfying

$$d(Tx,Ty) < \max\{d(x,Tx), d(y,Ty), d(x,y), d(x,Ty), d(y,Tx)\}$$

Let $x_0 \in X$ and the sequence $\{x_n\}$ generated by $x_{n+1} = Tx_n$, for all $n \ge 0$ and the sequence $\{x_n\}$ converges to *p*, the unique fixed point of T .Then the Picard iteration is w^2 -stable.

Proof. If $S = id$, the identity map, and $\alpha_n = \beta_n = 1$, then result follows from Theorem 3.5.

Theorem 3.6. Let (X,d) be a metric space and $S,T: X \to X$ such that $T(X) \subseteq S(X)$ and $S(X)$ or $T(X)$ is a complete subspace of X. Let z be a coincidence point of T and S, i.e., $Sz = Tz = p$. Let $x_0 \in X$ and the sequence $\{Sx_n\}$ generated by (5) converges to *p*. Let $\{Sx_n\} \subset X$ is an equivalent sequence and define

$$Ss_n = (1-\beta_n)Sy_n + \beta_n Ty_n,\ n \geq 0$$
$$\varepsilon_n = d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ts_n), \quad n \geq 0.$$

If the pair (S,T) satisfies

$$d(Tx,Ty) \leq ad(Sx,Sy) + Ld(Sx,Tx) \tag{14}$$

for $a \in (0,1)$ and $L \geq 0$. Then the Jungck-Ishikawa iteration is w^2 -stable.

Proof. Consider $\{Sy_n\}_{n=0}^{\infty}$ to be an equivalent sequence of $\{Sx_n\}$. Then, according to definition, if $\lim_{n\to\infty} d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ts_n) = 0$ implies that $\lim_{n\to\infty} Sy_n = p$, then the J-Ishikawa iteration is w^2-stable. In order to prove this, we suppose that $\lim_{n\to\infty} d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ts_n) = 0$.

Therefore, $\forall \varepsilon > 0,\ \exists\ n_0 = n(\varepsilon)$ such that $d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ts_n) < \varepsilon,\ \forall n \geq n_0$.

So,

$$d(p, Sy_{n+1},) \leq d(p, Sx_{n+1}) + d(Sx_{n+1}, Sy_{n+1})$$
$$\leq d(p, Sx_{n+1}) + d[(1-\alpha_n)Sx_n + \alpha_n Ty_n, Sy_{n+1}]$$
$$\leq d(p, Sx_{n+1}) + d[(1-\alpha_n)Sx_n + \alpha_n Ty_n,$$
$$(1-\alpha_n)Sy_n + \alpha_n Ts_n] + d[(1-\alpha_n)Sy_n + \alpha_n Ts_n, Sy_{n+1}]$$

$$\leq d(p, Sx_{n+1}) + (1-\alpha_n)d(Sx_n, Sy_n) +$$
$$\alpha_n d(Ty_n, Ts_n) + \varepsilon_n$$
$$\leq d(p, Sx_{n+1}) + (1-\alpha_n)d(Sx_n, Sy_n) +$$
$$\alpha_n[ad(Sy_n, Ss_n) + Ld(Sy_n, Ty_n)] + \varepsilon_n$$
$$\leq d(p, Sx_{n+1}) + (1-\alpha_n)d(Sx_n, Sy_n) +$$
$$a\alpha_n d(Sy_n, Ss_n) + L\alpha_n d(Sy_n, Ty_n) + \varepsilon_n$$
$$\leq d(p, Sx_{n+1}) + (1-\alpha_n)d(Sx_n, Sy_n) +$$
$$a\alpha_n d[(1-\beta_n)Sx_n + \beta_n Tx_n, (1-\beta_n)Sy_n + \beta_n Ty_n]$$
$$+ L\alpha_n d(Sy_n, Ty_n) + \varepsilon_n$$
$$\leq d(p, Sx_{n+1}) + (1-\alpha_n)d(Sx_n, Sy_n) +$$
$$a\alpha_n(1-\beta_n)d(Sx_n, Sy_n) +$$
$$a\alpha_n\beta_n d(Tx_n, Ty_n) + L\alpha_n d(Sy_n, Ty_n) + \varepsilon_n$$

$$\leq d(p, Sx_{n+1}) + (1-\alpha_n)d(Sx_n, Sy_n) +$$
$$a\alpha_n(1-\beta_n)d(Sx_n, Sy_n) +$$
$$a\alpha_n\beta_n[ad(Sx_n, Sy_n) + Ld(Sx_n, Tx_n)] +$$
$$L\alpha_n d(Sy_n, Ty_n) + \varepsilon_n$$

$$\leq d(p, Sx_{n+1}) + (1-\alpha_n)d(Sx_n, Sy_n) +$$
$$a\alpha_n(1-\beta_n)d(Sx_n, Sy_n) +$$
$$a^2\alpha_n\beta_n d(Sx_n, Sy_n) + aLd(Sx_n, Tx_n) +$$
$$L\alpha_n d(Sz_n, Ty_n) + \varepsilon_n$$

But,

$$d(Sx_n, Tx_n) \leq d(Sx_n, Sy_n) + d(Sy_n, p) + d(p, Tx_n)$$
$$= d(Sx_n, Sy_n) + d(Sy_n, p) + d(Tp, Tx_n)$$

$$\leq d(Sx_n, Sy_n) + d(Sy_n, p) + ad(Sp, Sx_n) +$$
$$Ld(Sp, Tp)$$

Similarly,

$$d(Sy_n, Ty_n) \leq d(Sy_n, Sx_n) + d(Sx_n, p) + d(p, Ty_n)$$

$$d(Sy_n, Ty_n) \leq d(Sy_n, Sx_n) + d(Sx_n, p) +$$
$$ad(Sp, Sy_n) + Ld(Sp, Tp)$$

Since Sy_n is an equivalent sequence of Sx_n, so we have $\lim_{n\to\infty}(Sx_n, Sy_n) = 0.$

This implies that $\lim_{n\to\infty}(p, Sy_{n+1}) = 0.$

This proves the theorem.

The convergence of the Jungck-Ishikawa iteration scheme is demonstrated in the following example

Example 3.1. To solve cubic equation $x^3 + 4x^2 - 5x - 10 = 0$, we rewrite the equation by splitting it into two parts $Sx = 5x$ and $Tx = x^3 + 4x^2 - 10.$

Following table illustrates the convergence of the iterative scheme.

n	Sx_{n+1}	Tx_n	x_{n+1}
0	-4	-5	-0.8
1	-7.5568	-7.952	-1.51136
2	-4.63956	-4.31543	-0.927913
3	-7.08334	-7.35487	-1.41667
4	-5.04219	-4.81539	-1.00844
5	-6.76619	-6.95774	-1.35324
.	.	.	.

.	.	.	.
89	-5.98443	-5.98444	-1.19689
90	-5.98443	-5.98443	-1.19689

It is noticed that neither Picard iteration nor the Jungck-Picard iteration scheme is converges towards the solution, in the case of Example 3.1.

4. CONCLUSIONS

The weak stability results obtained in Section 3 using Jungck Ishikawa iterative procedure enable us to solve numerous nonlinear problems arising in various disciplines where the usual iterative schemes (such as Picard iteration, Jungck-Picard iteration etc.) do not converge. Our results improve partially some of the results of Harder and Hicks [5-6] and Rhoades [13-14] in view of the remark of [19] stating that every stable iteration is w^2-weakly stable. Obviously, many results of Timish [19] are derived as special cases.

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