Channel Assignment Algorithms for Graphs in the Plane with Graceful Constraints

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ABSTRACT
An assignment of integer numbers to the vertices of a given graph under certain conditions is referred to as a graph labeling. The assignment of labels from the set $\{0,1,2,...,2q-1\}$ to the vertices of G (with $n=|V(G)|$ vertices and $q=|E(G)|$ edges) such that, when each edge has assigned a label defined by the absolute difference of its end-points, the resulting edge labels are $1,3\ldots,2q-1$ is referred to as an odd graceful labeling of the graph. In 2000, Kathiresan [13] used the notation $P_{n;m}$ to denote the graph (spider graph) obtained by identifying the end points of m paths each one has length n, we use the notation $C_{n;m}$ to denote the graph (closed spider) obtained by identifying the other end points of the graph $P_{n;m}$. In this article, we present three algorithms to show how to odd gracefully label the vertices and the edges of the following graphs; $P_{2r+1;\,m}$, $1\le r\le 5$, $m\ge 2$, the closed spider $C_{n;m}$, and the graphs obtained by joining one or two paths P_m to each vertex of the path P_n.

Keywords
Vertex labeling, edge labeling, odd graceful, algorithms.

1. INTRODUCTION
Let G is a finite simple graph, whose vertex set denoted $V(G)$, and the edge set denoted $E(G)$. The order of G is the cardinality $n=|V(G)|$ and the size of G is the cardinality $q=|E(G)|$. We write $uv\in E(G)$ if there is an edge connecting the vertices u and v in G. A path graph $P_n\,(v_1,v_2,v_3,...,v_n)$ simply denotes the graph that consists of a single line. In other words, it is a sequence of n vertices such that from each of its vertices there is an edge to the next vertex in the sequence. The first vertex is called the start vertex and the last vertex is called the end vertex. Both of them are called end or terminal vertices of the path. The other vertices in the path are internal vertices.
A cycle is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. The cycle graph has m vertex is denoted C_m. The graph $G=(V,E)$ consists of a set of vertices and a set of edges. If a nonnegative integer $f(u)$ is assigned to each vertex u, then the vertices are said to be "labeled." $G=(V,E)$ is itself a labeled graph if each edge e is given the value $f^*(e)=|f(u)-f(v)|$ where u and v are the endpoints of e. Clearly, in the absence of additional constraints, every graph can be labeled in infinitely many ways. Thus utilization of labeled graph models requires imposition of additional constraints which characterize the problem being investigated.

Gnanajothi [11] introduced a certain labeled graph model known as the odd-graceful labeling. An odd graceful labeling of the graph G with $n=|V(G)|$ vertices and $q=|E(G)|$ edges is a one-to-one function f of the vertex set $V(G)$ into the set $\{0,1,2,...,2q-1\}$ with this property: if we define, for any edge uv the function $f^*(uv)=|f(u)-f(v)|$ the resulting edge label are $1,3\ldots,2q-1$ .A graph is called odd graceful if it has an odd graceful labeling. When studying odd graceful labeling, we need only consider simple graphs without loops or parallel edges. A loop in a labeled graph would assume an edge label of 0. Parallel edges between a particular pair of vertices in labeled graph would always assume the same edge label, the edge labels be distinct in a graceful labeling. The odd graceful labeling problem is to find out whether a given graph is odd graceful, and if it is odd graceful, how to label the vertices. The common approach in proving the odd gracefulness of special classes of graphs is to either provide formulas for odd gracefully labeling the given graph, or construct desired labeling from combining the famous classes of odd graceful graphs.

