

Forcing Independent Spectrum in Graphs

A.P.Pushpalatha
Assistant Professor
Department of Mathematics
Thiagarajar College of
Engineering, Madurai-625015

G.Jothilakshmi
Assistant Professor
Department of Mathematics
Thiagarajar College of
Engineering, Madurai-625015

S.Suganthi,
V.Swaminathan
Ramanujan Research Centre,
Saraswathi Narayanan
College, Madurai-625022

ABSTRACT

Let $G = (V, E)$ be a simple graph. Let S be a maximum independent set of G . A subset T of S is called a forcing subset if T is contained in no other maximum independent subset in G . The independent forcing number of S denoted by $f_1(G, S)$ is the cardinality of a minimum forcing subset of S . The independent forcing number of G is the minimum of the independent forcing number of S , where S is a maximum independent subset in G . The independent forcing spectrum of G denoted by $\text{Spec}_1(G)$ is defined as the set $\text{Spec}_1(G) = \{k : \text{there exists a maximum independent set } S \text{ of } G \text{ such that } f_1(G, S) = k\}$. In this paper, a study of $\text{Spec}_1(G)$ is made..

Keywords

Forcing domination number of a graph, Forcing spectrum of a graph and Forcing independent spectrum of a graph.

1. INTRODUCTION

The forcing sets in a graph are a very interesting concept. In the management of an institution, the executive committee consists of senior members who have adequate rapport with other members of the institution. Some members of the executive committee may sit in other important committees also. Sometimes, restrictions are imposed on members that they can be part of exactly one committee. This precisely leads to the concept of forcing set. A subset of a minimum dominating set S is called a forcing subset with respect to S if this subset is contained in no other minimum dominating set of G . Many authors have studied this forcing concept with respect to several parameters like domination, matching, geodetic domination, chromatic partition, etc. This chapter studies the forcing concept with respect to maximum independence. A subset of a maximum independent set may be contained in other maximum independent sets also. For example, in C_5 , every vertex is contained in at least two maximum independent sets. The natural curiosity is to study such subsets which are constrained to remain only in one maximum independent set which is forced to remain only in that set. We consider only finite, simple and undirected graphs $G=(V,E)$. [2] Gary Chartered, Gavlas and Robert C.Vandell introduced the concept of Forcing domination number of a graph.

1.1 Definition

[2] A subset T of a minimum dominating set S is a forcing Subset for S if S is the unique minimum dominating set containing T . S is called the forcing dominating set of T . The minimum cardinality among the forcing subsets of S is called the forcing domination number of S and is denoted by $f(S, \gamma(G))$. The minimum forcing domination number among the minimum dominating sets of G is denoted by $f(G, \gamma)$. That is $f(G, \gamma) = \min f(S_i, \gamma)$, where S_i 's are the minimum dominating set of G . Clearly, for any graph G , $f(G, \gamma) \leq \gamma(G)$.

1.2 Defintion

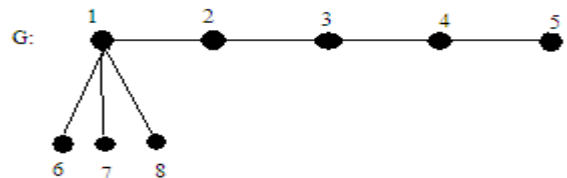
[1] Let G be a simple graph. The forcing spectrum of G denoted by $\text{Spec}_\gamma(G)$ is defined as the set $\text{Spec}_\gamma(G) = \{k : \text{there exists a minimum dominating set } S \text{ of } G \text{ such that } f(S, \gamma(G)) = k\}$.

The above two concepts are extended in the context of maximum independent sets.

1.3 Definition

Let S be a maximum independent set of G . A subset T of S is called a forcing subset if T is contained in no other maximum independent subset in G . The independent forcing number of S denoted by $f_1(G, S)$ is the cardinality of a minimum forcing subset of S . The independent forcing number of G is the minimum of the independent forcing number of S , where S is a maximum independent subset in G . The independent forcing spectrum of G denoted by $\text{Spec}_1(G)$ is defined as the set $\text{Spec}_1(G) = \{k : \text{there exists a maximum independent set } S \text{ of } G \text{ such that } f_1(G, S) = k\}$.

