

Generalized Algebraic Structure for Mathematical Morphology

Ramkumar P.B

Rajagir School of Engineering & Technology
 Rajagiri Valley
 Kochi

Pramod K.V

Cochin University of Science & Technology
 Department of Computer Applications
 Kochi

ABSTRACT

Mathematical morphology is a theory of image transformations and image functional which is based on set-theoretical, geometrical, and topological concepts. The methodology is particularly useful for the analysis of the geometrical structure in an image. The main goal of this paper is to introduce generalized algebraic structure for Mathematical Morphology. The following topics are discussed: introduction to mathematical morphology; generalization to algebraic structure; convex geometrical aspects of morphology. Some results presented in this paper is an extension to newly defined algebraic structure for Mathematical Morphology. . We hope that this generalization will be helpful for introducing new ideas in Morphological related works.

General Terms

Mathematical Morphology, Convex Analysis, Image Processing

Keywords

Adjunction, Morphogenetic field, Operator space ,Convex Morphological space, Image sequences ,Anchor.

1. INTRODUCTION

Mathematical morphology is a well-founded non-linear theory of image processing. Its geometry- oriented nature provides a framework for analyzing object shape characteristics such as size and connectivity, which are not easily accessed by linear methods. The initial form of mathematical morphology is applied to binary images. Mathematical morphology is theoretically founded on set theory. It contributes a wide range of operators to image processing, based on a few simple mathematical concepts. The operators are particularly useful for the analysis of binary images ,boundary detection, image enhancement, shape extraction, skeleton transforms and image segmentation. An image can be represented by a set of pixels. A morphological operation uses two sets of pixels, i.e., two images: the original data image to be analyzed and a structuring element (also called kernel) which is a set of pixels constituting a specific shape. A structuring element is characterized by a well- defined shape size, and origin. Its shape can be considered as a parameter to a morphological operation. Mathematical morphology uses tools of algebra and operates with point sets, their connectivity and shape. Therefore, a general theory for Mathematical Morphology is a need of the day. This paper is an attempt in that direction. Without stating any fundamental results, new definitions and generalized results are given below

2. DEFINITIONS

2.1 Morphogenetic field

Let $X \neq \emptyset$ and $W \subseteq P(X)$ such that i) $\phi, X \in W$, ii) If $B \in W$ then its complement $\bar{B} \in W$ iii) If $B_i \in W$ is a sequence of signals defined in X , then $\bigcup_{n=1}^{\infty} B_i \in W$.

$$\text{Let } A = \left\{ \begin{array}{l} \phi : W \rightarrow U / \phi(\cup A_i) = \vee \phi(A_i) \\ \& \phi(\cap A_i) = \wedge \phi(A_i) \end{array} \right\}.$$

Then W_U is called Morphogenetic field [9] where the family W_u is the set of all image signals defined on the continuous or discrete images Plane X and taking values in a set U . The pair (W_u, A) is called an operator space where A is the collection of operators defined on X .

2.2 Morphological space

The triplet (X, W_u, A) consisting of a set X , a morphogenetic field W_u and an operator A (or collection of operators) defined on X is called a Morphological space [9].

2.2.1 Discrete Morphological Spaces

If $X = Z^2$ then it is called Discrete Morphological space

2.2.2 Equal almost everywhere

Let (X, W_u, A) be a morphological space. Let

$x_c(\alpha), y_c(\alpha)$ be two functions (or signals or samples) in X . The two functions $x_c(\alpha), y_c(\alpha)$ are equal A - almost everywhere if $\exists W_1 \subseteq W_u$ such that $\Rightarrow x_c(\alpha) = y_c(\alpha) \quad \forall W_1 \subseteq W$ and $x_c(\alpha), y_c(\alpha) \in W_1$.

2.2.3 Clodum is an Operator Space

Let (X, W_u, A) be a morphological space. Let (X, W_u, A) be a morphological space and (W_U, A) be an operator space in (X, W_u, A) .

Let $X = E$ (E can be $R^m, Z^m, m = 1, 2, \dots$) and $U = V$ be a complete lattice. A consists of $\vee, \wedge, *, *'$. Then $(W_U, A) = (W_U, \vee, \wedge, *, *')$ is called a Clodum [5] (Complete lattice ordered double monoid).

2.3 Non negative Space

Let (W_R, A) be an operator space in a morphological space (X, W_R, A) . $L^+ = L^+(X, W_R)$ denote the collection of all non negative morphologically operator able functions (or images) on $X \rightarrow \bar{R}$.

2.4 Simple Function

A real valued function is simple if it has only a finite number of values.

