M/M/1 Retrial Queueing System with Two Types of Vacation Policies under Erlang – K Type Service

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ABSTRACT
Consider a single server retrial queueing system in which customers arrive in a Poisson process with arrival rate **λ** that which follows a Poisson process**.** Let k be the number of phases in the service station. The service time has Erlang k-type distribution with service rate kμ for each phase. Two types of vacation policies are discussed in this research paper that is **Bernoulli type vacation** and **exhaustive type vacation. T**he **vacation rate** follows an exponential distribution with parameter **α**. We assume that the services in all phases are independent and identical and only one customer at a time is in the service mechanism. If the server is **free** at the time of a primary call arrival, the arriving call begins to be served in Phase 1 immediately by the server then progresses through the remaining phases and must complete the last phase and leave the system before the next customer enters the first phase. If the server is **busy** or on **vacation**, then the arriving customer goes to orbit and becomes a source of **repeated calls**. This pool of sources of repeated calls may be viewed as a sort of queue. Every such source produces a Poisson process of repeated calls with intensity **σ**. If an incoming repeated call finds the server free, it is served in the same manner and leaves the system after service, while the source which produced this repeated call disappears. Otherwise, the system state does not change. We assume that the access from orbit to the service facility is governed by the **classical retrial policy.** This model is solved by using **Matrix geometric technique**. Numerical study have been done for Analysis of Mean number of customers in the orbit (MNCO),Truncation level (OCUT), Probabilities of server free, busy and in vacation for various values of **λ , μ ,k , p** , **α** and **σ** in elaborate manner and also various particular cases of this model have been discussed.

Keywords
Single Server, Erlang k-type service, Bernoulli vacation, Matrix geometric method, exhaustive vacation, classical retrial policy.

1. INTRODUCTION
Queuing systems in which arriving customers who find all servers and waiting positions (if any) occupied may retry for service after a period of time are called **Retrial queues [1, 2, 5, 6]** Because of the complexity of the retrial queueing models, analytic results are

generally difficult to obtain. There are a great number of numerical and approximations methods available. In this paper we will place more emphasis on the solutions by **Matrix geometric method [8].** In the literature, the analysis for queueing systems with vacations has been discussed through a considerable amount of work in recent years. Queues with server vacations occur in many engineering systems such as data switching systems, computer communication networks and telecommunication systems. Doshi **[3, 4]** has recorded prior work on vacation models and their applications in his survey paper. Keilson & Servi **[7]** have introduced a queueing system with Bernoulli vacation scheduling service that is clearly applicable to queueing systems involving communication systems. Many examples such as production system, bank services, computer and communication networks, etc., work with different vacation policies. In this work we study retrial queuing system with two types of vacation policies namely Bernoulli and Exhaustive type of vacations under Erlang – k type service by Matrix Geometric Method.

2. MODEL I

2.1 MODEL DESCRIPTION
Consider a single server retrial queueing system with Bernoulli type vacation scheduling introduced by Keilson & Servi [7] in which customers arrive in a Poisson process with arrival rate **λ**. These customers are identified as primary calls**.** Let k be the number of phases in the service station. Assume that the service time has **Erlang-k distribution** with service rate kμ for each phase. The **vacation rate** follows an exponential distribution with parameter **α**.

If the server is **free** at the time of a primary or repeated call arrival, then this arriving call begins to be served immediately and leaves the system after completion of the service. After completion of each service, the **server has an option to go on vacation** with probability **p** or continue to serve with probability **(1-p).** This type of vacation in queueing theory is called **Single vacation with Bernoulli schedule. The single vacation means after completion of vacation period he can once again go for vacation after completing atleast one service.** The server may return from the vacation at any time and is independent of number of customers in the system.

We assume that the services in all phases are independent and identical and only one customer at a time is in the service mechanism. If the server is **free** at the time of a primary call arrival, the arriving call begins to be served in Phase 1 immediately by the server then progresses through the remaining phases and must complete the last phase and leave the system before the next customer enters the first phase. If the server is **busy** or on **vacation**, then the arriving customer goes to orbit and becomes a source of repeated calls. This pool of sources of repeated calls may be viewed as a sort of queue. Every such source produces a Poisson process of repeated calls with intensity

σ. If an incoming repeated call finds the server free, it is served in the same manner and leaves the system after service, while the source which produced this repeated call disappears. Otherwise, the system state does not change.

We assume that the access from the orbit to the service facility follows the exponential distribution with rate **σ** which may depend on the current number n, ($n \geq 0$) the number of customers in the orbit. That is, the probability of repeated attempt during the interval (t, $t + \Delta t$), given that there are n customers in the orbit at time t is $\mathbf{n\sigma\ \Delta t}$. It is called the **classical retrial rate policy.** The input flow of primary calls, interval between repetitions and service time in phases are mutually independent.

2.2 MATRIX GEOMETRIC SOLUTIONS

Let N(t) be the random variable which represents the number of customers in orbit at any time t and S(t) be the random variable which represents the phase in which customer is getting the service at time t. The random process is described as

$\{<N(t), S(t)> / N(t) = 0,1,2,3\ldots; S(t) = 0,1,2,3,\ldots, k, k+1\}$

S(t) = 0 if the server being idle

S(t) = i for server being busy with the customer in the ith phase for i = 1,2,3,…k

S(t) = k+1 for the server to be on vacation.