1.4 Example



The maximum independent sets of G are $\{6,7,8,2,4\}$ (say S_1), $\{6,7,8,2,5\}$ (say S_2) and $\{6,7,8,3,5\}$ (say S_3). In S_1 , 4 is the only one vertex which is not contained in any other maximum independent set. Hence the spectrum of $S_1 = \{1\}$. In S_2 , $\{2,5\}$ is the subset of minimum cardinality which is not contained in any other maximum independent set. Hence the spectrum of $S_2 = \{2\}$. In S_3 , 3 is the only one vertex which is not contained in any other maximum independent set. Hence the spectrum of $S_3 = \{1\}$. Hence $\text{Spec}_1(G) = \{1,2\}$.

1.5 Remark

If G has a unique maximum independent set, then

$$\text{Spec}_1(G) = \{0\}.$$

$\text{Spec}_1(G)$ for some standard graphs

$$1. \text{Spec}_1(K_n) = \{1\}.$$

$$2. \text{Spec}_1(\bar{K}_n) = \{0\}.$$

$$3. \text{Spec}_1(K_{(1,n)}) = \{0\}.$$

$$4. \text{Spec}_1(K_{(m,n)}) = \begin{cases} \{0\}, & m \neq n \\ \{1\}, & m = n \end{cases}$$

$$5. \text{Spec}_1(C_n) = \begin{cases} \{1\}, & \text{if } n \text{ is even} \\ \{2\}, & \text{if } n \text{ is odd} \end{cases}$$

$$6. \text{Spec}_1(D_{m,n}) = 0, \text{ where } m < n.$$

$$7. \text{Spec}_1(P_n) = \begin{cases} \{2\}, & \text{if } n = 4 \\ \{0\}, & \text{if } n \text{ is odd} \\ \{1\}, & \text{if } n \text{ is even and } n \neq 4 \end{cases}$$

$$8. \text{Spec}_1(W_n) = \begin{cases} \{1\}, & \text{if } n \text{ is odd} \\ \{2\}, & \text{if } n \text{ is even} \end{cases}$$

2.1 Theorem

Let G be a graph with $\text{Spec}_1(G) = A$. Then for any integer k , there exists a graph H with $\text{Spec}_1(H) = \{x + k : x \in A\}$.

Proof

Let $H = G \cup kC_4$. $\text{Spec}_1(H) = \{x + k : x \in A\}$. Let $S = \{u_1, u_2, \dots, u_r\}$ be a maximum independent set of G whose forcing independent number is x . Let $v_{i1}, v_{i2}, v_{i3}, v_{i4}$ be the vertices of

i -th copy of C_4 in the cyclic order. Then $S \cup \{v_{i1}, v_{i3} : 1 \leq i \leq k\}$ is a maximum independent set of H , for which the minimum forcing set is $\{u_{j1}, u_{j2}, \dots, u_{jx}, v_i : 1 \leq i \leq k\}$, where $\{u_{j1}, u_{j2}, \dots, u_{jx}\}$ is an independent forcing set of S of minimum cardinality.

2.1 Theorem

There exists a graph G in which $\beta_0(G) \in \text{Spec}_1(G)$.

Proof

Let H be any connected graph. Let $G = H \circ K_1$. Let $|V(H)| = n$. Let $T = \{v_1, v_2, \dots, v_n\}$ be the set of all pendant vertices of G . Consider any proper subset $T_1 = \{v_{i1}, v_{i2}, \dots, v_{ij}\}$ of T . ($j < n$). Let $v_j \in T - T_1$. Then $T_2 = \{v_1, \dots, v_{j-1}, u_j, v_{j+1}, \dots, v_n\}$ is a maximum independent subset of G where u_j is the support of v_j in G . Thus T_1 is not in a forcing subset of T . Therefore T is the only independent forcing subset of T .

$$\text{Therefore } f_1(G, T) = n = \beta_0(G).$$

2.2 Observation

There exists a graph G for which $\text{Spec}_1(G) = \{1, \beta_0(G)\}$.

