Some Fixed Point Theorems on Meir Keeler Type under Strict Contractions

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ABSTRACT
This paper explores the common fixed point theorems involving two pairs of weakly compatible mappings. Also the property (E.A) is proved under a new contractive condition which is independent of the previous known contractive definitions.

Keywords:

Common fixed point, Compatible and Compatible mappings of type (A), (B), (C) and (P), Weakly compatible maps, property E.A.

1. INTRODUCTION

The study of common fixed point of mappings satisfying contractive type of conditions has emerged as a very active field of research activity during recent years. The most general form of the common fixed point theorem pertaining to four mappings A, B, S and T of a metric space (X, d) uses either a Banach type contractive condition if the form

$$d(Ax, By) \leq k\, M(x, y); \quad 0 \leq k < 1 \qquad (1.1)$$

where M(x, y) = max {d(Sx, Ty), d(Ax, Sx), d(By, Ty),

$$[d(Sx, By) + d(Ax, Ty)]\,/2\} \quad (1.2)$$

or a Meir Keeler type (ε, δ) contractive condition, that is given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon < M(x, y) < \varepsilon + \delta \implies d(Ax, Ay) \leq \varepsilon \qquad (1.3)$$

or a ψ contractive condition of the form

$$(Ax, Ay) \leq \psi\,(M(x, y)) \qquad (1.4)$$

involving a contractive gauge function $\psi : [0, \infty) \rightarrow [0, \infty)$ such that $\psi(t) < t$ for each t > 0.

Clearly Banach type contractive condition is a special case of both Meir Keeler (ε, δ) contractive and ψ contractive conditions. As ψ contractive type of condition does not guarantee the existence of fixed point unless some additional condition is assumed. So various authors have introduced different conditions on ψ

(i) $\psi(t)$ is non decreasing and $\dfrac{t}{(t - f(t))} < 0$ for every t > 0. (Carbone *et al.* [6])

(ii) $\psi(t)$ is non decreasing, for each t > 0, $\lim\limits_{n\to\infty} \psi^n(t) = 0$ which implies that $\psi(t) < t$. (Jachymski [7])

(iii) $\psi(t)$ is upper semi continuous (Boyd and Wong [5], Jachymski [7], Pant[16])

(iv) $\psi(t)$ is non decreasing, right continuous and $\psi(t) < t$ for every t > 0. (Park and Rhoades[19])

Besides this, we have deduction from Pant *et al.* [18] than a (ε, δ) contractive type of condition (1.2) neither assures the existence of fixed point nor implies an analogous ψ contractive condition (1.3). As a result of it two types of contractive conditions (1.2) and (1.3) are independent of each other thus to ensure the existence of common fixed point under contractive type of condition (1.2), δ takes on following conditions, used by various authors:

(v) δ is non decreasing. (Pant [16])

(vi) δ is lower semi continuous (Jungck [10], Jungck *et al.* [17])

It follows therefore that the known common fixed point theorems can be extended and generalized if instead of assuming one of the contractive condition (1.2) or (1.3) with additional conditions on δ and ψ, we assume contractive condition (1.2) together with the following condition of the form:

$$d(Ax, By) < \max\{\alpha_1[d(Sx, Ty) + d(Ax, Sx) + d(By, Ty)],$$
$$\alpha_2[d(Sx, By) + d(Ax, Ty)]/2\}$$

for $0 \leq \alpha_1 < 1, \ 1 \leq \alpha_2 < 2$ $\qquad (1.5)$

In this paper, we have proved the common fixed point theorems for four mappings adopting this approach.

2. PRELIMINARIES

Let S and T be the self mappings of a metric space (X, d), S and T are commuting if STx =TSx for all $x \in$ X. According to definition of Sessa [23], S and T are defined to be weakly commuting if for all $x \in$ X.

$$d(STx, TSx) \leq d(Tx, Sx) \quad (2.1)$$

Any two self maps of a metric space (X, d) are defined as compatible by Jungck [10], if

$$\lim_{n \to \infty} d(ST\, x_n, TS\, x_n) = 0 \quad (2.2)$$

Whenever { x_n } is a sequence in X, such that $\lim_{n \to \infty} S\, x_n =$ $\lim_{n \to \infty} T\, x_n$ = t for some t $\in$ X. Jungck [10] defined compatibility as generalization of commutativity and weak commutativity.