The possible state spaces for single server retrial queueing with Erlang K – phases service are

$\{ (i, j) \ / \ i = 0, 1, 2, 3,\ldots ;\ j = 0, 1, 2, 3,\ldots, k, k+1 \}$

The infinitesimal generator matrix **Q** for this model is given below

$$
Q = \begin{pmatrix}
A_{00} & A_0 & O & O & O & \ldots \\
A_{10} & A_{11} & A_0 & O & O & \ldots \\
O & A_{21} & A_{22} & A_0 & O & \ldots \\
O & O & A_{32} & A_{33} & A_0 & \ldots
\end{pmatrix}
$$

The matrices $\mathbf{A_{00}}$, $\mathbf{A_{01}}$, $\mathbf{A_{n\,n-1}}$, $\mathbf{A_{n\,n}}$ and $\mathbf{A_{n\,n+1}}$ for n = 1,2,3,… in the infinitesimal matrix generator **Q** are square matrices of order k+1.

We denote

$$S_1 = -(\lambda + k\mu) \quad S_2 = (1-p)\,k\mu$$

The matrix $\mathbf{A_{00}}$ is described as

$$
\begin{pmatrix}
-\lambda & \lambda & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & S_1 & k\mu & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & S_1 & k\mu & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & S_1 & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & S_1 & k\mu & 0 \\
S_2 & 0 & 0 & 0 & \ldots & 0 & S_1 & pk\mu \\
\alpha & 0 & 0 & 0 & \ldots & 0 & 0 & -(\lambda+\alpha)
\end{pmatrix}
$$

The Matrix $\mathbf{A_{nn-1} = (a_{ij})}$ where $\mathbf{a_{ij}} = \mathbf{n\sigma}$ if i = 1, j = 2

= **0** otherwise

The matrix $\mathbf{A_{n\,n}}$ for n = 1,2,3,… as

$$
\begin{pmatrix}
-(\lambda+n\sigma) & \lambda & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & S_1 & k\mu & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & S_1 & k\mu & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & S_1 & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & S_1 & k\mu & 0 \\
S_2 & 0 & 0 & 0 & \ldots & 0 & S_1 & pk\mu \\
\alpha & 0 & 0 & 0 & \ldots & 0 & 0 & -(\lambda+\alpha)
\end{pmatrix}
$$

The matrix $\mathbf{A_{nn+1} = A_0 = (a_{ij})}$ **for n = 0,1,2,…**

Where $\mathbf{a_{ij}} = \boldsymbol{\lambda}$ if i = j , i = 2,3,4,…,k+1

= **0** otherwise

If the capacity of the orbit is finite say M then

The matrix $\mathbf{A_{MM}}$ is given below

$$
\begin{pmatrix}
-(\lambda+M\sigma) & \lambda & 0 & 0 & \ldots & 0 & 0 \\
0 & -k\mu & k\mu & 0 & \ldots & 0 & 0 \\
0 & 0 & -k\mu & k\mu & \ldots & 0 & 0 \\
0 & 0 & 0 & -k\mu & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
(1-p)k\mu & 0 & 0 & 0 & \ldots & -k\mu & pk\mu
\end{pmatrix}
$$

Let **X** be a steady-state probability vector of **Q** and partitioned as **X**= (x(0),x(1),x(2), ….) and **X** satisfies

$$\mathbf{XQ = 0 , Xe = 1} \quad (1)$$

where $x(i) = (P_{i0}, P_{i1}, P_{i2}, \ldots, P_{ik}, P_{i\,k+1})$

In this paper we are applying the **Direct Truncation Method** to find the Steady state probability vector **X**. Let M denote the cut-off point for this truncation method. The steady state probability vector $\mathbf{X^{(M)}}$ is now partitioned as

$$\mathbf{X^{(M)}} = (x(0) , x(1), x(2) , \ldots x(M))$$

which satisfies

$$\mathbf{X^{(M)}\,Q = 0}\ ,\ \mathbf{X^{(M)}\,e = 1},$$

where $x(i) = (P_{i0}, P_{i1}, P_{i2}, \ldots, P_{ik+1})$ i = 0, 1, 2,3,…,M .

The above system of equations is solved by exploiting the special structure of the co-efficient matrix. It is solved by GAUSS-JORDAN elementary transformation method. Since there is no clear cut choice for M, we may start the iterative process by taking, say M=1 and increase it until the individual elements of **x** do not change significantly. That is, if M^* denotes the truncation point then $\|\mathbf{x}^{M^*}(\mathbf{i}) - \mathbf{x}^{M^*-1}(\mathbf{i})\|_\infty < \epsilon$, **w**here ϵ is an infinitesimal quantity.

2.3 STABILITY CONDITION

Theorem :

The inequality $\left(\frac{\lambda}{\mu}+\frac{p\lambda}{\alpha}\right)<1$ is the necessary and sufficient condition for system to be stable.