It is known that not every graph is odd graceful, for instance J. Gallian in his dynamic survey [17], he have collected everything on graph labeling, he observed that over thousand papers have been studied. In 1991, Gnanajothi [11] proved the following graphs are odd graceful: $P_n;C_m$ if and only if m is even. In 2000, Kathiresan [13] used the notation $P_{n;m}$ to denote the graph (spider graph) obtained by identifying the end points of m paths each one has length n. The spider graph is a tree with one vertex of degree at least three called the center of spider and all others with degree at most two (see Fig. 1A). Thus, a star with m legs is a spider of m legs. The length of a leg spider equals the number of vertices from the center to a vertex of degree one. The spider graph $P_{n;m}$ consisted of one central vertex v_0 connected with m number of paths P_{n-1} of the same length. In 1997 Eldergill [12] generalized Gnanajothi [11] result on stars by showing that the graph obtained by joining one end point from each of any odd number of paths of equal length is odd graceful graph. In 2002 Sekar [15] proved that the graphs; $P_{n;m}$ when $n\ge 2$ and m is odd, $P_{2;m}$ and $P_{4;m}$ such that $m\ge 2$, $P_{n;m}$ when n and m are even and n, $m\ge 4$, $P_{4r+1;4r+2}$, $P_{4r-1;4r}$, $r\ge 1$ and all n-polygonal snakes with n even are odd graceful. Sekar's study have produced estimates of $P_{n;m}$ but there are still some graphs in the form of $P_{n;m}$ did not prove to be odd graceful up to date, this graphs are $P_{3;m}$ for all $m \geq 2$, $P_{4r+1;4r}, P_{4r-1;4r-2}, P_{4r+1;4r-2}$ and $P_{4r-1;4r+2}, \forall r \geq 1$ In other words, Sekar proved the odd gracefulness of the graph $P_{n;m}$ if and only if, both of n and m are odd numbers, or both of n and m are even number or n is an even number and m is an odd number, but the unproved case appears when n is an odd number and m is an even number, which we tried to prove it in this paper. In 2009 Moussa and Bader [16] have presented the algorithms that showed the graphs obtained by joining n pendant edges to each vertex of C_m are odd graceful if and only if m is even. In 2010 Moussa [18] have presented the algorithms that showed the graph $C_m \cup P_n$ is odd graceful if m is even.

In this paper, we defined a new representation called the cycle representation denoted C -representation; it is similar to the π -representation made by Kotzig [8]. Secondly, we introduced algorithms that show how to label the vertices and the edges odd gracefully in some graphs viz.: the graphs obtained by joining one or two paths P_m $(v_1, v_2, v_3, ..., v_m)$ to each vertex of the path P_n, the graphs $P_{3;m}, P_{5;m}, P_{7;m}, P_{9;m}$ and $P_{11;m}$ for all $m \geq 2$ and the closed spider graph $C_{n;m}$. Finally, the correctness of these algorithms was proved. The remainder of this paper is organized as follows. In section 2, we mentioned a nice set of applications related with the odd graceful labeling. In section 3, we gave some assumptions and definitions related with the odd graceful labeling and graphs. Section 4 shows the assignment algorithms and contains the proofs of their correctness. Finally, section 5 is the conclusion of this research.

2. THE APPLICATION RANGES

The odd graceful labeling is one of the most widely used labeling methods of graphs. While the labeling of graphs is perceived to be a primarily theoretical subject in the field of graph theory and discrete mathematics, it serves as models in a wide range of applications.

- **The coding theory:** The design of certain important classes of good non periodic codes for pulse radar and missile guidance is equivalent to labeling the complete graph in such a way that all the edge labels are distinct. The node labels then determine the time positions at which pulses are transmitted.

- **The x-ray crystallography**: X-ray diffraction is one of the most powerful techniques for characterizing the structural properties of crystalline solids, in which a beam of X-rays strikes a crystal and diffracts into many specific directions. In some cases more than one structure has the same diffraction information. This problem is mathematically equivalent to determining all labeling of the appropriate graphs which produce a prespecified set of edge labels.

- **The communications network addressing**: It might be useful to assign each user terminal a “node label,” subject to the constrait that all connecting “edges” (communication links) receive distinct labels. In this way, the numbers of any two communicating terminals automatically specify (by simple subtraction) the link label of the connecting path; and conversely, the path label uniquely specifies the pair of user terminals which it interconnects.

The fields of graph labeling have a wide range of applications as we mentioned above; these applications are especially pervasive in channel assignment wireless networks and mobile computing [14]. For further information about other applications of labeled graphs, such applications include radar, circuit design, astronomy, data base management, on automatic drilling machine, determining configurations of simple resistor networks, and models for constraint programming over finite domains (see e.g., [1, 4, 5, 9 and 10]). Labeled graph also apply to other areas of mathematics (see e.g., [2, 6]).

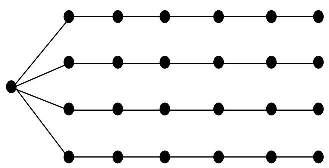
3. ASSUMPTIONS AND DEFINITIONS

Definition 1

A closed spider graph is a graph $C_{n;m}$ with two vertices v_0 and w of degree at least three called the end points of the closed spider and all others vertices with degree two (see Fig.1 B). The closed spider graph obtained from spider graph $P_{n;m}$ has the center vertex v_0 by connecting all the vertices of degree one to an addition vertex w. The vertex set $V(C_{n;m})$ of the closed spider graph is the union of three sets, the set of the most left vertex v_0, the set of the most right vertex w, and the set of all intermediate vertices in the paths $P^1_{n-2}, P^2_{n-2}, ..., P^m_{n-2}$. The total number of edges in $C_{n;m}$ is $q = (n-1)m$.