The commuting maps imply weakly commuting which on other hand implies compatibility. These are verified by examples given in literature. See [10] and [24]

Jungck *et al.* [12] defined pair {S, T} of self maps of metric space (X, d) as compatible of type A if

$$\lim_{n \to \infty} d(ST\, x_n, T^2\, x_n) = 0 \text{ and } \lim_{n \to \infty} d(ST\, x_n, S^2\, x_n) = 0$$
$$(2.3)$$

whenever { x_n } is a sequence in X, such that $\lim_{n \to \infty} S\, x_n =$ $\lim_{n \to \infty} T\, x_n$ = t for some t $\in$ X.

Clearly weakly commuting implies compatible of type (A). but by [12] its converse is not true.

Pathak and Khan [20] defined the S and T to be compatible mapping of type (B) as a generalization of compatible mappings of type (A) if

$$\lim_{n \to \infty} d(TSx_n, SSx_n) \leq \frac{1}{2}[\lim_{n \to \infty} d(TSx_n, Tt) + \lim_{n \to \infty} d(Tt, TTx_n)]$$
and
$$\lim_{n \to \infty} d(STx_n, TTx_n) \leq \frac{1}{2}[\lim_{n \to \infty} d(STx_n, St) + \lim_{n \to \infty} d(St, SSx_n)]$$
$$(2.4)$$

whenever { x_n } is a sequence in X, such that $\lim_{n \to \infty} S\, x_n =$ $\lim_{n \to \infty} T\, x_n$ = t for some t $\in$ X.

The compatible mappings of type (A) are compatible of type (B), but its converse is not true. See [10]. However compatibility of type (A) and compatibility of type (B) are equivalent if S and T are given continuous.

Pathak *et.al.* [21] defined selfmaps S and T of metric space (X, d) as compatible mapping of type (P) if

$$\lim_{n \to \infty} d(SS\, x_n, TT\, x_n) = 0 \quad (2.5)$$

whenever { x_n } is a sequence in X, such that $\lim_{n \to \infty} S\, x_n =$ $\lim_{n \to \infty} T\, x_n$ = t for some t $\in$ X.

Again compatibility of type (A) and compatibility of type (P) are equivalent if S and T are given continuous, See [21].

Pathak *et.al.* [22] defined self maps S and T of metric space (X, d) to be compatible mapping of type (C), as generalization of compatible mappings of type (A) if

$$\lim_{n \to \infty} d(TSx_n, SSx_n) \leq \frac{1}{3}[\lim_{n \to \infty} d(TSx_n, Tt) + \lim_{n \to \infty} d(Tt, SSx_n)$$
$$+ \lim_{n \to \infty} d(Tt, TTx_n)] \text{ and}$$

$$\lim_{n \to \infty} d(STx_n, TTx_n) \leq \frac{1}{3}[\lim_{n \to \infty} d(STx_n, St) + \lim_{n \to \infty} d(St, TTx_n)$$
$$+ \lim_{n \to \infty} d(St, SSx_n)] \quad (2.6)$$

whenever { x_n } is a sequence in X, such that $\lim_{n \to \infty} S\, x_n =$ $\lim_{n \to \infty} T\, x_n$ = t for some t $\in$ X.

So, we see that compatibility of type (A) and compatibility of type (C) are equivalent if S and T are continuous. See [14].

Definition 1. [14] S and T are said to be weakly compatible if they commute at their coincidence points i.e. if St = Tt for some t $\in$ X then STt = TSt.

Lemma 1. [10, 12, 20, 21 and 22] If S and T are compatible, or compatible of type (A) or compatible of type (P), or compatible of type (B), or compatible of type (C), then they are weakly compatible.

The following example shows that converse is not true;

Example 1. Let X = [0, 20] with usual metric. Defined S, T: X $\to$ X by

$$Sx = \begin{cases} 0 & if \ x = 0 \\ x + 15 & if \ 0 < x \leq 5 \\ x - 5 & if \ 5 < x \leq 20 \end{cases} \quad \text{and}$$

$$Tx = \begin{cases} 0 \ if \ x \in \{0\} \cup \{5, 20\} \\ 3 \ if \ 0 < x \leq 5 \end{cases}$$

Let x_n be the sequence defined by $x_n = 5 + \dfrac{1}{n}$, n$\in$ N, then

$$S\, x_n = x_n \text{ - } 5 \to 0, \quad T\, x_n = 0 \to 0.$$

As $n \to \infty$, S(0) = 0 = T(0) and ST(0) = 0 = TS(0)