Proof:

Let Q be an infinitesimal generator matrix for the queueing system (without retrial)

The stationary probability vector X satisfying

$$\mathbf{XQ = 0 \quad and \quad Xe=1} \qquad (2)$$

Let R be the rate matrix and satisfying the equation

$$A_0+RA_1+R^2A_2 = 0 \qquad (3)$$

The system is stable if sp(R) <1

We know that the Matrix R satisfies sp(R) <1 if and only if

$$\Pi A_0 e < \Pi A_2 e \qquad (4)$$

where $\Pi = (\pi_1,\ldots,\pi_{k+1})$ and satisfies

$$\Pi A = 0 \text{ and } \Pi e = 1 \qquad (5)$$

and

$$A=A_0+A_1+A_2 \qquad (6)$$

Here $\mathbf{A_0}$, $\mathbf{A_1}$ **and** $\mathbf{A_2}$ are square matrices of order **k** and

$\mathbf{A_0} = \lambda \mathbf{I}$, **I** the identity matrix of order k+1

The matrix $\mathbf{A_1}$ is given below

$$\begin{pmatrix} S_1 & k\mu & 0 & \ldots & 0 & 0 \\ 0 & S_1 & k\mu & \ldots & 0 & 0 \\ 0 & 0 & S_1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & S_1 & k\mu \\ \alpha & 0 & 0 & \ldots & 0 & -(\lambda+\alpha) \end{pmatrix}$$

The matrix $\mathbf{A_2}$ is described as

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ (1-p)k\mu & 0 & 0 & \ldots & 0 & k\mu \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

By substituting $\mathbf{A_0}$, $\mathbf{A_1}$, $\mathbf{A_2}$ in equations **(4), (5)** and **(6)**, we get $\left(\frac{\lambda}{\mu}+\frac{p\lambda}{\alpha}\right)<1$. The inequality $\left(\frac{\lambda}{\mu}+\frac{p\lambda}{\alpha}\right)<1$ is also a sufficient condition for the retrial queueing system to be stable. Let Q_n be the number of customers in the orbit after the departure of n^{th} customer from the service station. We first prove the embedded Markov chain $\{Q_n, n\geq 0\}$ is ergodic if $\left(\frac{\lambda}{\mu}+\frac{p\lambda}{\alpha}\right)<1$. $\{Q_n, n\geq 0\}$ is irreducible and aperiodic. It remains to be proved that $\{Q_n, n\geq 0\}$ is positive recurrent. The irreducible and aperiodic Markov chain $\{Q_n, n\geq 0\}$ is positive recurrent if $|\psi_m| <\infty$ for all m and $\lim_{m\to\infty} \sup \psi_m < 0$, where

$$\psi_m = E((Q_{n+1} - Q_n) / Q_n = m) \quad (m = 0,1,2,3,4,5,\ldots)$$

$$\psi_m = \left(\frac{\lambda}{\mu}+\frac{p\lambda}{\alpha}\right) - \left(\frac{m\sigma}{\lambda+m\sigma}\right)$$

If $\left(\frac{\lambda}{\mu}+\frac{p\lambda}{\alpha}\right)<1$, then $|\psi_m| <\infty$ for all **m** and

$\lim_{m\to\infty} \sup \psi_m < 0$. Therefore the embedded Markov chain $\{Q_n, n\geq 0\}$ is ergodic.

2.4 SPECIAL CASES

1. If K=1 and p = 0 , then this model becomes the Single server retrial queueing model and our numerical results coincide with the following closed form of Number of customers in the orbit in the steady state [6]

$$\text{Mean Number of Customers in the orbit} = \frac{\rho(\lambda+\rho\sigma)}{(1-\rho)\sigma}$$

2. As $\sigma\to\infty$ and p = 0, the closed form of number of customers in the orbit tends to length of the queue in standard queueing system with Erlang-k service [4]

$$L_q = \left(\frac{k+1}{2k}\right)\left(\frac{\rho^2}{1-\rho}\right)$$

For many values of λ, μ, k and very high values of σ (>10000), the above result coincides with our numerical results.

3. If k=1 , p=0 and σ→∞ ,then mean number of customers in the orbit coincides with

$$Lq = \left(\frac{\rho^2}{1-\rho}\right)$$

2.5 SYSTEM PERFORMANCE MEASURES

In this section some important performance measures along with formulas and their qualitative behaviour for this queueing model are studied. Numerical study has been dealt in very large scale to study these measures. Defining

P(n , 0) = Probability that there are n customers in the orbit and server is free

P(n , i) = Probability that there are n customers in the orbit, server is busy with customer in the i^{th} phase for i = 1,2,3,…k.

P(n , k+1) = Probability that there are n customers in the orbit and server is on vacation.

We can find various probabilities for various values of λ, μ, p, α, k and σ and the following parameters can be easily studied with these probabilities

a. The probability mass function of Server state

Let S(t) be the random variable which represents the phase in which customer is getting service at time t .

Prob (The server is idle) = $\sum_{i=0}^{\infty} p(i,0)$

Prob (The server busy with customer in the j^{th} phase)

$$= \sum_{i=0}^{\infty} p(i,j)$$

Prob (The server is in vacation) = $\sum_{i=0}^{\infty} p(i,k+1)$

b. The probability mass function number of customers in the orbit

Let X(t) be the random variable representing the number of customers in the orbit.

