Colourings in Bipartite Graphs

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ABSTRACT
The concept of X-chromatic partition and hyper independent chromatic partition of bipartite graphs were introduced by Stephen Hedetniemi and Renu Laskar. We find the bounds for X-chromatic number and hyper independent chromatic number of a bipartite graph. The existence of bipartite graph with $\chi_h(G)$=a and $\gamma_Y(G)$=b-1, $\chi_h(G)$=a and $\chi_X(G)$=b where a ≤b are proved. We also prove the existence of bipartite graphs for any three positive integers a, b, c such that c ≥ 2(b-a)+1, there exists a graph G such that $\chi_X(G)$=a, $\chi_X d(G)$=b and |Y|=c. The bipartite theory of Dominator colouring is introduced.

Keywords
X.Chromatic number, hyper independent chromatic number, X-dominator X-colouring of a graph.

1. INTRODUCTION
Bipartite theory of graph introduced by Hedetniemi [4,5] and Renu Laskar states that given any graph problem, say P, there is corresponding problem, say Q, on bipartite graph whose solution gives a solution for the problem P. The concept of X-chromatic and hyper independent chromatic partition introduced in [4,5] are studied. Varities of domination is discussed in the two books of T.W.Haynes[2,3]. The concept of X-dominator X-colouring which is bipartite version of dominator colouring [1] is introduced. Unless otherwise stated, by a graph we mean bipartite graph G=(X, Y, E) with |X|=p without isolates.

2. X-COLOURING IN GRAPHS
In this section, we give the definition of X-colouring of a bipartite graph G as given in [4] and find its bounds.

Definition 2.1: [4] Let G be a graph. Two vertices u, v ∈ X are X-adjacent if they are adjacent to a common vertex in Y. Let $d_Y(x)$ denote the number of vertices X-adjacent to x.

Definition 2.2: [4] Two vertices u, v ∈ X are X-independent if there does not exist a vertex y ∈ Y adjacent to both u and v. A subset S of X is X-independent if every pair of vertices u and v in S of X is X-independent. The maximum cardinality of a X-independent set is called X-independence number of G and is denoted by $\beta_X(G)$.

Definition 2.3: [4] A X-colouring of a bipartite graph G is a partition $\{X_1, X_2, \ldots, X_k\}$ of X into X-independent sets. The X-chromatic number $\chi_X(G)$ of a bipartite graph is the smallest order of an X-colouring of G.

Theorem 2.4: Given positive integers k and p, 2 ≤ k ≤ p, there exists a graph with $\chi_X(G) = k$, |X|=p and |Y| = q (q ≥p).

Proof: Let G be a bipartite graph with bipartition X and Y. The bipartite graph G is defined as follows: X = $\{u_1, u_2, \ldots, u_p\}$, Y= $\{y_1, y_2, \ldots, y_q\}$, k vertices $u_1, u_2, \ldots, u_k$ are X-adjacent to each other through same $y_1 \in Y$. Let u_{k+1} be X-adjacent to each other through different y ∈ Y- $\{y_1\}$ say y_2. u_{k+2} is X-adjacent to u_2 through y_3 where $y_3 \in Y-\{y_1, y_2\}$ and so on. If p-k is greater than k then u_{2k+1} is made X-adjacent to u_1 and so on.

Since k vertices in X are X-adjacent, $\chi_X(G) \geq k$. Let X_i be the set with elements u_i and X-neighbours of u_{i+1} $1 \leq i \leq k-1$. Let X_k be the set with vertices u_k and X-neighbours of u_1. Then, $\{X_1, X_2, \ldots, X_k\}$ forms a X-chromatic partition. Hence, $\chi_X(G) \leq k$. Therefore, $\chi_X(G) = k$ with |X|=p.

Theorem 2.5: In a bipartite graph G, $\chi_X(G) = p$ if and only if G is a graph with every vertex in X is (p-1) X-regular.