For example, $\text{Spec}_1(K_n \circ K_1) = \{1, n\}$.

2.3 Observation

For any positive integer 'a', there exists a graph G with

$$\text{Spec}_1(G) = \{a, a + 1\}. \text{ For example, take } G = P_4 \cup (a - 1)C_4.$$

2.4 Definition

Let k, n be two positive integers, such that $2 \leq k \leq n$. Let M be a set with n elements. The Kneser graph $H(n, k)$ is defined as the graph whose vertex set V is the set of all subsets of a n -set of cardinality k and two vertices of $H(n, k)$ are adjacent if and only if the corresponding sets are disjoint.

2.4 Theorem

$$\beta_0(H(n, 2)) = n - 1 \text{ and } \text{Spec}_1(H(n, 2)) = \{2\}$$

Proof

Let $S = \{1, 2, \dots, n\}$. Then $V(H(n, 2)) = \{(x_i, x_j) : x_i, x_j \in S, x_i \neq x_j\}$. For any $x \in S$, $T = \{(x, y) : y \in S, y \neq x\}$ is an independent set in $H(n, 2)$. Therefore $\beta_0(H(n, 2)) \geq |T| = n - 1$. Let T be any independent set of $H(n, 2)$. Suppose $(x, y) \in T$. Then $(u, v) \notin T$, for all $u, v \notin \{x, y\}$. There are $(n-2)$ -

elements in $S - \{x, y\}$. Thus $(n-2)C_2$ vertices in $H(n, 2)$ can not be in T . Therefore $|T| \leq nC_2 - \frac{(n-2)(n-3)}{2} = 2n - 3$. Let $(x, y) \in T$. Then any element of T is either (x, z) or (y, z) . Let $(x, z) \in T$.

Case(i): Suppose $(y, z) \in T$. Then $|T| = 3$.

Case(ii): Suppose $(y, z) \notin T$. Then $(y, w) \notin T$, for all $w \neq \{x, y\}$. That is, T does not contain $n - 2$ elements. Therefore $|T| \leq (2n - 3) - (n - 2) = n - 1$. Thus $|T| \leq \max \{3, n - 1\}$. Therefore $\beta(H(n, 2)) \leq n - 1$. Therefore $\beta_0(H(n, 2)) = n - 1$. From the above, we get that the only maximum independent sets of $H(n, 2)$ are $\{u_i, u_j : 1 \leq i \leq n, i \neq j, u_r \text{ fixed}\}$.

Therefore, $\text{Spec}_1(H(n, 2)) = \{2\}$.

2.5 Observation

$$\text{Spec}_1(P_m \square P_n) = \begin{cases} \{0\}, & \text{if } mn \equiv 1 \pmod{2} \\ \{1\}, & \text{otherwise} \end{cases}$$

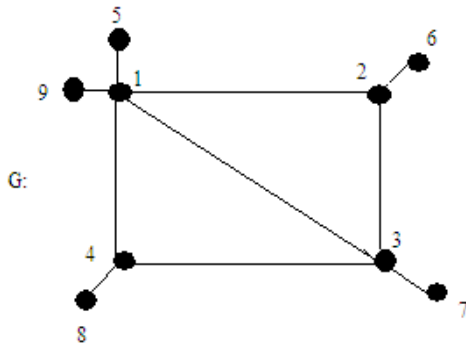
Proof:

If $mn \equiv 1 \pmod{2}$, then $P_m \square P_n$ contains a unique maximum independent set of cardinality $\lfloor \frac{mn}{2} \rfloor$. Therefore $\text{Spec}_1(P_m \square P_n) = \{0\}$. If $mn \equiv 0 \pmod{2}$, then $P_m \square P_n$ contains exactly two maximum independent sets of cardinality $\frac{mn}{2}$ and each vertex is contained in a unique maximum independent set. Therefore $\text{Spec}_1(P_m \square P_n) = \{1\}$. $\text{Spec}_1(P_m \square P_n) = \{0\}$.