Prob (No customers in the orbit) = $\sum_{j=0}^{k+1} p(0,j)$

Prob (i customers in the orbit) = $\sum_{j=0}^{k+1} p(i,j)$

c. The Mean number of customers in the orbit

$$\text{MNCO} = \sum_{i=0}^{\infty} i(\sum_{j=0}^{k+1} p(i,j))$$

d. **The probability that the orbiting customer is blocked**

$$\text{Blocking Probability} = \sum_{i=1}^{\infty}\sum_{j=1}^{k+1} p(i,j)$$

e. **The probability that an arriving customer enter into service immediately**

$$\text{PSI} = \sum_{i=0}^{\infty} p(i,0)$$

2.6 NUMERICAL STUDY

MNCO : Mean Number of Customers in the Orbit

P_0 : Probability that the server is idle

P_1 : Probability that the server is busy

P_2 : Probability that the server is in vacation

Table 1 and Table 2 show the effect of retrial rate σ over the system. As σ increases, mean number of customers in the orbit decreases and this model becomes classical queueing system with Bernoulli vacation if σ >5000. Probabilities P_0, P_1 and P_2 **are independent of σ**

Table 1: System Measures for λ= 4 μ =10 α =100 p = 0.5

σ	Ocut	MNCO	P_0	P_1	P_2
10	12	0.4703	0.58	0.4	0.02
20	11	0.3255	0.58	0.4	0.02
30	11	0.2772	0.58	0.4	0.02
40	11	0.2531	0.58	0.4	0.02
50	11	0.2386	0.58	0.4	0.02
60	11	0.229	0.58	0.4	0.02
70	11	0.2221	0.58	0.4	0.02
80	11	0.2169	0.58	0.4	0.02
90	11	0.2129	0.58	0.4	0.02

100	11	0.2097	0.58	0.4	0.02
200	11	0.1952	0.58	0.4	0.02
300	10	0.1903	0.58	0.4	0.02
400	10	0.1879	0.58	0.4	0.02
500	10	0.1865	0.58	0.4	0.02
600	10	0.1855	0.58	0.4	0.02
700	10	0.1848	0.58	0.4	0.02
800	10	0.1843	0.58	0.4	0.02
900	10	0.1839	0.58	0.4	0.02
1000	10	0.1836	0.58	0.4	0.02
2000	10	0.1821	0.58	0.4	0.02
3000	10	0.1817	0.58	0.4	0.02
4000	10	0.1814	0.58	0.4	0.02
5000	10	0.1813	0.58	0.4	0.02
6000	10	0.1812	0.58	0.4	0.02
7000	10	0.1811	0.58	0.4	0.02

Table 2 : System Measures for $\lambda = 8$ $\mu = 10$ $\alpha = 100$ $p = 0.5$

σ	Ocut	MNCO	P_0	P_1	P_2
10	62	6.82	0.16	0.8	0.04
20	56	4.72	0.16	0.8	0.04
30	54	4.02	0.16	0.8	0.04
40	53	3.67	0.16	0.8	0.04
50	52	3.46	0.16	0.8	0.04
60	51	3.32	0.16	0.8	0.04
70	51	3.22	0.16	0.8	0.04
80	51	3.145	0.16	0.8	0.04
90	51	3.0866	0.16	0.8	0.04
100	50	3.04	0.16	0.8	0.04
200	50	2.83	0.16	0.8	0.04
300	49	2.76	0.16	0.8	0.04
400	49	2.725	0.16	0.8	0.04
500	49	2.704	0.16	0.8	0.04
600	49	2.69	0.16	0.8	0.04
700	49	2.68	0.16	0.8	0.04
800	49	2.6725	0.16	0.8	0.04
900	49	2.6666	0.16	0.8	0.04
1000	49	2.662	0.16	0.8	0.04
2000	49	2.641	0.16	0.8	0.04
3000	49	2.634	0.16	0.8	0.04
4000	49	2.6305	0.16	0.8	0.04
5000	49	2.6284	0.16	0.8	0.04
6000	49	2.627	0.16	0.8	0.04
7000	49	2.626	0.16	0.8	0.04

Table 3 shows the effect of number of phases (k) over the system. As k increases, mean number of customers in the orbit decreases. Probabilities P_0, P_1 and P_2 **are independent of k.**

Table 4 shows the effect of probability of going for vacation (p) over the system. As p increases, mean number of customers in the orbit increases.

Table 3 : System Measures for $\lambda = 5$ $\mu = 10$ $\alpha = 100$ $p = 0.5$ $\sigma = 100$

k	Ocut	MNCO	P_0	P_1	P_2
1	20	0.6105	0.475	0.5	0.025
2	16	0.4789	0.475	0.5	0.025
3	15	0.4351	0.475	0.5	0.025
4	14	0.4132	0.475	0.5	0.025
5	14	0.4	0.475	0.5	0.025
6	14	0.3912	0.475	0.5	0.025
7	14	0.385	0.475	0.5	0.025
8	14	0.3803	0.475	0.5	0.025
9	13	0.3766	0.475	0.5	0.025
10	13	0.3737	0.475	0.5	0.025
11	13	0.3713	0.475	0.5	0.025
12	13	0.3693	0.475	0.5	0.025
13	13	0.3676	0.475	0.5	0.025
14	13	0.3662	0.475	0.5	0.025
15	13	0.3649	0.475	0.5	0.025
16	13	0.3638	0.475	0.5	0.025
17	13	0.3628	0.475	0.5	0.025
18	13	0.362	0.475	0.5	0.025
19	13	0.3612	0.475	0.5	0.025
20	13	0.3605	0.475	0.5	0.025
21	13	0.3599	0.475	0.5	0.025
22	13	0.3593	0.475	0.5	0.025
23	13	0.3588	0.475	0.5	0.025
24	13	0.3583	0.475	0.5	0.025
25	13	0.3579	0.475	0.5	0.025
26	13	0.3575	0.475	0.5	0.025
27	13	0.3571	0.475	0.5	0.025
28	13	0.3568	0.475	0.5	0.025
29	13	0.3564	0.475	0.5	0.025
30	13	0.3561	0.475	0.5	0.025
31	13	0.3559	0.475	0.5	0.025
32	13	0.3556	0.475	0.5	0.025
33	13	0.3553	0.475	0.5	0.025
34	13	0.3551	0.475	0.5	0.025
35	13	0.3549	0.475	0.5	0.025