Proof: If every vertex in X is (p-1) X-regular then $\{\{x_1\}, \{x_2\}, \ldots, \{x_p\}\}$ is a partition of X into X-independent sets. Therefore, $\chi_X(G) = p$.

Conversely, $\chi_X(G) = p$ then $\{\{x_1\}, \{x_2\}, \ldots, \{x_p\}\}$ is a partition of X into X-independent sets. If there exists a vertex x_i with $d_Y(x_i) < p-1$. Let x_k be the vertex not X-adjacent with x_i. Then, $\{\{x_1\}, \{x_2\}, \ldots, \{x_{i-1}\}, \{x_i, x_k\}, \ldots, \{x_{k-1}\}, \{x_k\}, \ldots, \{x_p\}\}$ is a partition of X into X-independent sets and $\chi_X(G) < p$, a contradiction. Therefore, every vertex is of X-degree (p-1). Hence, every vertex in X is (p-1) X-regular.

Theorem 2.6: Let G be a bipartite graph on |X|=p vertices. Then $\frac{p}{\beta_X} \leq \chi_X(G) \leq p - \beta_X + 1$.

Proof: Let $\{X_1,X_2,\ldots,X_{\chi X}\}$ be a X-chromatic partition of X(G). Then, $|X_i| \le \beta_X(G).$ Therefore, $p = \sum_{i=1}^{\chi_X} |X_i| \le \chi_X(G)\beta_X(G).$ Hence, $\frac{p}{\beta_X} \le \chi_X(G).$ Let D be a β_X-set of G. Let $D=\{x_1,x_2,\ldots,x_{\beta X}\}$. Then, $\Pi = \{D,\{x_{\beta X+1}\},\ldots,\{x_p\}\}$ is a X-chromatic partition of G. Therefore, $\chi_X(G) \le p - \beta_X + 1.$

Hence, $\frac{p}{\beta_X} \le \chi_X(G) \le p - \beta_X + 1.$

Theorem 2.7: Let G be a connected bipartite graph, $\chi(G) = \chi_X(G)$ if and only if G is G_1, P_n, C_{2n}, $n\neq 3$, where G_1 is the graph $G_1=(X_1 \cup X_2, Y, E)$, vertices in X_i i=1,2 are not X-adjacent. At least one vertex in X_1 is X-adjacent to vertices of X_2 through different y in Y.

Proof: If G is G_1, P_n, C_{2n}, $n\neq 3$, then $\chi(G) = \chi_X(G)$

Let $\chi(G) = \chi_X(G)$. Let X_1, X_2 be partition of X into X-independent sets. The X-neighbours of X_1 are in X_2 and vice versa. Since, G is connected, there is a vertex in one partition X-adjacent to all the vertices in other partition, we get G_1 or two vertices in X_1 has same X-neighbour in X_2 which gives P_n, C_{2n}, $n\neq 3$.

3. HYPER COLOURING IN GRAPHS

In this section, we define hyper independent colouring in graphs as given in [2] and find its bounds.

Definition 3.1: [4]A subset S of X is hyper independent set if there does not exist a vertex y in Y such that N(y) is contained in S.

Definition 3.2: [4]A hyper colouring of a graph is a partition of X into hyper independent sets, the hyper independent chromatic number $\chi_h(G)$, is the smallest order of a hyper independent colouring of G.

Theorem 3.3: Given two positive integers, k and p such that $2 \le k \le p$, then there exists a graph G with $\chi_h(G) = k; |Y| = k-2$ and $|X| = p.$

Proof: Let $X = \{x_1, x_2, \ldots ,x_p\}$ be vertices of X. Let the vertices $x_1,x_2,\ldots,x_{p-k+2}$ be adjacent to y_1 in Y. Let x_i be X-adjacent to all other vertices of X through different y in Y (i= p-k+3 to p). Then (k-2) vertices in X is of X-degree (p-1) and (p-k+2) are X-adjacent through same y in Y. Since each of the (k-2) vertices with X-degree (p-1) are X-adjacent through different vertices of Y, no two of them can be in a hyper independent set. Also $x_1,x_2,\ldots,x_{p-k+2}$ cannot be all in the same hyper independent set. Therefore, there are at least two hyper independent set from $x_1,x_2,\ldots,x_{p-k+2}$ and (k-2) hyper independent sets from $x_{p-k+3},x_{p-k+4},\ldots,x_p$. Therefore, $\chi_h(G) \ge k$.