Clearly S ant T are weakly compatible maps, since they commute at their coincidence point t=0. On other hand, we have

$$ST\, x_n = S(0) = 0, \quad SS\, x_n = S(x_n \text{-}5) = x_n + 10$$

$$TS\, x_n = T(x_n \text{-}5) = 3 \qquad TT\, x_n = T(0) = 0.$$

Consequently $\lim_{n \to \infty} \left| ST\, x_n \text{ - } TS\, x_n \right| = 3 \neq 0$ that proves S and T are not compatible. Moreover we have $\lim_{n \to \infty} \left| TS\, x_n \text{ - } SS\, x_n \right| =$

$$\lim_{n \to \infty} \left| 3 \text{ - } x_n \text{ - } 10 \right| = 12 \neq 0$$

Thus S and T are not compatible of type (A). Furthermore,

$$12 = \lim_{n \to \infty} \left| TS\, x_n - SS\, x_n \right| \not\leq \frac{1}{2} \left[\lim_{n \to \infty} \left| TS\, x_n - Tt \right| \right.$$

$$\left. + \lim_{n \to \infty} \left| Tt - TT\, x_n \right| \right] = \frac{3}{2}$$

So, $12 \not\leq \dfrac{3}{2}$ which shows that S and T are not compatible of type (B).

Again $12 = \lim_{n \to \infty} \left| TS\, x_n - SS\, x_n \right| \not\leq \dfrac{1}{3} \left[\lim_{n \to \infty} \left| TS\, x_n - Tt \right| \right.$

$$\left. \lim_{n \to \infty} \left| Tt - SS\, x_n \right| + \lim_{n \to \infty} \left| Tt - TT\, x_n \right| \right] = \frac{18}{6} = 3$$

Thus, $12 \not\leq 3$, which shows that S and T are not compatible of type (C). Also we have

$$\lim_{n \to \infty} \left| SS\, x_n - TT\, x_n \right| = 15 \neq 0$$

therefore S and T are not compatible of type (P).

Property (E.A) [1]. Let A and S be two self maps of a metric space (X, d), then they are said to satisfy *property (E.A.)*, if there exists a sequence $\{ x_n \}$ in X, such that $\lim_{n \to \infty} A\, x_n = \lim_{n \to \infty} S\, x_n$ = t, for some t $\in$ X.

If two maps are noncompatible, then they satisfy the property (E.A). However, its converse is not true. To support our assertion, we quote some examples from [1].

Example 2 ([1]). Let X = [0, ∞). Define T, S: X $\longrightarrow$ X by

$$Tx = \frac{x}{4}, \quad Sx = \frac{3x}{4}, \ \forall\ x \in X.$$

Consider the sequence $\{ x_n \} = \dfrac{1}{n}$. Clearly $\lim_{n \to \infty} T\, x_n = \lim_{n \to \infty}$

S x_n = 0. Then T and S satisfy property (E.A).

Example 3 ([1]). Let X = [2, ∞). Define T, S: X $\longrightarrow$ X by

Tx = x +1, $\quad$ Sx = 2x + 1, $\forall$ x $\in$ X.

Suppose that property (E.A.) holds; then there exists in X a sequence $\{ x_n \}$ satisfying

$$\lim_{n \to \infty} A\, x_n = \lim_{n \to \infty} S\, x_n = t, \text{ for some t } \in X.$$

Therefore, $\lim_{n \to \infty} x_n = t - 1$ and $\lim_{n \to \infty} x_n = \dfrac{(t-1)}{2}$. Then

t = 1, which is a contradiction since 1 $\notin$ X. Hence, T and S do not satisfy property E.A.

Lemma 2. (Jachymski [7]) Let *A, B, S* and *T* be self mappings of a metric space (X, *d)*, such that $A(X) \subset T(X)$, $B(X) \subset S(X)$. Assume further that for given $\varepsilon > 0$ there exists $\delta > 0$ such that for all x , y in X.

$$\varepsilon < M(x, y) < \varepsilon + \delta \implies d(Ax, By) \leq \varepsilon$$

and $\qquad$ d(Ax, By) < M(x, y) whenever M(x, y) > 0

where

M(x, y) = max {d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Sx, By) +
$\qquad\qquad\qquad\qquad\qquad\qquad$ d(Ax, Ty)] /2}

Then for each x_0 in X, the sequence $\{y_n\}$ in X is defined by the rule $y_{2n} = Ax_{2n} = Tx_{2n+1},\ y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$
is a Cauchy sequence.