Table 4 : System Measures for $\lambda = 5$ $\mu = 10$ $\alpha = 100$ $k = 5$ $\sigma = 100$

p	Ocut	MNCO	P_0	P_1	P_2
0.02	13	0.3519	0.499	0.5	0.001
0.04	13	0.3538	0.498	0.5	0.002
0.06	13	0.3557	0.497	0.5	0.003
0.08	13	0.3577	0.496	0.5	0.004

0.1	13	0.3596	0.495	0.5	0.005
0.12	13	0.3615	0.494	0.5	0.006
0.14	13	0.3635	0.493	0.5	0.007
0.16	14	0.3654	0.492	0.5	0.008
0.18	14	0.3674	0.491	0.5	0.009
0.2	14	0.3694	0.49	0.5	0.01
0.22	14	0.3714	0.489	0.5	0.011
0.24	14	0.3734	0.488	0.5	0.012
0.26	14	0.3754	0.487	0.5	0.013
0.28	14	0.3774	0.486	0.5	0.014
0.3	14	0.3794	0.485	0.5	0.015
0.32	14	0.3814	0.484	0.5	0.016
0.34	14	0.3834	0.483	0.5	0.017
0.36	14	0.3855	0.482	0.5	0.018
0.38	14	0.3875	0.481	0.5	0.019
0.4	14	0.3896	0.48	0.5	0.02
0.42	14	0.3916	0.479	0.5	0.021
0.44	14	0.3937	0.478	0.5	0.022
0.46	14	0.3958	0.477	0.5	0.023
0.48	14	0.3979	0.476	0.5	0.024
0.5	14	0.4	0.475	0.5	0.025
0.52	14	0.4021	0.474	0.5	0.026
0.54	14	0.4042	0.473	0.5	0.027
0.56	14	0.4064	0.472	0.5	0.028
0.58	14	0.4085	0.471	0.5	0.029
0.6	14	0.4106	0.47	0.5	0.03
0.62	14	0.4128	0.469	0.5	0.031
0.64	14	0.415	0.468	0.5	0.032
0.66	14	0.4171	0.467	0.5	0.033
0.68	14	0.4193	0.466	0.5	0.034
0.7	14	0.4215	0.465	0.5	0.035
0.72	14	0.4237	0.464	0.5	0.036
0.74	15	0.4259	0.463	0.5	0.037
0.76	15	0.4281	0.462	0.5	0.038
0.78	15	0.4304	0.461	0.5	0.039
0.8	15	0.4326	0.46	0.5	0.04

MODEL II

3.1 MODEL DESCRIPTION

Consider a single server retrial queueing system with exhaustive type vacation in which customers arrive in a Poisson process with arrival rate **λ**. These customers are identified as primary calls**.** Let k be the number of phases in the service station. Assume that the service time has **Erlang-k distribution** with service rate kμ for each phase. The **vacation rate** follows an exponential distribution with parameter **α**. If the server is **free** at the time of a primary or repeated call arrival, then this arriving call begins to be served immediately and leaves the system after completion of the service. The server goes for vacation compulsorily after servicing to all customers and the system becomes empty. This type of vacation in queueing theory is called **exhaustive service type single vacation. The single vacation means after completion of vacation period he can again go for vacation once again after servicing atleast one customer.** The server may return from the vacation at any time and is independent of number of customers in the system. We assume that the services in all phases are independent and identical and only one customer at a time is in the service mechanism. If the server is **free** at the time of a primary call arrival, the arriving call begins to be served in Phase 1 immediately by the server then progresses through the remaining phases and must complete the last phase and leave the system before the next customer enters the first phase. If the server is **busy** or on **vacation**, then the arriving customer goes to orbit and becomes a source of repeated calls. This pool of sources of repeated calls may be viewed as a sort of queue. Every such source produces a Poisson process of repeated calls with intensity **σ**. If an incoming repeated call finds the server free, it is served in the same manner and leaves the system after service, while the source which produced this repeated call disappears. Otherwise, the system state does not change. We assume that the access from the orbit to the service facility governed by **classical retrial rate policy.** The input flow of primary calls, interval between repetitions and service time in phases are mutually independent.

3.2 MATRIX GEOMETRIC SOLUTIONS

The random process is described as

{<N(t), S(t) > / N(t) = 0,1,2,3…; S(t) = 0,1,2,3,…, k, k+1 }

S(t) = 0 if the server being idle

S(t) = i for server being busy with the customer in the i^{th} phase for i = 1,2,3,…,k

S(t) = k+1 for the server to be on vacation.