2.6 Theorem

Given any three positive integers $a + 1, a + 2, a + 3, a \geq 0$, there exists a graph G with $\text{Spec}_1(G) = \{a + 1, a + 2, a + 3\}$.

Proof:



Consider the graph $H = G \cup aC_4$, where $a \geq 0$. Then $\text{Spec}_1(G) = \{a + 1, a + 2, a + 3\}$. The maximum independent sets of G are $S_1 = \{3, 5, 6, 8, 9\}, S_2 = \{4, 5, 6, 7, 9\}, S_3 = \{5, 6, 7, 8, 9\}$,

$$f_1(G, S_1) = 1, f_1(G, S_2) = 2, f_1(G, S_3) = 3$$

Hence $\text{Spec}_1 G = \{1, 2, 3\}$.

2.7 Result

Let G be the graph obtained by joining exactly two vertices one each from two vertex disjoint complete graphs. Then $\text{Spec}_1(G) = \{2\}$.

Proof

Let K_n and K_m be any two complete graphs with vertex sets $\{u_1, u_2, \dots, u_n\}$ and $\{v_1, v_2, \dots, v_m\}$ respectively. Join a vertex of K_n say $u_i, 1 \leq i \leq n$ with any one vertex of K_m say $v_j, 1 \leq j \leq m$. Let the resulting graph be denoted by G . Now the maximum independent sets are $\{u_1, v_j\}, 1 \leq j \leq m$ or $\dots, \{u_n, v_j\}, 1 \leq j \leq m$ or $\{u_i, v_1\}, 1 \leq i \leq n$ or $\{u_i, v_2\}, 1 \leq i \leq n, \dots, \{u_i, v_m\}, 1 \leq i \leq n$. Hence $\text{Spec}_1(G) = \{2\}$.

2.8 Definition

From a graph G , by Mycielski's construction, one can get a graph $\mu(G)$, with $V(\mu(G)) = V \cup U \cup \{w\}$, where: $V = V(G) = \{v_1, v_2, \dots, v_n\}$; $U = \{u_1, u_2, \dots, u_n\}$ and $E(\mu(G)) = E(G) \cup \{u_i, v : v \in N_G(v_i)\} \cup \{u_i, w : 1 \leq i \leq n\}$. For each $0 \leq i \leq n$, v_i and u_i are called the corresponding vertices of $\mu(G)$.

2.9 Theorem

Let $|V(G)| = n \geq 3$. Then $\beta_0(\mu(G)) > \max \{|V(G)|, 2\beta_0(G)\}$ if and only if G is constructed as follows: Start with $\overline{K_n}$. Choose a subset S of vertices of $\overline{K_n}$ of cardinality s and a subset R of S of cardinality $r \leq s-3$ such that $r + s > n$. Draw at least one edge from each vertex of $V-S$ to the vertices of R . Draw edges such that $\beta_0(G) < \frac{n+t}{2}$ where $t = r + s - n$.