3. MAIN RESULTS

Theorem 1. Let A, B, S and T be the self maps defined on a metric space (X, d) satisfying the following conditions:

(1) Ax $\subset$ Tx, Bx $\subset$ Sx $\qquad\qquad\qquad\qquad$ (3.1)

(2) Given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all x, y in X,

$$\varepsilon \leq M(x, y) < \varepsilon + \delta \implies d(Ax, By) < \varepsilon \text{ and } (3.2)$$

(3) $d\ Ax, By\ < \max\{\alpha_1[\ Sx, Ty\ + d\ Ax, Sx\ + d\ By, Ty\ ]$

$$\alpha_2[d\ Sx, By\ + d\ Ax, Ty\ ]\}$$

for $0 \leq \alpha_1 < 1, \quad 1 \leq \alpha_2 < 2$ $\qquad\qquad\qquad$ (3.3)

If one of A(x), B(x), S(x) or T(x) is a complete subspace of X, then

$\qquad$ (I) $\quad$ A and S have a coincidence

$\qquad$ (II) $\quad$ B and T have coincidence

$\qquad$ Moreover, if A and S, as well as, B and T are weakly compatible, then A, B, S and T have a unique common fixed point.

Proof: Let $x_0 \in$ X be any arbitrary point of X. From (3.1), we can construct a sequence $\{ y_n \}$ in X as follows

$y_{2n} = Ax_{2n} = T\, x_{2n+1}$ and $y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$,
$\qquad\qquad\qquad n = 0,1,2,\ldots\ldots\ldots..$ $\qquad\qquad$ (3.4)

Suppose d($y_{2n,}\ y_{2n+1}$) = 0 for some n. Then $y_{2n} = y_{2n+1}$ implies that $Ax_{2n} = T\, x_{2n+1} = Bx_{2n+1} = S\, x_{2n+2}$ and T and B have a coincidence point. Further if d($y_{2n+1,}\ y_{2n+2}$) = 0 for some n then $Ax_{2n+2} = T\, x_{2n+3} = Bx_{2n+1} = S\, x_{2n+2}$ and A and S have a coincidence point.

$\qquad$ Moreover if we assume that d($y_n,\ y_{n+1}$) $\neq$ 0 for each n, then we have M(x, y) > 0, otherwise d($y_n,\ y_{n+1}$) = 0 $\quad$ (3.5) which is a contradiction.

$\qquad$ We now show that $\{y_n\}$ is a Cauchy sequence. For this it is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}$ is a Cauchy sequence. Then there exists $\in$ >0, such that for each even integer 2k, there exists even integers 2m (k) > 2n(k) > 2k such that d($y_{2n(k)}, y_{2m(k)}$) $\geq\ \in$ $\qquad\qquad$ (3.6)

$\qquad$ For every even integer 2k, let 2m(k) be the least positive integer exceeding 2n(k) satisfying (3.6) such that

$\qquad$ d($y_{2n(k)}, y_{2m(k)-2}$) < $\in$ $\qquad\qquad\qquad\qquad$ (3.7)

Further $\delta > 0 \implies$ d($y_{2n(k)}, y_{2m(k)}$) $\leq$ d($y_{2n(k)}, y_{2m(k)-2}$) + $d_{2m(k)-2}$ +
$\qquad\qquad\qquad\qquad d_{2m(k)-1}$ $\qquad\qquad\qquad\qquad$ (3.8)

Then by (3.5), (3.6) and (3.7) it follows that

$$\lim_{n \to \infty} \text{d}(y_{2n(k)}, y_{2m(k)}) = \delta \qquad\qquad (3.9)$$

Also, using triangular inequality

$$\left| \text{d}(y_{2n(k)}, y_{2m(k)-1}) - \text{d}(y_{2n(k)}, y_{2m(k)}) \right| \leq d_{2m(k)-1}, \qquad (3.10)$$

$\left| d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)}) \right| \le d_{2m(k)-1} + d_{2n(k)},$

and so, on using (3.9) we get

$$\lim_{n \to \infty} d(y_{2n(k)}, y_{2m(k)-1}) \to \delta \qquad (3.11)$$

$$\lim_{n \to \infty} d(y_{2n(k)+1}, y_{2m(k)-1}) \to \delta \qquad (3.12)$$

as k $\to \infty$. Thus

$d(y_{2m(k)}, y_{2m(k)}) \le d_{2n(k)} + d(y_{2n(k)+1}, y_{2m(k)}) \le d_{2n(k)}$