A simple operatorable function ψ can be represented in the form $\psi = \bigvee_{\forall i} a_i \chi_{E_i}$ where $a_i \in R$ and ψ_{E_i} is the characteristic function of a set E_i or a structuring element in X in (X, W_R, A)

2.5 Proposition

Let (X, W_u, A) be a morphological space and let A_\vee be the upper slope transform. Then $A_\vee(\bigvee_{\forall \alpha} g_\alpha) = \bigvee_{\forall \alpha} (A_\vee g_\alpha)$, $g_\alpha \in L^+$.

2.6 Proposition

If ψ is a simple function in $L^+(X, W_R)$ then the transform of ψ with respect to A_\vee is $A_\vee(\psi) = \bigvee_{\forall i} a_i A_\vee(E_i)$.

2.7 Proposition

If λ is defined for E in W_u by $\lambda(E) = A_\vee(\psi \chi_E)$ then λ is a slope transform on W_u .

3. TRANSLATION INVARIANCE

3.1 A_j Translation invariance

Let (X, W_u, A) be a morphological space and (W_U, A) be an operator space in (X, W_u, A) . An operator $A_i \in A$ is called A_j translation invariant (or T_{A_j} operator) if it commutes with any A_j translation. i.e $A_i A_j = A_j A_i \forall j$.

3.2 General Non Linear Super Position Principle

Let (X, W_u, A) be a morphological space and (W_U, A) be an operator space in (X, W_u, A) .

An operator $B \in A$ is A_j translation invariant iff

$$B(\bigvee_{i \in J} d_i * x_i) = \bigvee_{i \in J} d_i * B(x_i), d_i \in U, x_i \in W_U.$$

3.2.1 Proposition

Let (X, W_u, A) be a morphological space and (W_U, A) be an operator space in (X, W_u, A) .

An operator A_i on (W_U, A) is dilation invariant to A_j translation iff it can be expressed as $A_i(F)(x) = \bigvee_{y \in X} F(y) * B_y(x)$ where B_y is defined as $B_y(x) = A_i(\gamma_y)(x)$.

3.2.2 A_j Translation invariant system ($A_j TI$)

Let (X, W_u, A) be a morphological space and (W_U, A) be an operator space in (X, W_u, A) .

If an operator obeys the general linear super position principle and is A_j translation invariant then the operator

defines a **A_j translation invariant system ($A_j TI$)**. If

A_j is horizontal translation and dilation then it is called a Dilation translation invariant system (DTI)[1]. Similarly Erosion translation invariant system can also be defined.

3.2.3 Proposition

Let (X, W_u, A) be a morphological space and (W_U, A) be an operator space in (X, W_u, A) .

A Signal operator Δ on (W_U, A) is a translation invariant dilation [1] if and only if it can be represented as the supremal * convolution of the input signal with the system's response $H = \Delta(q)$.

Similarly, we can define it for erosion also.

3.3 Slope Transforms

3.3.1 Proposition

Let (X, W_u, A) be a morphological space and (W_U, A) be an operator space in (X, W_u, A) .

Let $A = A_\vee$, the upper slope transform. If $x(t) \in L^+$, then the upper slope transform[1] of $x(t)$ with respect to A_\vee is $A_\vee(x(t)) = \text{Sup}_{y(t) \leq x(t)} A_\vee(y(t))$ where the supremum is

extended over all simple functions in L^+ , satisfying $0 \leq y(t) \leq x(t), \forall t \in X$.

3.3.2 Proposition

Let (X, W_u, A) be a morphological space and (W_U, A) be an operator space in (X, W_u, A) .

If $f \in L^+$ and $E \subseteq W$ then $f_{\chi_E} \in L^+$ and the slope transform of f over E with respect to A_\vee is $A_{\vee_E} f = A_\vee f_{\chi_E}$.

With reference to the next result [8], we give a lemma for slope transforms in a Morphological space.

4. IMAGE SEQUENCES

Let E, T be non empty sets. Denote functions from E into T by $\text{Fun}(E, T)$. If T is a complete lattice, then $\text{Fun}(E, T)$ is a complete lattice too. Given an index set P , denote set of all image sequences [8] indexed by P , by $\text{Fun}(E, T)^P$. Denote the elements in $\text{Fun}(E, T)^P$ by \mathbf{F} and the value of \mathbf{F} at index $p \in P$ by F_p . Then $\mathbf{F} \leq \mathbf{G} \iff F_p \leq G_p, p \in P \iff F_p(x) \leq G_p(x), p \in P, x \in E$.

4.1 Lemma for Slope Transforms

Lemma Let (X, W_u, A) be a morphological space and (W_U, A) be an operator space in (X, W_u, A) .

Let $A = A_\vee$ the upper slope transform. If $f, g \in L^+$ and $f \leq g$, then

i) $A_\vee(f) \leq A_\vee(g)$ ii) If $f \in L^+$ and $E, F \in W_U$ and if $E \subseteq F$ then $A_{\vee_E}(f) \leq A_{\vee_F}(g)$.