Definition 2

β : $1, 2, 3, 4...\ \rightarrow\ 1, 3, 5, 7....$. The expression $\beta(i) = 2i - 1$ describes the function β of a variable i, which is an integer, then this function relates each input, i, with a single output, $2i - 1$.


(A)

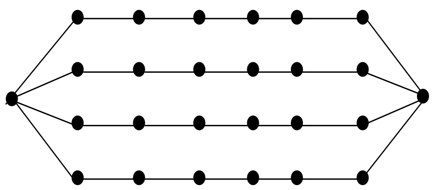

(B)

Figure 1: The graph $P_{n;m}$ and the graph $C_{n;m}$

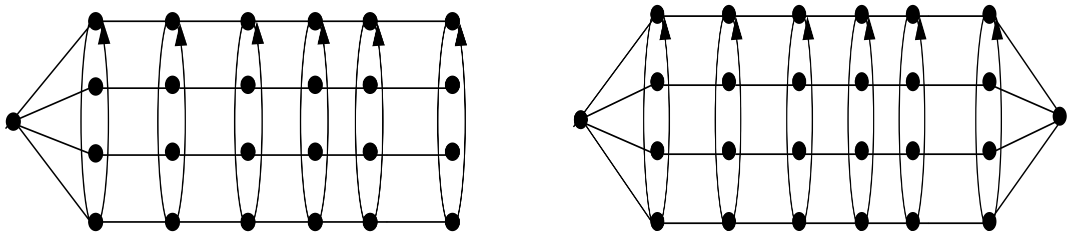


Figure 2: The C-representation of the graphs $P_{n;m}$ and $C_{n;m}$

Definition 3

We define C-representation of $P_{n;m}$ and $C_{n;m}$ as the following; the central vertex v_0 is surrounded by $n-1$ nonrealistic cycles C_m, these mean that, for any two vertices $v_i^j \in V(P_i)$ and $v_s^j \in V(P_s)$ in different legs P_i and P_s respectively are nonadjacent vertices. In other words, each cycle C_j, $j=1,2...,n-1$ has a vertex set $V_j=\{v_j^1, v_j^2,...,v_j^m\}$ and has an empty edge set. Let the vertex set of the path P_i is demonstrated by listing them in the order $V^i=\{v_0\}\cup\{v_1^i,v_2^i,...,v_{n-1}^i\}$, and v_i^j be the intersection point between the path P_i and the cycle C_j where $i=1,2...m$ and $j=1,2...,n-1$ respectively (see Figure 2). C-representation of $C_{n;m}$ is defined similarly; there are $n-1$ nonrealistic cycles C_m draw between the two vertices v_0 and w. Let the vertex set of the path P_i is demonstrated by listing them in the order $V_j^i=\{v_0\}\cup\{v_1^i,v_2^i,...,v_{n-1}^i\}\cup\{w\}$, and v_i^j be the intersection point between the path P_i and the cycle C_j where $i=1,2...m$ and $j=1,2...,n-1$, (see Figure 2). For all $j=1,2...,n-1$, we call the cycle C_j is odd cycle if j is odd and C_j is even cycle if j is even.

Definition 5

We use the notation $P_n \circ P_m$ to denote the graph obtained by joining one end point of n internally disjoint path P_m each of length m to each vertex of the path P_n, see figure7A. The graph $P_n \circ P_m$ has a vertex set with cardinality $n(m+1)$ and an edge set with cardinality $q=2nm+(n-1)$. The graph obtained by adding two leaves to each vertex of the path P_n then adding a path P_m to each leaf, denoted $(P_n \circ \overline{K}_2)\circ P_m$, see figure 8A. The graph $(P_n \circ \overline{K}_2)\circ P_m$ has a vertex set with cardinality $2mn+3n$ and an edge set with cardinality $q=2nm+3n-1$.

4. VARIATION OF ODD GRACEFUL LABELING OF $P_{n;m}$ GRAPHS

4.1 The graph $P_{4r-1;\mathrm{m}}, r=1,2,3$ (See Figure3)

The graph $P_{4r-1;\mathrm{m}}$ has a vertex set $V(P_{4r-1;\mathrm{m}})$ with cardinality equals $m(4r-2)+1$ and an edge set $E(P_{4r-1;m})$ with cardinality equals $q=m(4r-2)$. For any j, let the cycle C_j is demonstrated by listing the vertices in the order $v_j^1, v_j^2,...,v_j^m, v_j^1$. The algorithm alternatively labels the vertices on the odd cycles C_j for all $j=1,3,5\ldots,4r-3$, and the vertices on the even cycles C_j for all $j=2,4,\ldots,4r-2$, as illustrated in Algorithm2. The edge's labeling induced by the absolute value of the difference of the vertex's labeling, as illustrated in Algorithm3. At the beginning, the algorithm starts with the initial procedure;