Proof

Let $V(G) = \{u_1, u_2, \dots, u_n\}$ and $V(\mu(G)) = \{u_1, u_2, \dots, u_n, u'_1, u'_2, \dots, u'_n, v\}$. Let $\beta_0(G, \mu(G)) = n + t$ where $2\beta_0(G) < n + t$. Since $|V(G)| \geq 3, v$ does not belong to any β_0 set of $\mu(G)$, then $S \cap V(G) \neq \emptyset$. Let $S \cap V(G) = r \geq 1$. Let $S \cap V'(G) = s \geq 1$. Suppose $S_1 = S \cap V(G)$ and $T' = S \cap V'(G)$. Let T be the set of corresponding vertices of T' . Suppose $T \cap S_1 = \emptyset$. Then $|T \cup S_1| = |T| + |S_1| = |T'| + |S_1| = s + r \leq n$ (since $T \cup S_1$ is a subset of $V(G)$). Therefore $n + t \leq n$ implies $t \leq 0$, a contradiction. Suppose S_1 is not contained in T . Let $T = \{u_{i_1}, u_{i_2}, \dots, u_{i_r}\}$ where $S_1 - T \neq \emptyset$. Let without loss of generality $u_1 \in S_1 - T$. Let $W = T' \cup \{u'_1\}$. Therefore, $W \cup T$ is independent in $\mu(G)$ of cardinality $r + s + 1$, a contradiction. Therefore, $S_1 - T = \emptyset$. That is, $S_1 \subseteq T$. Suppose $r \geq s$. $r + s = n + t \leq 2r$. That is, $n + t \leq 2\beta_0(G)$ (Since $r \leq \beta_0(G)$) .a

contradiction, since $2\beta_0(G) \leq n + t$. Therefore, $r < s$. Thus, $2\beta_0(G) < r + s < 2s$. Therefore, $\beta_0(G) < s$ implies $r \leq \beta_0(G) < s$. Let $T = S \cap V'(G) = \{u_1', u_2', \dots, u_s'\}$. Thus $T = \{u_1, u_2, \dots, u_s\}$. Since $S_1 \subseteq T$, let without loss of generality, $S_1 = \{u_1, u_2, \dots, u_r\}$, $r < s$. Let $U = \{u_{r+1}, \dots, u_s\}$. Then $|U| = s - r$. Let H be the sub graph of G induced by U . Every vertex of U is not adjacent to any vertex of S_1 , for otherwise, S will not be independent. Let $\beta_0(\langle U \rangle) = \alpha \geq 1$. Therefore, $\beta_0(G) \geq \alpha + r$. Therefore, $2\beta_0(G) \geq 2(\alpha + r) < r + s$ implies $\alpha < \frac{s-r}{2}$. But $\alpha \geq 1$. Therefore, $s - r > 2r \geq 2$. Therefore, $s \geq r + 3$. Therefore, $|T - S_1| \geq 3$. since $r \leq s - 3$, $r + s = n + t$ gives $r = n + t - s \leq s - 3$. Therefore, $n + t \leq 2s - 3$ implies $2s \geq n + t + 3$. Therefore, $s \geq \frac{n+t+3}{2}$. But $s \leq n$. Therefore, $\frac{n+t+3}{2} \leq s \leq n$. Therefore, $t + s \leq n$ implies $t \leq n - 3$. Thus $n + t \leq 2n - 3$. Therefore, $\beta_0(\mu(G)) \leq 2n - 3$. Choose any subset R' of $V'(G)$ such that $|R'| = s$ and $s \leq n$. Let R be the corresponding vertices of R' in $V(G)$. let X be a subset of R of cardinality r such that $r + s = n + t$, $t \geq 1$. Since $r \leq s - 3$, there are at least three vertices in $R - X$. Let $U = R - X$ and $Y = V - R$. There is no edge between Y and U . Every vertex of Y is adjacent to at least one vertex of R .

For : otherwise, if there exist a vertex $y \in Y$ such that y is not adjacent to any vertex of R , then $T' \cup \{y\} \cup R$ is an independent set of $\mu(G)$ of cardinality $s + 1 + r$, a contradiction. Draw edges between R and U and between R and Y such that $\beta_0(G) < \frac{n+t}{2}$. Then $\beta(\mu(G)) > \max\{|V(G)|, 2\beta_0(G)\}$. It is easily seen that the construction described in the theorem gives rise to a graph for which $\beta(\mu(G)) > \max\{|V(G)|, 2\beta_0(G)\}$ and if this condition is satisfied then G comes out from K_n in the manner in which the construction is done. Hence the theorem.