$\qquad + d(S x_{2n(k)}, T x_{2m(k)-1}) \qquad (3.13)$

Thus, on letting k $\to \infty$, on both sides of the last inequality, we have

$$\delta \le d(S x_{2n(k)}, T x_{2m(k)-1}) \le \psi (M(x_{2n(k)}, x_{2m(k)-1})) \qquad (3.14)$$

where

$M(x_{2n(k)}, x_{2m(k)-1}) = \max\{ \alpha_1 [d(y_{2n(k)}, y_{2m(k)-1}) + d_{2n(k)} +$

$\qquad d_{2m(k)-1}] + \alpha_2 [d(y_{2m(k)+1}, y_{2m(k)-1}) + d(y_{2n(k)}, y_{2m(k)})]/2\}$

$\qquad (3.15)$

Combining (3.5), (3.6), (3.7), (3.9), (3.11) and (3.12) yields the following contradiction from (3.14)

$\delta \le \psi(\delta) < \delta \qquad (3.16)$

Thus {y_{2n}} is a Cauchy sequence and so {y_n} Cauchy sequence.

Suppose that TX is a complete subsequence of X, then the sub sequence $y_n = T x_{2n+1}$ is a Cauchy sequence in TX and so have a limit u. Let $v \in T^{-1}u$, then Tv =u. since {y_{2n}} is convergent, so {y_n} converges to u and hence {y_{2n+1}} also converges to u. Now on setting x = x_{2n} and y = v, we get

$d(Ax_{2n}, Bv) < \max\{ \alpha_1 [d(Sx_{2n}, Tv) + d(Ax_{2n}, Sx_{2n}) + d(Bv, Tv)] +$

$\qquad \alpha_2 [d(Sx_{2n}, Bv) + d(Ax_{2n}, Tv)]/2\} \qquad (3.17)$

at n$\to \infty$, we have d(u, Bv) < d(u, Bv) which implies that Bv = u. Also since BX $\subset$ SX, so u = Bv implies that $w \in S^{-1}u$, then

$d(Aw, Bx_{2n+1}) < \max\{ \alpha_1 [d(Sw, Tx_{2n+1}) + d(Aw, Sw) + d(Bx_{2n+1},$

$Tx_{2n+1})] + \alpha_2 [d(Sw, Bx_{2n+1}) + d(Aw, Tx_{2n+1})]/2\} \qquad (3.18)$

at n$\to \infty$, we have d(Aw, u) < d(u, Aw) which implies that u =Aw . This makes clear that $u = Tv = Bv = Aw = Sw$ (3.19)

Now since u = Tv = Bv, so by weak compatibility of (B, T) it follows that TBv = BTv and so we get Bu = BTv = TBv = Tu. Also since u = Aw = Sw so by weak compatibility of A and S it follows that SAw = ASw and so we get Au = ASw = SAw = Su. Thus from (3.3), we have

$d(Aw, Bu) < \max\{ \alpha_1 [d(Sw, Tu) + d(Aw, Sw) + d(Bu, Tu)] +$

$\qquad \alpha_2 [d(Sw, Bu) + d(Aw, Tu)]/2\} \qquad (3.20)$

which gives d(u, Bu) < d(u, Bu) which is a contradiction for $0 \le \alpha_1 < 1, 1 \le \alpha_2 < 2$, this implies that u = Bu. Similarly (3.3) can be used to show Au = u. Therefore we have u = Bu = Tu = Au = Su. Hence u is the common fixed point of all A, B, S and T.

Now, if we assume SX is complete the argument analogue to the previous completeness argument, which proves the theorem. If AX is complete then $u \in$ BX $\subset$ SX. So the theorem is established. The uniqueness of the common fixed point follows easily from condition (3.3). This completes the proof of the theorem.

Lemma 3.1. [10, 12, 20, 21, 22] Let S and T are compatible, compatible either of type (A), (type (P), type (B) or type (C)) self mappings of a metric space (X, d). If Su = Tu for some $u \in$ X, it follows that STu = TSu.
From the previous theorem, our result is immediate.

Corollary 3.1. Let A, B, S and T be any self map of a metric space (X, d) satisfying the conditions (1), (2) and (3) of Theorem 1. Now if pairs {A, S} are compatible i.e. compatible of type (A),(type (P), type (B) or type (C)), then the four maps having a unique common fixed point $u \in$ X.