The possible state spaces are

{ (i , j) / i = 0, 1, 2, 3,… ; j = 0, 1, 2, 3,…, k ,k+1 }

The matrices $\mathbf{A_{00}}$, $\mathbf{A_{01}}$, $\mathbf{A_{n\,n-1}}$, $\mathbf{A_{n\,n}}$ and $\mathbf{A_{n\,n+1}}$ for n = 1,2,3,… in the infinitesimal matrix generator **Q** are square matrices of order k+1.

The matrix $\mathbf{A_{00}}$ is described as

$$
\begin{pmatrix}
-\lambda & \lambda & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & S_1 & k\mu & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & S_1 & k\mu & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & S_1 & \dots & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & \dots & S_1 & k\mu & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & S_1 & k\mu \\
\alpha & 0 & 0 & 0 & \dots & 0 & 0 & -(\lambda+\alpha)
\end{pmatrix}
$$

$\mathbf{A_{nn-1} = (a_{ij})}$ **for n = 1,2,3,…**

where $a_{ij} = n\sigma$ if i = 1, j = 2

$= 0$

The matrix $\mathbf{A}_{nn}$ for n = 1,2,3,… are

$$\begin{pmatrix} -(\lambda+n\sigma) & \lambda & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & S_1 & k\mu & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & S_1 & k\mu & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & S_1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & S_1 & k\mu & 0 \\ k\mu & 0 & 0 & 0 & \dots & 0 & S_1 & 0 \\ \alpha & 0 & 0 & 0 & \dots & 0 & 0 & -(\lambda+\alpha) \end{pmatrix}$$

$\mathbf{A_{nn+1} = A_0 = (a_{ij})}$ **for n = 0,1,2…**

where, $a_{ii} = \lambda$ if i = 2,3,4,…,k+1

$= 0$ otherwise

If the capacity of the orbit is finite say M then

The matrix A_{MM} is given below

$$\begin{pmatrix} -(\lambda+M\sigma) & \lambda & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -k\mu & k\mu & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -k\mu & k\mu & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & -k\mu & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -k\mu & k\mu & 0 \\ k\mu & 0 & 0 & 0 & \dots & 0 & -k\mu & 0 \\ \alpha & 0 & 0 & 0 & \dots & 0 & 0 & -\alpha \end{pmatrix}$$

Let **X** be a steady-state probability vector of **Q** and partitioned as **X**= (x(0),x(1),x(2), ….) and **X** satisfies

$$\mathbf{XQ = 0 , Xe = 1}$$

where $x(i) = (P_{i0}, P_{i1}, P_{i2}, \dots\dots P_{ik}, P_{ik+1})$

In this paper we are applying the **Direct Truncation Method** to find the Steady state probability vector **X**. Let M denote the cut-off point for this truncation method. The steady state probability vector $\mathbf{X}^{(M)}$ is now partitioned as

$$\mathbf{X}^{(M)} = (x(0), x(1), x(2), \dots x(M))$$

which satisfies

$$\mathbf{X}^{(M)}\mathbf{Q} = 0, \mathbf{X}^{(M)}\mathbf{e} = 1,$$

where $x(i) = (P_{i0}, P_{i1}, P_{i2}, \dots, P_{ik+1})$ i =0, 1, 2,3,…,M .

The above system of equations is solved by exploiting the special structure of the co-efficient matrix. It is solved by GAUSS-JORDAN elementary transformation method. Since there is no clear cut choice for M, we may start the iterative process by taking, say M=1 and increase it until the individual elements of **x** do not change significantly. That is, if M^* denotes the truncation point then

$\|x^{M^*}(i) - x^{M^*-1}(i)\|_\infty < \epsilon$, where ϵ is an infinitesimal quantity.

3.3 STABILITY CONDITION

Theorem :

The inequality $\left(\dfrac{\lambda}{\mu}\right) < 1$ is the necessary and sufficient condition for system to be stable.

Proof:

Let Q be an infinitesimal generator matrix for the queueing system (without retrial)

The stationary probability vector X satisfying

$$\mathbf{XQ = 0 \quad and \quad Xe=1} \quad \textbf{(8)}$$

Let R be the rate matrix and satisfying the equation

$$\mathbf{A_0+RA_1+R^2A_2 = 0} \quad \textbf{(9)}$$

The system is stable if sp(R) <1

We know that the Matrix R satisfies sp(R) <1 if and only if

$$\mathbf{\Pi A_0 e < \Pi A_2 e} \quad \textbf{(10)}$$

where $\mathbf{\Pi} = (\pi_1, \dots, \pi_k)$ and satisfies

$$\mathbf{\Pi A = 0 \text{ and } \Pi e = 1} \quad \textbf{(11)}$$

and $\mathbf{A=A_0+A_1+A_2}$ **(12)**

Here $\mathbf{A_0}$, $\mathbf{A_1}$ **and** $\mathbf{A_2}$ are square matrices of order **k** and

$\mathbf{A_0} = \lambda \mathbf{I}$, **I** is corresponding identity matrix

The matrix $\mathbf{A_1}$ is given below

$$\begin{pmatrix} S_1 & k\mu & 0 & \dots & 0 & 0 & 0 \\ 0 & S_1 & k\mu & \dots & 0 & 0 & 0 \\ 0 & 0 & S_1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & S_1 & k\mu & 0 \\ 0 & 0 & 0 & \dots & 0 & S_1 & 0 \\ \alpha & 0 & 0 & \dots & 0 & 0 & -(\lambda+\alpha) \end{pmatrix}$$

The matrix $\mathbf{A_2}$ is described as

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ k\mu & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