4.2 Fenchel conjugate

The conjugate x^* of x is defined by $x^*(\alpha) = \bigvee_{v \in R^d} \langle \alpha, v \rangle - x(v), \wedge (x(\alpha)) = x^*$. The operator \wedge is known as Fenchel conjugate[1].

4.3 Proposition

Let (X, W_u, A) be a morphological space and (W_U, A) be an operator space in (X, W_u, A) .

For any $x(\alpha) \in W_U$, the conjugate of $x(\alpha)$ is convex and lower semi continuous.

5. CONVEX MORPHOLOGY

5.1 Convex Morphological Space

Let (X, W_u, A) be a morphological space and (W_U, A) be an operator space in (X, W_u, A) .

If X is a class of concave functions then (X, W_u, A) is called concave morphological space. If X is a class of convex functions then (X, W_u, A) is called **Convex morphological space**.

In next proposition we give a particular property of Convex morphological space.

5.1.1 Proposition

If $X = k$ is a convex function and if each k_n denote the segmented smaller linearly – sloped convex structuring components of k with the sizes $s_1, (s_2 - s_1) \dots (s_n - s_{n-1})$, respectively then X can be written as $X = k_1 \oplus k_2 \oplus \dots \oplus k_n$.

5.2 Anchor

The following definition of the operator ‘Anchor’ can be generalized to Morphological Structures.

Given a signal f and an operator ψ , a point x in the domain of f is an anchor [8] for f with respect to ψ if $\psi(f)(x) = f(x)$. $A(f, \psi)$ is the set of anchors for f with respect to ψ where $A(f, \psi)$ is a subset of the domain of f .

Anchors are useful in the computation of erosions and openings.

Let ψ be an opening by B , then the geometrical interpretation of an opening is that the lower bound of a function f is an anchor. If f has a finite domain D then there is at least one such global minimum, and therefore at least one anchor point. Proof of the following two results are obtained from the above argument.

5.2.1 Theorem

If γB is a morphological opening on a finite domain and B is finite, then the set of opening anchors is always non-empty: $A(f, \gamma B) \neq \emptyset$

5.2.2 Theorem

In a concave Morphological space, there exists at least one anchor point.

Proof: Let (X, W_u, A) be a convex morphological space and (W_U, A) be an operator space in (X, W_u, A) . Given a signal f belongs to X and an operator ψ belongs to A , a point x in the domain of f is an anchor for f with respect to ψ if $\psi(f)(x) = f(x)$.

Let ψ be an opening by B . Then the lower bound of a function f is an anchor. Since f is convex, there is at least one such global minimum, and therefore at least one anchor point.

6. GENERALIZED ADJUNCTIONS

6.1 Definition

Let (X, W_u, A) & (Y, W_u, \bar{A}) be morphological spaces.

The pair (A, \bar{A}) is called an adjunction iff

$A(X) \leq Y \iff X \leq \bar{A}(Y)$ where \bar{A} is an inverse operator of A .

6.1.1 Proposition

Let $(X, W_u, \delta) \& (Y, W_u, \varepsilon)$ be morphological spaces with operators dilation and erosion on A. Then $\delta(X) \leq Y \Leftrightarrow X \leq \varepsilon(Y)$.

6.1.2 Proposition (For Lattice)

Let $(X, W_u, A) \& (Y, W_u, \bar{A})$ be morphological spaces.

The pair (A, \bar{A}) is called an adjunction iff $\forall u, v \in X, \exists$ an adjunction $(l_{u,v}, m_{v,u})$ on U such that

$$\bar{A}(x(u)) = \bigvee_{v \in X} m_{v,u}(x(v)) \text{ and}$$

$$A(y(v)) = \bigwedge_{u \in X} l_{u,v}(y(u)), \forall u, v \in X, x, y \in W_U.$$

6.2 Morphological Closure

The operator $\phi = \varepsilon \circ \delta$ defines a closure called morphological closure [7] and $\phi^* = \delta \circ \varepsilon$ defines a kernel, called morphological kernel.

6.3 Separation

Let $(X, W_u, A), (X, W_u, \bar{A})$ be morphological spaces.

Let (A, \bar{A}) be adjunctions. (X, A) is separated in a primary sense if A verifies the following two properties.

Let $x \in X, A(x) \subseteq \bigcup_{\forall x_i \in W_U} A(x_i) \Rightarrow \exists j \in I$ such that

$$A(x) \subseteq A(x_j), A(x) = A(y) \Rightarrow x = y \forall x, y \in X$$

and $\phi = A \circ \bar{A}$ defines a morphological closure.

7. CONCLUSION

Mathematical morphology uses concepts from set theory, geometry and topology to analyze geometrical structures in an image. A series of operators is applied to an image in order to make certain features clearer, distinguishing meaningful information from irrelevant distortions. Therefore this attempt to generalize the theory of Mathematical Morphology is more relevant. We hope that this generalization may help researchers to find out new operators.

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