2.10 Theorem

Let $n = 10$. To construct a graph G of order 10 such that $\beta_0(\mu(G)) = 12$ and $\beta_0(G) = 5$. consider K_{10} with $V(K_{10}) = \{u_1, u_2, \dots, u_{10}\}$. let $R = \{u_1, u_2, \dots, u_8\}$ and $X = \{u_1, u_2, u_3, u_4\}$. $U = R - X = \{u_5, u_6, u_7, u_8\}$. $Y = V - R = \{u_9, u_{10}\}$. Draw at least one edge from each of u_9, u_{10} to R . Draw edges between X and U and among U such that β_0 of the resulting graph is 5. Then $\{u_1', u_2', \dots, u_8', u_1, u_2, u_3, u_4\}$ is a maximum independent set of $\mu(G)$. $\beta(\mu(G)) = 12 > \max\{|V(G)|, 2\beta_0(G)\}$.

2.11 Theorem

Let G be a graph of order n . Let $\beta_0(\mu(G)) = \max\{2\beta_0(G), n\}$ and let $\beta_0(G) = \frac{n}{2}$. Then $\text{Spec}_1(\mu(G)) = \text{Spec}_1(G) \cup \{\alpha\}$ where α is 1 or 2α is 1 iff there exists a vertex in G which is not contained in any maximum independent set.

Proof:

Suppose $\beta_0(G) = \frac{n}{2}$ (that is n is even). For any maximum independent set S of G , $S \cup S'$ is a maximum independent set of $\mu(G)$, where $S' = \{x' : x \in S\}$. Also V' is a maximum independent set of $\mu(G)$. Since $\beta_0(G) = \frac{n}{2}$, $G \neq \overline{K_n}$. Therefore there exists at least one edge in G .

Case (i): Any vertex in G is contained in a maximum independent set of G . Then any $u' \in V'(\mu(G))$ is contained in two maximum independent sets of $\mu(G)$ namely V' and $I - \{u\} \cup \{u'\}$, where I is the maximum independent set containing u . Also, if u, v are adjacent in V , then u', v' are contained in the unique maximum independent set of $\mu(G)$, namely V' . Therefore the forcing number of V' is 2.

Case (ii): There exists a vertex u in G which is not contained in a maximum independent set of G . Then u' is contained in a unique maximum independent set of $\mu(G)$, namely V' . Therefore the forcing number of V' is 1. Let I be any maximum independent set of $\mu(G)$ different from V' . Then $I = S \cup S'$, where S is a maximum independent set of G . Any forcing independent subset of S in G is also a forcing independent subset of I in $\mu(G)$. Any subset of S' is not a forcing independent subset of I since S' is contained in V' . Let $T \subseteq S$ and $T_1 \subseteq S'$. Suppose $T \cup T_1$ is a forcing independent subset of $S \cup S'$ in $\mu(G)$. Then T is a forcing independent subset of S in G and hence a forcing independent subset of $S \cup S'$ in $\mu(G)$. But $|T| < |T \cup T_1|$, a contradiction. Therefore, the forcing independence number of $S \cup S'$ in $\mu(G)$ coincides with the forcing independence number of $S \in G$. Hence the theorem

2.11 Theorem

$4 \notin \text{Spec}_1(C_5 \circ k_1)$.

Proof:

Any maximum independent set of $C_5 \circ k_1$ is one of the following types

- (i) All pendant vertices.
- (ii) Single vertex from C_5 and 4 pendant vertices attached to the other vertices of C_5 .
- (iii) Two vertices from C_5 and 3 pendant vertices attached to the other vertices of C_5 .

The maximum independent set formed by type (i) has forcing independence number 5. The maximum independent set formed by type (ii) has forcing independence number 2.

The maximum independent set formed by type (iii) has forcing independence number 3. Therefore $\text{Spec}_i(C_5 \circ K_1) = \{1, 3, 5\}$.

Hence $4 \notin \text{Spec}_i(C_5 \circ K_1)$.

2.12 Theorem

$$\text{Spec}_i(C_{2n} \circ K_1) = \{n, n + 1, \dots, (2n - 2), 2n\}.$$

Proof

Consider $C_{2n} \circ K_1$. Any β_0 -set of $C_{2n} \circ K_1$ is one of the following types.