Theorem 2: Let A, B, S and T be the selfmaps of a metric space (X, d) satisfying the following conditions;
(1)AX $\subset$ TX, BX $\subset$ SX $\qquad (3.21)$
(2) d(Ax,By) < max{ α_1 [d(Sx,Ty) + d(Ax,Sx) + d(By,Ty)]+

$\qquad \alpha_2$ [d(Sx,By)+d(Ax,Ty)]/2}

for $0 \le \alpha_1 < 1, 1 \le \alpha_2 < 2.$ $\qquad (3.22)$

Let one of the mappings (A,S) or (B,S) be weakly compatible, satisfying property (E.A.). If the range of one of the mappings be a complete subspace of X, then A, B, S and T have a unique common fixed point.

Proof: Let B and T satisfy property E.A. Then there exists a sequence { x_n } in X such that B$x_n \to$ t and T$x_n \to$ t for some t in X. Since BX $\subset$ SX, for each x_n, there exists y_n in X, such that Bx_n = S y_n . Thus B$x_n \to$ t, T$x_n \to$ t and S $y_n \to$ t. We claim that A $y_n \to$ t. If not, there exists a sub sequence {A y_m } of {A y_n), a positive integer M and a number r> 0 such that for each m$\ge$M, we have

d(A y_m , t) $\ge$ r, d(A y_m ,B x_m) $\ge$ r, $\qquad (3.23)$

d(A y_m ,B x_m) < max{ α_1 [d(S y_m ,T x_m) + d(A y_m ,S y_m) +

d(B x_m ,T x_m)] + α_2 [d(A y_m ,T x_m) + d(B x_m ,S y_m)] /2}

$\qquad (3.24)$

$\qquad$ < d (A y_m ,S y_m) $\qquad (3.25)$

a contradiction. Hence A $y_n \to$ t. Now suppose that SX is a complete subspace of X. Then, since S $y_n \to$ t, there exists a point u in X. Such that t = Su. If A$u \ne$ Su, the inequality.

d(Au, B x_n) < max{ α_1 [d(Su,Tx_n)+d(Au,Su)+d(B x_n ,Tx_n)]+

$\qquad \alpha_2$ [d(Au,Tx_n) +d(B x_n ,Su)] /2} $\qquad (3.26)$

On taking n$\to \infty$, yields d(Au, Su) < d(Au, Su), a

contradiction. Hence Au= Su. Since A and S are weakly compatible so it implies that ASu = SAu and so Au = ASu = SAu = Su.

On the other hand, since AX $\subset$ TX, there exists a point w $\in$ X, such that Au = Tw. We assert that Tw = Bw. If B$w \neq$ Tw, then by (2) we get

d(Au,Bw) < max{ α_1 [d(Su,Tw)+d(Au,Su)+d(Bw,Tw)]+

$$\alpha_2 \text{ [d(A}u\text{,T}w\text{)+ d(B}w\text{,S}u\text{)]/2}}$$

< d(Bw, Au)   (3.27)

a contradiction. Hence Au = Bw = Tw = Su, which shows that the pairs (A, S) and (B, T) have a point of coincidence u and w, respectively. The proof is similar if we consider the case when pair (A, S) enjoys prorperty (E.A).

Now by weak compatibility property of B and T, it implies that BTw = TBw and. BBw = BTw = TBw = TTw. Suppose that A$u \neq$ AAu. So we have from (3.22),

d(Au, AAu) = d(AAu, Bw) < max { α_1 [d(SAu, Tw) +

d(AAu, SAu) + d(Bw, Tw)] + α_2 [d(AAu, Tw)+d(Bw, SAu)]/2}

< d(AAu, Au).   (3.28)

which is a contradiction. Thus Au = AAu = SAu and Au is a common fixed point of A and S. Similarly Au = Bw is a common fixed point of B and T. The proof is similar when TX is assumed to be complete subspace of X. The cases in which AX or BX is a complete subspace of X are similar to the cases in which TX or SX respectively be complete since AX $\subset$ TX and BX $\subset$ SX. The uniqueness of the common fixed point follows easily from (3.22). Hence the theorem.

We now illustrate the above theorem by the way of the following example.

Example 4. Let X = [0, 1] and d be the usual metric on X. Define A, B, S and T: X $\rightarrow$ X by

Ax = Bx = 0, if x = 0 or x = 1, Ax = Bx = $\dfrac{1}{n+2}$, if

$\dfrac{1}{n+2} \leq x < \dfrac{1}{n}$, n $\in$ N; and SX = Tx = x $\forall$ x $\in$ X.