1. Put the graph $P_{4r-1;m}$ in the C -representation which consists of the central vertex and the cycles $C_1, C_2 \ldots C_{4r-2}$.
2. Number the central vertex v_0 with the value $f(v_0)=0$

Algorithm 1: Procedure Initialization

Beginning by the odd cycles C_j, performing an action on the cycles (referred to as "numbering" the vertices) , if $r=1$ or 2; there are at most six cycles in the graph $P_{4r-1;\mathrm{m}}$, number the vertices on the cycle C_1 according the function $f(v_1^i)=(8r-4)m-\beta$ i , traversing to the cycle C_3 and number its vertices according the function $f(v_3^i)=(8r-6)m-\beta$ i , and traversing to the cycle C_5 and number its vertices according the function $f(v_5^i)=(8r-10)m-\beta$ i . The vertices on the even cycles would be numbered consecutively: $2,6,10...,4m(2r-1)-4)+2$. If $r=3$ in this case, the graph $P_{4r-1;\mathrm{m}}$ has ten cycles $C_1, C_2,..., C_{10}$. The vertices on the even cycles $C_2, C_4, C_6, C_8, C_{10}$ would be numbered according to step 2.2, and the vertices on the odd cycles C_1, C_3, C_5, C_7, C_9 would be numbered using step 1.2.

1. j is odd

1.1. If $r=1$ or 2; number the vertices on the induced cycles C_1, C_3, C_5 respectively as follows:

$f(v_1^i) = (8r-4)m - \beta\ i$,

$f(v_3^i) = (8r-6)m - \beta\ i$,

$f(v_5^i) = (8r-10)m - \beta\ i$.

1.2. If $r=3$; number the vertices on the induced cycles C_1, C_3, C_5, C_7, C_9 respectively as follows:

$f(v_1^i) = 20m - \beta\ i$, $f(v_3^i) = 2m - \beta\ i$,

$f(v_5^i) = 2m - \beta\ i$, $f(v_7^i) = 12m - \beta\ i$,

$f(v_9^i) = 18m - \beta\ i$.

2. j is even

2.1 .If $r=1$ or 2;

$$f(v_j^i) = 2jm - 2\beta\ i \qquad i = 1, 2, \ldots, m$$

.

2.2 If $r=3$; number the vertices on the induced cycle $C_2, C_4, C_6, C_8, C_{10}$ respectively as follows:

$f(v_2^i) = 20m - 2\beta(i)$, $f(v_4^i) = 16m - 2\beta(i)$,

$f(v_6^i) = 8m - 2\beta(i)$, $f(v_8^i) = 4m - 2\beta(i)$,

and $f(v_{10}^i) = 12m - 2\beta(i)$.

Algorithm 2

Algorithm for Edge Labelings

The label's distribution of the edges between the central vertex v_0 and the cycle C_1, and the label's distribution of the edges between the cycles $C_1, C_2 \ldots C_{4r-2}$ is induced by the absolute value of the difference of their end-vertices's labeling, the following algorithm gives the odd edge labelings.

1. The edge's labels between the central vertex v_0 and the cycle C_1, induced by $f^*(v_0 v_1^i) = (8r-4)m - \beta\ i$, for all $i = 1, 2, \ldots, m$, $r = 1, 2, 3$.
2. The label's distribution of the edge between the cycles $C_1, C_2 \ldots C_{4r-2}$ is defined consecutively by:

   2.1. If $r=1$; $\beta\ i$ for all $i = 1, 2, \ldots, m$.

   2.2. If $r=2$; $8m+\beta\ i$, $6m+\beta\ i$, $2m+\beta\ i$, $2m-\beta\ i$, and $6m-\beta\ i$ for all $i = 1, 2, \ldots, m$.

   2.3. If $r=3$; $\beta\ i$, $18m-\beta\ i$, $14m-\beta\ i$, $12m-\beta\ i$, $4m-\beta\ i$, $4m+\beta\ i$, $8m+\beta\ i$, $14m+\beta\ i$, and $6m+\beta\ i$ for all $i = 1, 2, \ldots, m$.
3. The resulting labeling is odd graceful.

Algorithm 3

Theorem 1 The graphs $P_{4r-1;m}$ are odd graceful for all $m \geq 2$ and $r = 1, 2, 3$.