By substituting A_0, A_1, A_2 in equations **(10)**, **(11)** and **(12)**, we get $\left(\frac{\lambda}{\mu}\right) < 1$. The inequality $\left(\frac{\lambda}{\mu}\right) < 1$ is also a sufficient condition for the retrial queueing system to be stable. Let Q_n be the number of customers in the orbit after the departure of n^{th} customer from the service station. We first prove the embedded Markov chain $\{Q_n, n \geq 0\}$ is ergodic if $\left(\frac{\lambda}{\mu}\right) < 1$. $\{Q_n, n \geq 0\}$ is irreducible and aperiodic. It remains to be proved that $\{Q_n, n \geq 0\}$ is positive recurrent. The irreducible and aperiodic Markov chain $\{Q_n, n \geq 0\}$ is positive recurrent if $|\psi_m| < \infty$ for all m and $\lim_{m \to \infty} \sup \psi_m < 0$, where

$$\psi_m = E((Q_{n+1} - Q_n) / Q_n = m), \quad m=0,1,2,3,4,5,\ldots$$

$$\psi_m = \left(\frac{\lambda}{\mu}\right) - \left(\frac{m\sigma}{\lambda + m\sigma}\right)$$

If $\left(\frac{\lambda}{\mu}\right) < 1$, then $|\psi_m| < \infty$ for all **m** and $\lim_{m \to \infty} \sup \psi_m < 0$

Therefore the embedded Markov chain $\{Q_n, n \geq 0\}$ is ergodic.

3.4 SYSTEM PERFORMANCE MEASURES

In this section some important performance measures along with formulas and their qualitative behaviour for this queueing model are studied. Numerical study has been dealt in very large scale to study these measures. We can find various probabilities for various values of λ, μ, α, k, and σ. The formulas for system measures which are discussed in **section 2.5** hold for this model also.

3.5 NUMERICAL STUDY

MNCO : Mean Number of Customers in the Orbit

P_0 : Probability that the server is idle

P_1 : Probability that the server is busy

P_2 : Probability that the server is in vacation

Tables 5 and Table 6 show the effect of retrial rate σ over the system. As σ increases, mean number of customers in the orbit decreases and this model becomes classical queuing system with Exhaustive vacation if σ >5000.

Table 5 : System Measures for λ= 4 μ =10 α =100 k=5

σ	Ocut	MNCO	P_0	P_1	P_2
10	12	0.4405	0.5811	0.4000	0.0189
20	11	0.3018	0.5787	0.4000	0.0213
30	11	0.2553	0.5779	0.4000	0.0221
40	10	0.2319	0.5774	0.4000	0.0226
50	10	0.2179	0.5772	0.4000	0.0228
60	10	0.2085	0.5770	0.4000	0.0230
70	10	0.2018	0.5768	0.4000	0.0232
80	10	0.1968	0.5767	0.4000	0.0233
90	10	0.1929	0.5767	0.4000	0.0233
100	10	0.1898	0.5766	0.4000	0.0234
200	10	0.1757	0.5763	0.4000	0.0237
300	10	0.1710	0.5762	0.4000	0.0238
400	10	0.1686	0.5762	0.4000	0.0238
500	10	0.1672	0.5762	0.4000	0.0238
600	10	0.1663	0.5761	0.4000	0.0239
700	10	0.1656	0.5761	0.4000	0.0239
800	10	0.1651	0.5761	0.4000	0.0239
900	10	0.1647	0.5761	0.4000	0.0239
1000	10	0.1644	0.5761	0.4000	0.0239
2000	10	0.1630	0.5761	0.4000	0.0239
3000	10	0.1625	0.5761	0.4000	0.0239
4000	10	0.1623	0.5761	0.4000	0.0239
5000	10	0.1622	0.5760	0.4000	0.0240
6000	10	0.1621	0.5760	0.4000	0.0240
7000	10	0.1620	0.5760	0.4000	0.0240
8000	10	0.1619	0.5760	0.4000	0.0240

Table 6 : System Measures for λ= 8 μ =10 α =100 k = 5

σ	Ocut	MNCO	P_0	P_1	P_2
10	49	5.1337	0.1969	0.8000	0.0031
20	44	3.5368	0.1930	0.8000	0.0070
30	43	3.0026	0.1908	0.8000	0.0092
40	42	2.7347	0.1895	0.8000	0.0105
50	41	2.5737	0.1886	0.8000	0.0114

60	41	2.4662	0.1879	0.8000	0.0121
70	40	2.3893	0.1874	0.8000	0.0126
80	40	2.3316	0.1871	0.8000	0.0129
90	40	2.2867	0.1868	0.8000	0.0132
100	40	2.2508	0.1865	0.8000	0.0135
100	40	2.2508	0.1865	0.8000	0.0135
200	39	2.0888	0.1854	0.8000	0.0146
300	39	2.0347	0.1849	0.8000	0.0151
400	39	2.0076	0.1847	0.8000	0.0153
500	39	1.9914	0.1846	0.8000	0.0154
600	39	1.9805	0.1845	0.8000	0.0155
700	39	1.9728	0.1845	0.8000	0.0155
800	39	1.9670	0.1844	0.8000	0.0156
900	39	1.9625	0.1844	0.8000	0.0156
1000	39	1.9589	0.1844	0.8000	0.0156
1000	39	1.9589	0.1844	0.8000	0.0156
2000	39	1.9426	0.1842	0.8000	0.0158
3000	39	1.9372	0.1842	0.8000	0.0158
4000	39	1.9345	0.1842	0.8000	0.0158
5000	39	1.9328	0.1841	0.8000	0.0159
6000	39	1.9318	0.1841	0.8000	0.0159
7000	39	1.9310	0.1841	0.8000	0.0159
8000	39	1.9304	0.1841	0.8000	0.0159
9000	39	1.9300	0.1841	0.8000	0.0159
10000	39	1.9296	0.1841	0.8000	0.0159

Table 7 shows the effect of number of phases (k) over the system. As k increases, mean number of customers in the orbit decreases.