Consider $\Pi = \{\{x_1, x_2,\ldots, x_{p-k+1}\}, \{x_{p-k+2}\},\ldots,\{x_p\}\}$. Then Π is a partition of X into k hyper independent sets. Therefore, $\chi_h(G) \le k$. Hence, $\chi_h(G) = k.$

Definition 3.4: [4] A subset D of X is a Y-dominating set if every y in Y is adjacent to at least one vertex in D. The minimum cardinalilty of a Y-dominating set is called Y-domination number and is denoted by $\gamma_Y(G)$.

Theorem 3.5: [6] Let G be a bipartite graph. A subset D of X is Y-dominating set if and only if X-D is hyper independent set.

Theorem 3.6: Let G be a bipartite graph on |X|=p vertices. Then, $\frac{p}{p-\gamma_Y} \le \chi_h \le \gamma_Y + 1.$

Proof: Let S be a minimum Y-dominating set of G. Then, (X-S) is a hyper independent set. Therefore, $\{X-S,\{x_1\},\{x_2\},\ldots,\{x_{|S|}\}\}$ is a partition of X into hyper independent sets. Hence, $\chi_h \le \gamma_Y + 1.$

Let $\Pi = \{X_1,X_2,\ldots,X_{\chi h}\}$ be a minimum hyper independent partition of X(G). Since, $\gamma_Y(G) + \beta_h(G) = p$, $|X_i| \le \beta_h(G) = p - \gamma_Y(G).$ Therefore, $|X_i| = \sum_{i=1}^{\chi_h} X_i$. That is, $p \le \chi_h(G)(p - \gamma_Y(G))$. Therefore, $\frac{p}{p-\gamma_Y} \le \chi_h$.

Theorem 3.7: Given two positive integers a and b with $2 \le a \le b,$ there exists a graph G with $\chi_h(G) = a,$ $|Y| = (b-a)a + \frac{(a-3)(a-2)}{2} + 1$ and $\gamma_Y(G) = b-1$.

Proof: Let X={$x_1,x_2,\ldots,x_{b-a},u_1,u_2,\ldots,u_{a-2},w_{11},w_{12},w_{21},w_{22},\ldots,w_{(b-a)1},w_{(b-a)2}$}. Let x_1, $x_2,\ldots,x_{b-a}$, be X-adjacent through same y in Y. Let u_i be X-adjacent to $x_2,\ldots,x_{b-a},u_1,u_2,\ldots,u_{i-1}$ through different y in Y (i=2 to a-2). Attach P_4 to every vertex x_i (i=1 to b-a) and let the vertices of this P_4 X-adjacent to x_i be w_{i1}, w_{i2}. Let G be the resulting graph. Clearly, $\chi_h(G) \geq a$.

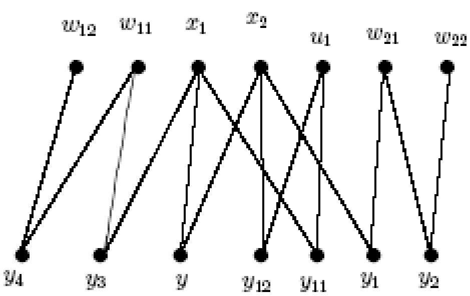
Π={{ $x_1,x_2,\ldots,x_{b-a-1},w_{12},w_{22},\ldots,w_{(b-a-1)2},w_{(b-2)1}$},

{$x_{b-a},w_{(b-a)2},w_{11},w_{21},\ldots,w_{(b-a-1)1}$},{$u_1$},{$u_2$},…,{$u_{a-2}$}} is a partition of X into hyper independent sets. Therefore, $\chi_h(G) \leq a$. Therefore, $\chi_h(G) = a$.