Proof

The graphs $P_{4r-1;m}$ are trees, so they can be drawn as bipartite graphs, in two partite sets denoted as ODD-SET (O) and EVEN-SET (E), with edges only between the vertices in O and E. Any two vertices have only one edge between any two vertices in the partite sets O and E or only one path. All vertices in the two sets O and E can be thought of as the vertices on the odd cycles and the vertices on the even cycles plus the central vertex v_0 respectively. All the vertices of the set O are labeled consecutively with odd values from the set $\{1, 3, 5 \ldots 2m(4r-2)-1\}$, and all the vertices in the set E are labeled consecutively with even values from the set $\{2\ 4, \ldots, 2m(4r-2)-2\}$. Thus, the reader can find out easily that the vertex labels are assigned uniquely as shown in the above algorithms, and the absolute values of the difference of the vertex's labelings are odd value. This guarantees that the edge labelings are odd values. The result is the edge labels being the set $1, 3 \ldots 2m(4r-2)-1$ **.** To prove that all edges' labels are different we have to consider the following cases:

(i) If $r=1$ (case 2.1 Algorithm 3), *the* edges' labels are odd, distinct and they are $1, 3, 5, \ldots, 2q-1$.

(ii) If $r=2$ (case 2.2 Algorithm 3), the edges have the labels; $8m+\beta\ i$, $6m+\beta\ i$, $2m+\beta\ i$, $2m-\beta\ i$, and $6m-\beta\ i$ are clearly distinct odd values. Because the maximum value of the function $\beta\ i$ is $2m-1$ which implies that the difference between the minimum value of $6m-\beta\ i$, and the maximum value of $2m+\beta\ i$ equals $4m-2\beta\ i = 2$, thus the induced edge labels will *never be* the same

(iii) If $r=3$ (case 2.3 Algorithm 3), the proof in this case go by the same way as in case (ii)

Therefore, the above algorithm proves that the graph $P_{4r-1;m}$ is odd graceful for all $m \geq 2$ and $r = 1, 2, 3$. .■

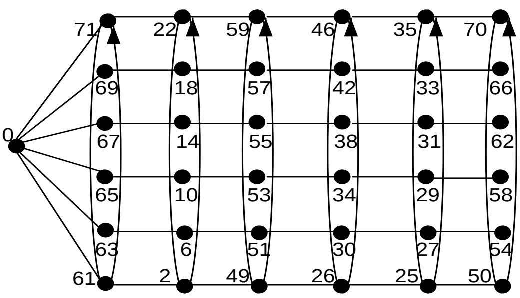


Figure 3: shows the graph $P_{7;6}$ is odd graceful

4.2 The graph $P_{4r+1;m}$, $r = 1, 2$ (See Figure4)

The labeling behaves of the graph $P_{4r+1;m}$, $r = 1, 2$ totally similarly and is less structured than the algorithms in section 3.1. Put the graph in the C-representation and number the central vertex with the value $f(v_0)=0$. The following algorithm gives the labeling of $P_{4r+1;m}$, $r=1,2$.

1. Put the graph $P_{4r+1;m}$ in the C -representation which consists of the central vertex and the cycles $C_1, C_2 ... C_{4r}$.
2. Number the central vertex v_0 with the value $f(v_0)=0$
3. If j is odd value less than or equal to $4r-1$ and $r=1$ or 2
   3.1 Number the vertices on the odd cycles C_1, C_3, C_5, C_7 respectively as follows: $f(v_1^i)=8rm-\beta(i)$, $f(v_3^i)=(8r-2)m-\beta(i)$, $f(v_5^i)=(8r-8)m-\beta(i)$, $f(v_7^i)=(8r-14)m-\beta(i)$.
4. If j is even value less than or equal to $4r$ and $r=1$ or 2 $f(v_j^i)=2jm-2\beta(i)$ $i=1,2,\ldots,m$
5. The edge labels induced by taking the absolute value of the difference of incident vertex labels.
6. The resulting labeling is odd graceful.

Algorithm 4

Theorem 2

The graphs $P_{4r+1;m}$ is odd graceful for all $m \geq 2$ and $r=1,2$.

Proof

The proof has the same steps as in Theorem 1.

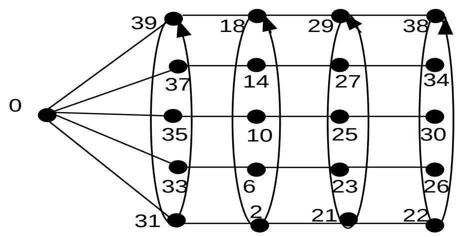


Figure 4: shows the graph $P_{5;5}$ is odd graceful

4.3 Variation of odd graceful labeling $C_{n;m}$

The odd graceful labeling behaves partially differently is less structured than the algorithms in sections 3.1 and 3.2 hence the algorithm has to be flexible to handle that. The following algorithm gives the odd graceful labeling for a closed spider.