Table 8 shows the effect of vacation rate (α) over the system. As α increase, mean number of customers in the orbit decreases.

Table 7 : System Measures for λ= 5 μ =10
α =100 σ = 100

k	O_cut	MNCO	P1	P0	P2
1	21	1.0190	0.4827	0.5000	0.0173
2	18	0.8935	0.4831	0.5000	0.0169
3	16	0.8517	0.4833	0.5000	0.0167
4	16	0.8308	0.4834	0.5000	0.0166
5	15	0.8182	0.4834	0.5000	0.0166
6	15	0.8099	0.4835	0.5000	0.0165
7	15	0.8039	0.4835	0.5000	0.0165
8	15	0.7994	0.4835	0.5000	0.0165
9	15	0.7959	0.4835	0.5000	0.0165
10	15	0.7931	0.4835	0.5000	0.0165
11	15	0.7908	0.4835	0.5000	0.0165
12	15	0.7889	0.4835	0.5000	0.0165
13	15	0.7873	0.4836	0.5000	0.0164
14	14	0.7859	0.4836	0.5000	0.0164
15	14	0.7847	0.4836	0.5000	0.0164
16	14	0.7837	0.4836	0.5000	0.0164
17	14	0.7828	0.4836	0.5000	0.0164
18	14	0.7820	0.4836	0.5000	0.0164
19	14	0.7812	0.4836	0.5000	0.0164
20	14	0.7806	0.4836	0.5000	0.0164
21	14	0.7800	0.4836	0.5000	0.0164
22	14	0.7794	0.4836	0.5000	0.0164
23	14	0.7789	0.4836	0.5000	0.0164
24	14	0.7785	0.4836	0.5000	0.0164
25	14	0.7780	0.4836	0.5000	0.0164
26	14	0.7777	0.4836	0.5000	0.0164
27	14	0.7773	0.4836	0.5000	0.0164
28	14	0.7770	0.4836	0.5000	0.0164
29	14	0.7767	0.4836	0.5000	0.0164
30	14	0.7764	0.4836	0.5000	0.0164
31	14	0.7761	0.4836	0.5000	0.0164
32	14	0.7758	0.4836	0.5000	0.0164
33	14	0.7756	0.4836	0.5000	0.0164
34	14	0.7754	0.4836	0.5000	0.0164
35	14	0.7752	0.4836	0.5000	0.0164
36	14	0.7750	0.4836	0.5000	0.0164
37	14	0.7748	0.4836	0.5000	0.0164
38	14	0.7746	0.4836	0.5000	0.0164
39	14	0.7744	0.4836	0.5000	0.0164
40	14	0.7743	0.4836	0.5000	0.0164

Table 8 : System Measures for λ= 5 μ =10
k = 5 σ = 100

α	Ocut	MNCO	P_0	P_1	P_2
10	15	0.5736	0.2968	0.5000	0.2032
20	13	0.4181	0.3866	0.5000	0.1134
30	13	0.3837	0.4222	0.5000	0.0778
40	13	0.3706	0.4410	0.5000	0.0590
50	13	0.3642	0.4526	0.5000	0.0474
60	13	0.3606	0.4603	0.5000	0.0397
70	13	0.3583	0.4659	0.5000	0.0341
80	13	0.3567	0.4701	0.5000	0.0299

90	13	0.3556	0.4734	0.5000	0.0266
100	13	0.3548	0.4761	0.5000	0.0239
100	13	0.3548	0.4761	0.5000	0.0239
200	13	0.3518	0.4880	0.5000	0.0120
300	13	0.3511	0.4920	0.5000	0.0080
400	13	0.3507	0.4940	0.5000	0.0060
500	13	0.3506	0.4952	0.5000	0.0048
600	13	0.3505	0.4960	0.5000	0.0040
700	13	0.3504	0.4966	0.5000	0.0034
800	13	0.3503	0.4970	0.5000	0.0030
900	13	0.3503	0.4973	0.5000	0.0027
1000	13	0.3503	0.4976	0.5000	0.0024
1000	13	0.3503	0.4976	0.5000	0.0024
2000	13	0.3501	0.4988	0.5000	0.0012
3000	13	0.3501	0.4992	0.5000	0.0008
4000	13	0.3501	0.4994	0.5000	0.0006
5000	13	0.3500	0.4995	0.5000	0.0005
6000	13	0.3500	0.4996	0.5000	0.0004
7000	13	0.3500	0.4997	0.5000	0.0003
8000	13	0.3500	0.4997	0.5000	0.0003
9000	13	0.3500	0.4997	0.5000	0.0003
10000	13	0.3500	0.4998	0.5000	0.0002

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