Clearly S={$w_{11},w_{21},\ldots,w_{(b-a)1},x_1,u_1,u_2,\ldots,u_{a-2}$} is a Y-dominating set with $\gamma_Y(G) = b - a + 1 + a - 2 = b - 1$.

Example 3.8: Construction of graph with $\chi_h(G) = 3$ and $\gamma_Y(G) = 4$.

The partition Π={{x_1,w_{12},w_{22}},{x_2,w_{11},w_{12}},{u_1}} is a hyper independent set of G and S={w_{11},w_{12},x_1,u_1} is a Y-dominating set of G.



Observation 3.9: Every X-independent set is a hyper independent set. Therefore, every X-chromatic partition is a hyper independent chromatic partition of G. Therefore, $\chi_h(G) \leq \chi_X(G)$.

Theorem 3.10: Given positive integers a>1 and b$\geq$a, there exists a graph G with $\chi_h(G) = a$ and $\chi_X(G) = b$.

Proof: Let us assume a<b. Let G be a graph with bipartition X and Y. Let X={$x_1,x_2,\ldots,x_b$} and Y ={$y_1,y_2,\ldots,y_q$}. Let the vertices $x_1,x_2,\ldots,x_b$ be X-adjacent through same y in Y. Therefore, $\chi_X(G) = b$, Let x_i be X-adjacent to all x_j, b-a+2 $\leq$ i $\leq$b; i$\neq$j; 1$\leq$j$\leq$b through different y in Y. Clearly, $\chi_h(G) \geq a$. Let Π = {{ $x_1,x_2,\ldots,x_{b-a+1}$}, {x_{b-a+2}},…,{x_b}}. Then, Π is a partition of X into hyper independent set of G. Therefore, $\chi_h(G) \leq a$.

Let a=b. Let X={$x_1,x_2,\ldots,x_b$}. Make every vertex in X, (b-1) X-regular through different y in Y. Therefore, $\chi_h(G) = a$ and $\chi_X(G) = b$.

4. X-DOMINATOR COLOURING OF A BIPARTITE GRAPH

Domination and colouring have nice interactions in graphs. Partitioning the vertex set of a graph into subsets with desired property is an interesting problem. Colouring problem is also a partition problem. We initiate the study of X-dominator X-colouring of graphs.

Definition 4.1: [1] A dominator colouring of G is defined to be a proper colouring in which every vertex dominates a colour class. The dominator chromatic number, $\chi_d(G)$ is the minimum number of colours that allows a dominator colouring of G.

Definition 4.2: [4] A subset S of X is an X-dominating set if every vertex in X-D is X-adjacent to at least one vertex in D. A X-dominating set S is a minimal X-dominating set if no proper subset of S is X-dominating set. The minimum cardinality of a minimal X-dominating set is called the X-domination number of G and is denoted by $\gamma_X(G)$.

Definition 4.3: Let G=(X,Y,E) be a graph. A X-independent partition of X, $\Pi = (X_1, X_2, \ldots, X_\chi)$ is called X-dominator X-colouring of G if every vertex x in X, X-dominates some colour class in Π. We assume x in X X-dominates {x}. The smallest cardinality of X-dominator X-colouring of G is called X-dominator, X-colouring number of G and is denoted by $\chi_X d(G)$.

Example 4.4: For any bipartite graph G=(X,Y,E) with vertex X={$x_1,x_2,x_3,\ldots,x_p$} then Π={{x_1},{x_2}, …, {x_p}} is clearly a X-dominator,X-colouring of G. Therefore, X-dominator X-colouring of every graph exists.

Remark 4.5: $1 \leq \chi_X d(G) \leq$ p.

Remark 4.6: Since any X-dominator, X-colouring of G is a X-colouring of G, we have $\chi_X(G) \leq \chi_X d(G)$.

Theorem 4.7: Let G=(X,Y,E) be a bipartite graph with |X|=p, |Y|=q. Then, $\chi_X d(G) = p$ if and only if there exists a vertex in Y of degree p or $q \geq p-1$.