1. Put the graph $C_{n;m}$ in the C-representation which consists of the two end points v_0, w and the cycles $C_1, \ldots, C_{n-2}$. The cycle c_j is demonstrated by listing the vertices in the order $v_j^1, v_j^2, \ldots, v_j^m, v_j^1$ where $j=1,2\ldots,n-2$.
2. Number the most right vertex v_0 with the value $f(v_0)=0$.
3. If the total number of edges $q=(n-1)m$ is odd, number the vertex w with the value $f(w)=(n-2)m+1$,
4. If the total number of edges $q=(n-1)m$ is even, and both of n and m are even, number the vertex w with the value $f(w)=nm-1$; otherwise number the vertex w with the value $f(w)=(n+1)m$.
5. Consecutively number the vertices on the odd cycles c_j as $(2n-j-1)m-\beta(i)$,and consecutively number the vertices on the even cycles c_j as $(j+2)m-2\beta(i)$
6. The edges' labels between the vertex v_0 and the cycle C_1 induced by $f^*(v_0v_1^i)=2(n-1)m-\beta(i)$, for all $i=1,2,\ldots,m$.
7. For $j=1,2,\ldots,n-3$ $f^*(v_j^i v_{j+1}^i)=2m(n-(j+2))+\beta(i)$ $i=1,2,\ldots,m$
8. If the total number of edges $q=(n-1)m$ is odd $f^*(wv_{n-2}^i)=2(\beta(i)-m)-1$, for all $i=1,2,\ldots,m$.
9. If q is even and both of n and m are even numbers $f^*(wv_{n-2}^i)=2\beta(i)-1$, for all $i=1,2,\ldots,m$.
10. If q is even and n is odd, and m is odd or even number. $f^*(wv_{n-2}^i)=\beta(i)$,for all $i=1,2,\ldots,m$.
11. The resulting labeling is odd graceful

Algorithm 5

Theorem 3

The closed spider graph is odd graceful.

Proof

The graph $C_{n;m}$ can be drawn as bipartite graphs, in two partite sets denoted as ODD-SET (O) and EVEN-SET (E), with edges only between the vertices in O and E. Any two vertices have only one edge between any two vertices in the partite sets O and E or only $m-1$ paths. If n is even, all the vertices in the two sets O and E can be thought of as the vertices on the odd cycles with the vertex w and the vertices on the even cycles with the central vertex v_0 respectively, If n is odd, all the vertices in the set O can be thought of as the vertices on the odd cycles with the vertex W and the central vertex v_0, and all the vertices in the set E can be thought of as the vertices on the even cycles. If n is odd, the vertices of the set O are labeled consecutively with odd values from the set $\{2m(n-1)-1, 2m(n-1)-3, 2m(n-1)-5, \ldots\}$. The vertices in the set E are labeled consecutively with even values from the set $\{2, 4, \ldots, mn-2\}$. Thus, the absolute values of the difference of the vertices' labelings are odd value. This guarantees that the edge labelings are odd values. The result is the edge labels being the set $\{1,3\ldots,2q-1\}$. The vertices' labels are assigned uniquely as given in algorithm 5. To show that all edge labels are different we have to consider the following cases.

(i) The edges $v_0v_1^i$, $i=1,2,\ldots,m$ (step 6, Algorithm 5) are numbered uniquely from $\{2nm-2m-1, \ldots, 2nm-4m-1\}$.

(ii) The edges $v_j^i v_{j+1}^i, i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n-2$ (step 7, Algorithm 5) are numbered uniquely from the set $\{2nm - 6m + 1, \ldots, 4m - 1\}$.

(iii) The edges $wv_{n-2}^i,\ i = 1,2,\ \ldots, m$ (steps 8, 9, 10 Algorithm 5) are numbered uniquely from $\{2m\text{-}1,\ldots,1\}$.
Therefore the edges' labels are odd, distinct and numbered consecutive form $\{1,\ldots,2q-1\}$. ■