Here (A,S) satisfies property (E.A) . Taking { $\dfrac{1}{n+2}$ } $\subseteq$

[0, 1], we get $\displaystyle\lim_{n\to\infty}$A($\dfrac{1}{n+2}$) = $\displaystyle\lim_{n\to\infty}$($\dfrac{1}{n+2}$) = 0 =

$\displaystyle\lim_{n\to\infty}$S($\dfrac{1}{n+2}$) = $\displaystyle\lim_{n\to\infty}$($\dfrac{1}{n+2}$).

Similarly, (B, T) satisfies property (E.A.).

Then A, B, S and T satisfy all the conditions of above Theorem **2.** It can be verified in this example that B and T are weakly compatible type, since Ax = Sx, occurs if x = 0 or x =

$\dfrac{1}{n+2}$, n $\in$ N .Here we have, AS(0) = A0 = 0 =S0 = SA(0) and

AS($\dfrac{1}{n+2}$) = A($\dfrac{1}{n+2}$) = $\dfrac{1}{n+2}$ = S($\dfrac{1}{n+2}$), n $\in$ N.

Similarly (B, T) is weakly compatible.

Theorem 3: Let A, B, S and T be weakly compatible selfmaps of a metric space (X, d) satisfying the following conditions;

(1)   AX $\subset$ TX, BX $\subset$ SX   (3.29)

(2)   d(Ax,By) < max{ α_1 [d(Sx,Ty) + d(Ax,Sx) + d(By,Ty)]

$$+ \alpha_2 \text{ [d(Sx,By)+d(Ax,Ty)] /2}}$$

for 0 $\leq \alpha_1$ <1, 1 $\leq \alpha_2$ < 2.   (3.30)

Let one of the mappings (A, S) or (B, S) be noncompatible, satisfying property (E.A). If the range of one of the mappings be a complete subspace of X, then A, B, S and T have a unique common fixed point and the fixed point is a point of discontinuity.

Proof: Suppose that B and T be noncompatible maps, so there exists a sequence { x_n } in X such that

$$\lim_{n\to\infty} \text{B}\,x_n = \text{t and } \lim_{n\to\infty} \text{T}\,x_n = \text{t}$$ (3.31)

for some t $\in$ X, but $\displaystyle\lim_{n\to\infty}$d(BT$x_n$, TB$x_n$) is either nonzero or nonexistent. Since BX $\subset$ SX, for each x_n , there exists a y_n $\in$ X, such that Bx_n = Sy_n. Thus Bx_n $\rightarrow$ t, Tx_n. $\rightarrow$ t and Sy_n $\rightarrow$ t. We claim that Ay_n $\rightarrow$ t. If not, there exists a subsequence {Ay_m} of {Ay_n), a positive integer M and a number r > 0 such that for each m$\geq$M, we have

d(Ay_m, t) $\geq$ r, d(Ay_m,Bx_m) $\geq$ r,   (3.32)

d(Ay_m, Bx_m) < max { α_1 [d(Sy_m,Tx_m)+ d(Ay_m,Sy_m)

+ d(Bx_m,Tx_m)]+ α_2 [d(Ay_m,Tx_m) +d(Bx_m,Sy_m)] /2}

(3.33)

< d(Ay_m, Sy_m)   (3.34)

a contradiction. Hence Ay_n $\rightarrow$ t. Suppose that SX is a complete subspace of X. Then since Sy_n $\rightarrow$ t, there exists a point u in X. Such that t = Su. If A$u \neq$ Su, the inequality.

d(Au,Bx_n) < max{ α_1 [d(Su,Tx_n) + d(Au,Su) + d(Bx_n,Tx_n)]

$$+ \alpha_2 \text{ [d(A}u\text{,T}x_n\text{)+d(B}x_n\text{,S}u\text{)]/2}}$$ (3.35)

on taking n$\rightarrow \infty$, yields d(Au, Su) < [d(Au, Su)] a contradiction. Hence Au = Su. Since A and S are weakly compatible so it implies that ASu= SAu and then AAu = ASu = SAu = SSu.

On the other hand, since AX $\subset$ TX, there exists a point w $\in$ X, such that Au = Tw. We assert that Tw = Bw. If B$w \neq$ Tw, then by (3.30) we get

d(Au,Bw) < max{ α_1 [d(Su,Tw) + d(Au,Su) + d(Bw,Tw)]

$$+ \alpha_2 \text{ [d(A}u\text{,T}w\text{)+d(B}w\text{,S}u\text{)]/2}}$$

< d(Bw, Au)   (3.36)

a contradiction. Hence Au = Su = Bw = Tw, which sh
pairs (A, S) and (B, T) have a point of coincidenc
respectively. The proof is similar if we consider the case when
pair (A, S) enjoys prorperty (E.A).