- (i) All pendant vertices of $C_{2n} \circ K_1$.
- (ii) k independent vertices from C_{2n} and $n - k$ pendant vertices attached to other vertices of C_{2n} , ($1 \leq k \leq n$).

The maximum independent set formed by type(i) has forcing independence number $2n$. The maximum independent set of type(ii) has forcing independence number $2n - (k + 1)$, if $k \leq n - 1$ and n if $k = n$.

2.13 Remark

$$\text{Spec}_i(C_{(2n+1)} \circ K_1) = \{n, n + 1, \dots, (2n - 1), (2n + 1)\}.$$

2.14 Remark

$$|\text{Spec}_i(C_n \circ K_1)| = \left\lceil \frac{n}{2} \right\rceil$$

2.15 Result

Let $G = G_1 \cup G_2$. Then $\text{Spec}_i(G_1 \cup G_2) = \{x : x = y + z, y \in \text{Spec}_i(G_1), z \in \text{Spec}_i(G_2)\}$.

Proof:

Let $y \in \text{Spec}_i(G_1)$ and $z \in \text{Spec}_i(G_2)$. Let $\{u_1, u_2, \dots, u_t\}$ be a maximum independent set of G_1 with forcing independent number y and $\{v_1, v_2, \dots, v_s\}$ be a maximum independent set of G_2 with forcing independence number z . Then $\{u_1, u_2, \dots, u_t, v_1, v_2, \dots, v_s\}$ is a maximum independent set of $G_1 \cup G_2$ with forcing independence number $y + z$. Therefore $y + z \in \text{Spec}_i(G_1 \cup G_2)$.

Any maximum independent set I of $G_1 \cup G_2$ is the union of maximum independent set I_1 of G_1 and a maximum independent set I_2 of G_2 . That is $I = I_1 \cup I_2$. Let k be the forcing independence number of I . Any forcing independent subset of I_1 combined with any forcing independent set of I_2 is a forcing independent set of I . Let $I = \{u_1, u_2, \dots, u_{t_1}, v_1, v_2, \dots, v_{t_2}\}$,

where $u_1, u_2, \dots, u_{t_1} \in I_1$ and $v_1, v_2, \dots, v_{t_2} \in I_2$. Let $\{u_{i_1}, u_{i_2}, \dots, u_{i_{r_1}}, v_{j_1}, v_{j_2}, \dots, v_{j_{r_2}}\}$ is a forcing independent subset of I of cardinality k . Clearly $\{u_{i_1}, u_{i_2}, \dots, u_{i_{r_1}}\}$ is forcing in I_1 and $\{v_{j_1}, v_{j_2}, \dots, v_{j_{r_2}}\}$ is forcing in I_2 . Therefore $k = r_1 + r_2$, where $r_1 \in \text{Spec}_i(G_1)$ and $r_2 \in \text{Spec}_i(G_2)$.

2.16 Result

$$\text{Spec}_i(C_n) \cup \text{Spec}_i(\overline{C}_n) = \begin{cases} \{0, 1\} & \text{if } n = 3 \\ \{1, 2\} & \text{if } n \geq 4, n \text{ even} \\ \{2\} & \text{if } n \geq 5 \text{ and } n \text{ odd} \end{cases}$$

Proof:

$$\text{Spec}_i(C_n) = \begin{cases} \{1\} & \text{if } n \text{ is even, } n \geq 4 \text{ (or) } n = 3. \\ \{2\} & \text{if } n \geq 5, n \text{ odd} \end{cases}$$

Let $n \geq 5$. Then $\beta_0(\overline{C}_n) = 2$. Any vertex of \overline{C}_n is contained in two maximum independent set of C_n . Therefore $\text{Spec}_i(C_n) = \{2\}$. Also, $\text{Spec}_i(\overline{C}_4) = \{2\}$.

$$\text{Hence } \text{Spec}_i(C_n) \cup \text{Spec}_i(\overline{C}_n) = \begin{cases} \{0, 1\} & \text{if } n = 3 \\ \{1, 2\} & \text{if } n \geq 4, n \text{ even} \\ \{2\} & \text{if } n \geq 5 \text{ and } n \text{ odd} \end{cases}$$

2.17 Remark

Given any positive integer k , there exists a graph G such that $\beta_0(G) - |\text{Spec}_i(G)| = k$. For: $\beta_0(C_{(2k+2)}) = k + 1$. $|\text{Spec}_i(C_{(2k+2)})| = 1$.