4.4 Odd gracefulness of $P_n \circ P_m$

Algorithm for odd graceful Labeling of the graph $P_n \circ P_m$

We used an injection ψ from $V(P_n \circ P_m)$ to the order set $v_1, v_2, \ldots, v_{n(m+1)}$. The injection ψ can be defined formally as; start from the most bottom-right vertex v_m of a path n^{th} P_m rename it as $\psi(v_m) = v_1$ then move up on the path n^{th} P_m and rename its vertices consecutively as $v_2, v_3, \ldots, v_m$ till the most top-right vertex v_n and rename it as $\psi(v_n) = v_{m+1}$, traversing to the left adjacent vertex v_{n-1} and rename it as $\psi(v_{n-1}) = v_{m+2}$, then move down through the $(n\text{-}1)^{\text{th}}$ path P_m and rename its vertices consecutively as $v_{m+3}, v_{m+4}, \ldots, v_{2(m+1)}$ traversing to the left non-adjacent vertex v_m on the path $(n\text{-}2)^{\text{th}}$ path P_m and repeat the previous processes on the rest paths until reach the most top-left (bottom-left) vertex of path P_1. In the end the vertex set of the graph $P_n \circ P_m$ is demonstrated by listing the vertices in the order $v_1, v_2, \ldots, v_{n(m+1)}$. The odd graceful labeling function $f(v_i)$ defines as follows:

$$f(v_i) = \begin{cases} (i-1) & i\,odd \\ 4nm + 2n - i - 1 & i\,even \end{cases}$$

The edge labels induced by taking the absolute value of the difference of incident vertex labels, the edge labelling function

$$f^* : E(P_n \circ P_m) \rightarrow \{1,3,5,\ldots,2q-1\}$$

defined by

$f^*(v_i v_{i+1}) = 4mn + 2n - 2i - 1$. Therefore the edges' labels are odd, distinct and numbered consecutive form

$\{1,3,5,\ldots,4mn + 2n - 3\} = \{1,3,5,\ldots,2q-1\}$.