Now by weak compatibility property of B and T, it implies
that BTw = TBw and BBw = BTw = TBw = TTw. Now, suppose
that Au ≠ AAu. So we have from (3.30)
d(Au, AAu) = d(AAu, Bw)

$$< \max \{ \ \alpha_1 \ [d(SAu, Tw) + d(AAu, SAu) + d(Bw, Tw)]$$

$$+ \ \alpha_2 \ [d(AAu, Tw)+d(Bw, SAu)]/2\} \qquad (3.37)$$

$$< \ d(AAu, Au). \qquad (3.38)$$

which is a contradiction. Thus Au = AAu = SAu, then Au is a
common fixed point of A and S. Similarly Au = Bw is a common
fixed point of B and T. The proof is similar when TX is assumed
to be complete subspace of X. The cases in which AX or BX is a
complete subspace of X are similar to the cases in which TX or
SX respectively be complete since AX $\subset$ TX and BX $\subset$ SX.
Uniqueness of the common fixed point follows easily.

We have to show now that the mappings are discontinuous at
the common fixed point. Let us suppose that B is continuous at
common fixed point t, such that t = Au = Bw. So on taking the
sequence { x_n } as taken in (3.30), we have $\lim\limits_{n\to\infty} BB x_n = Bt = t$

and $\lim\limits_{n\to\infty} BT x_n = Bt = t$. By weak compatibility property of B

and T, it follows that BTx_n = TBx_n. On letting $n \to \infty$, this

gives us $\lim\limits_{n\to\infty} BT x_n = \lim\limits_{n\to\infty} TB x_n = Bt = t$. Thus, we get d(BT$x_n$,

TBx_n) = d(Bt, Bt) = 0, which contradicts the fact that

$\lim\limits_{n\to\infty}$ d(BTx_n, TBx_n) is either nonzero or nonexistent for the

sequence { x_n } of (3.30). Hence B is discontinuous at the fixed

point. Now, suppose that T is continuous, then for the sequence

{ x_n } of (3.30), we get $\lim\limits_{n\to\infty} TB x_n = Tt = t$ and $\lim\limits_{n\to\infty} TT x_n = Tt = t$.

t. Hence, the inequality, in view of these limits, gives us;

d(At, BT x_n) < max { α_1 [d(St, TT x_n) + d(At, St) +

$$d(BT x_n, TT x_n)] + \alpha_2 \ [d(At, TT x_n) + d(BT x_n, St)] /2\}$$
$$(3.39)$$

which gives contradiction, unless $\lim\limits_{n\to\infty} BT x_n = TT x_n = Tt = t$.

But $\lim\limits_{n\to\infty} BT x_n = Tt = t$. and $\lim\limits_{n\to\infty} TB x_n = Tt = t$ which contradicts

the fact that d(BTx_n, TBx_n) is either nonzero

or nonexistent. Hence, both B and T are discontinuous at their
common fixed point. Likewise, it can be shown that A and S are
also discontinuous at the common fixed point. Thus all the
selfmaps A, B, S and T are discontinuous at the common fixed
point. Hence, the theorem is established.
Our example is motivated by example of Pant [17].

Example 5. Let X= [1, 20] and d be the usual metric on X.
Define A, B, S, T:X $\to$ X by

$$Ax = \begin{cases} 1 \ if, x = 1 \\ 3 \ if, x > 1 \end{cases} \qquad Bx = \begin{cases} 1 \ if, x = 1 \ or \geq 5 \\ 6 \ if, 1 < x < 5 \end{cases}$$

$$Sx = \begin{cases} 1 \ if, x = 1 \\ 6 \ if, x > 1 \end{cases} \qquad Tx = \begin{cases} 7+x \ if, 1 < x < 5 \\ x+1 \ if, x \geq 5 \end{cases}$$

T1 = 1

Then A, B, S and T satisfy all the conditions of above Theorem **2**
and have a unique common fixed point x = 1. It can be verified in
this example that B and T satisfy the property (E.A). Also, it can
be seen that all the mappings A, B, S and T are discontinuous at
the common fixed point.

4. CONCLUSION

In this paper, we have proved the common fixed point
theorems for four mappings by removing the assumption of
continuity, relaxing the compatibility to weak compatibility
property and also replacing the completeness of space. So our
result generalizes and improves various other similar results of
fixed point. Besides this, we have used a new property,
introduced by Aamri and Moutawakil [1], satisfying selfmaps of
a metric space, which generalize the notion of noncompatible
maps.

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