2.18 Remark

$$\text{Spec}_i(G) \leq \beta_0(G) \leq n - \chi(G) + 1 \text{ (since } \chi(G) \leq n - \beta_0(G) + 1).$$

2.19 Theorem

Let G_1 and G_2 be two simple graphs. Let without loss of generality, $\beta_0(G_1) \leq \beta_0(G_2)$. Then $\text{Spec}_i(G_1 + G_2) = \text{Spec}_i(G_2)$.

Proof:

Since $\beta_0(G_1 + G_2) = \max(\beta_0(G_1), \beta_0(G_2))$ and $\beta_0(G_1) \leq \beta_0(G_2)$, we have $\text{Spec}_i(G_1 + G_2) = \text{Spec}_i(G_2)$.

2.20 Result

$\beta_0(G) \in \text{Spec}_i(G)$ if and only if there exists a maximum independent set I in G such that for every vertex u in I , there

exists a vertex $u' \notin I$ such that $(I - \{u\}) \cup u'$ is independent.

Proof:

Suppose there exists a β_0 -set I satisfying the hypothesis. Then given any proper subset J of I , there exists $w \in I - J$ and for this w , there exists $w' \notin I$ such that $I' = (I - \{w\}) \cup \{w'\}$ is a β_0 -set. Clearly J is contained in I as well as I' . Therefore no proper subset of I is a forcing independent set of I . Therefore $\beta_0(G) \in \text{Spec}_1(G)$. Conversely, let $\beta_0(G) \in \text{Spec}_1(G)$. Then there exists a β_0 -set I such that any proper subset of I is not a forcing independent set of I . Let $u \in I$. Then $I - \{u\}$ is not a forcing independent set of I . Therefore $I - \{u\}$ is contained in a β_0 -set I' . Since $|I - \{u\}| = \beta_0 - 1$ and $|I'| = \beta_0$, there exists $u' \in I'$, $u' \notin I$ such that $(I - \{u\}) \cup \{u'\} = I'$.

2.21 Corollary

If $\beta_0(G_1) \in \text{Spec}_1(G)$ and $\beta_0(G_2) \in \text{Spec}_1(G_2)$, then $\beta_0(G_1 \cup G_2) \in \text{Spec}_1(G_1 \cup G_2)$.

2. REFERENCES

[1] P. Afshania, H. Hatamia and E.S. Mahmoodian, On the spectrum of the forced matching number of graphs, Australas. J. Combin. 30(2004), 147.

- [2] Gary Chartrand, Heather Gavlas and Robert C. Vandell, The Forcing domination number of a graph, JCMCC 25 (1997), 161-174.
- [3] Terasa W. Haynes, Stephen T. Hedetniemi, Peter J. Slater, "Fundamentals of Domination in Graphs", Marcel Dekker Inc. (1998).
- [4] P. Adams, M. Mahdian and E.S. Mahmoodian, On the forced matching member of graphs, preprint".
- [5] G. Chartrand and P. Zhang, The forcing geodetic number of a graph". Discuss. math. graph theory, 19(1999), pp:45-58.
- [6] M.E. Riddle, The minimum forcing number for the forus and hyper cube, preprint.
- [7] T.W. Heynes, S.T. Hedetniemi, P.J. Slater, Domination in graphs, Advanced topics, Marcel Dekker., Inc., (1998).
- [8] T.W. Heynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in graphs, Marcel Dekker., Inc., (1998).
- [9] F. Harray, Graph theory, Addition Wesley, Reading Mass (1972).