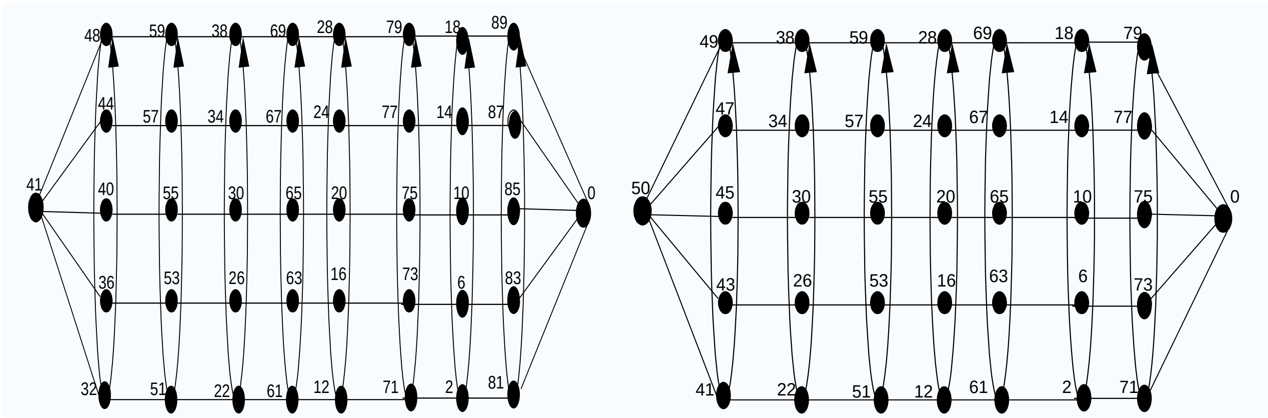


Figure 5: The graph $C_{10;5}$ and the graph $C_{9;5}$

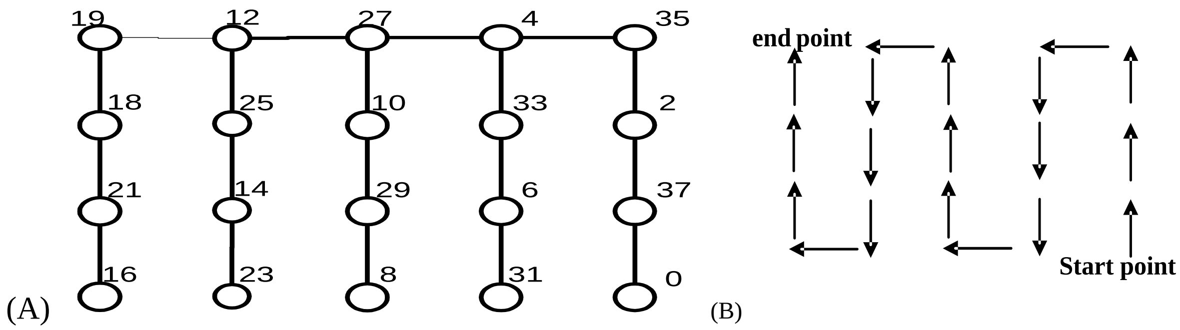
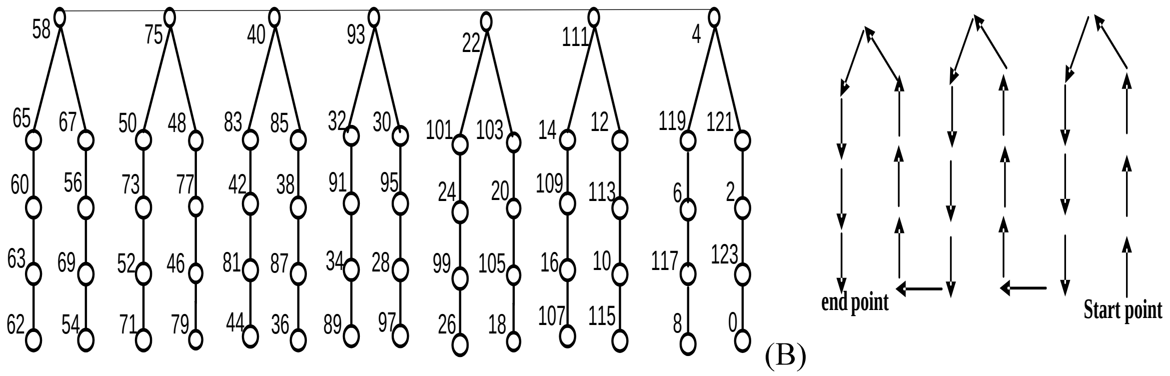


Figure 6: (A) The labeled graph $P_5 \circ P_3$ (B) The path track we follow to label the graph $P_5 \circ P_3$.



(A) (B)

Figure 7: (A) The labeled graph $(P_7 \circ \overline{K}_2) \circ P_3$, (B) The path track we follow to label the graph $(P_7 \circ \overline{K}_2) \circ P_3$

4.5 The graph $(P_n \circ \overline{K}_2) \circ P_m$ is odd graceful Algorithm for odd graceful Labeling of the graph $(P_n \circ \overline{K}_2) \circ P_m$

The injection ψ can be defined formally as; start from the most bottom-right vertex v_{2nm+3n} of a path $2n^{th}$ P_m rename it as $\psi(v_{2nm+3n}) = v_1$ then move up on the path $2n^{th}$ P_m till the right adjacent leaf to the most top- right vertex and rename these vertices consecutively as $v_2, v_3, ..., v_m, v_{m+1}$ till the most top- right vertex and rename it as v_{m+2}, traversing to the left adjacent leaf to the most top- right vertex and rename it as v_{m+3}, then move down through $(2n-1)^{th}$ path and rename its vertices consecutively as $v_{m+4}, v_{m+5}, \dots$, traversing to the left non-adjacent vertex on the $(2n-2)^{th}$ path and repeat the previous processes on the rest paths until reach the bottom-left vertex of the first path P_m. In the end, the vertex set of the graph $(P_n \circ \overline{K}_2) \circ P_m$ is demonstrated by listing the vertices in the order $v_1, v_2, ..., v_{2nm+3n}$. The graph $(P_n \circ \overline{K}_2) \circ P_m$ labeled in a similar way as $P_n \circ P_m$, the vertex labeling function $f(v_i)$ defines as follows:

$$f(v_i) = \begin{cases} (i-1) & i\ odd \\ 4nm + 6n - i - 1 & i\ even \end{cases}$$

The edge labels induced by taking the absolute value of the difference of incident vertex labels. The edge labeling function

$$f^* : E((P_n \circ \overline{K}_2) \circ P_m) \rightarrow \{1,3,5,.., 2q-1\}$$

defined by

$$f^*(v_i v_{i+1}) = 4mn + 6n - 2i - 1, i = 1,2,3,..., 2mn + 3n$$

Therefore the edges' labels are odd, distinct and numbered consecutive form $\{1, \dots, 2q-1\}$.

5. CONCLUSION

The following conclusions can be drawn from the present study: a graph model can be used for the channel assignment problem, the nodes of the graph correspond to cells or their base stations and the edges represent cell adjacency, channel assignment algorithms are often built on a basic assignment of one channel per node, known as a graph labelling. A major contribution of these study was that, three algorithms were developed to generate all possible odd graceful labelings of the vertices and the edges of spider graph, closed spider graph and the graphs obtained by joining one or two paths P_m to each vertex of the path P_n. These algorithms traversed exactly once each vertex and each edge in the graphs mentioned above, since the number of vertices in the given graph G equals $n = |V(G)|$ and the number of edges in G equals $q = |E(G)|$, then at most $O(n+q)$ time is spent in total labeling of the vertices and edges, thus the total running time of the developed algorithms is $O(n+q)$.

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