Proof: If there exists a vertex in Y of degree p then $\chi_X d(G) = p$. If $q \geq p-1$ then $\sum_{y \in Y} d(y) \geq p(p-1)$. Every vertex in X is X-adjacent to other vertices in X. Therefore, $\chi_X d(G) = p$.

Conversely, suppose $\chi_X d(G) = p$. Let every point y in Y be such that d(y) < p and q < p-1. Let $x_1, x_2, \ldots, x_r$ be the vertices in X which are adjacent to y where r = d(y) < p. There exists x_{r+1} which is not X-adjacent to any x_i $1 \leq i \leq r$ say x_r. Then $\{\{x_1, x_2, \ldots, x_{r-1}\}, \{x_r, x_{r+1}\}, \ldots, \{x_p\}\}$ is a X-dominator, X-colouring of G. A contradiction to $\chi_X d(G) = p$ Therefore, there exists a vertex in Y of degree p or $q \geq p-1$.

Theorem 4.8: Let G=(X,Y,E) be a bipartite graph with |X|=p, |Y|=q. Then $\chi_X d(G) = 1$ if and only if $G \cong pK_{1,a}$.

Proof: If $G \cong pK_{1,a}$ then $\chi_X d(G) = 1$.

Conversely, if $\chi_X d(G) = 1$ then every vertex in X are X-independent. Therefore, $G \cong pK_{1,a}$.

Theorem 4.9: Let G be a connected bipartite graph. Then $\max\{\chi_X(G), \gamma_X(G)\} \leq \chi_X d(G) \leq \chi_X(G) + \gamma_X(G)$.

Proof: A X-dominatorX-colouring must be a proper X-colouring, whe have $\chi_X(G) \leq \chi_X d(G)$. Also, let C be a minimum X-dominator, X-colouring of G. For each colour class of X, let x_i be a vertex in the class I, with $1 \leq i \leq \chi_X d(G)$. Let $S = \{x_i : 1 \leq i \leq \chi_X d(G)\}$. Let v belong to X(G). Then v X-dominates a colour class i, for some i $(1 \leq i \leq \chi_X d(G).)$ Then v is X-dominated by the colour class i, in particular x_i. Therefore, $\gamma_X(G) \leq |S| = \chi_X d(G)$. Hence, $\max\{\chi_X(G), \gamma_X(G)\} \leq \chi_X d(G)$.

Let C be a proper X-colouring of X with $\chi_X(G)$ colours. Assign colours $\chi_X(G)+1$, $\chi_X(G)+2, \ldots, \chi_X(G) + \gamma_X(G)$ to the vertices of a minimum X-dominating set of G leaving the rest of the vertices coloured as before. This is a X-dominator, X-colouring of G, since it is still a proper X-colouring and the X-dominating set provides the colour class that every vertex X-dominates.

Theorem 4.10: Given positive integers a, b, c such that $c \geq 2(b-a)+1$, there exists a graph G such that $\chi_X(G) = a; \chi_X d(G) = b$ and $|Y| = c$.

Proof: Let G=(X,Y,E) be the graph. Let X= $\{x_{11}, x_{12}, \ldots, x_{1a}, x_{21}, x_{22}, \ldots, x_{2a}, \ldots, x_{(b-a+1)1}, \ldots, x_{(b-a+1)a}\}$. Let Y=$\{y_1, y_{12}, y_2, \ldots, y_{(b-a)(b-a+1)}, y_{b-a+1}\}$. The edges E(G)=$\{x_{11}y_1, x_{12}y_1, \ldots, x_{1a}y_1, x_{1a}y_{12}, \ldots, x_{(b-a+1)a}y_{b-a+1}\}$. Then, $\{\{x_{11}, x_{21}, \ldots, x_{(b-a+1)}\}, \{x_{12}, x_{22}, \ldots, x_{(b-a+1)2}, \ldots, \{x_{1a}, \ldots, x_{(b-a+1)a}\}\}$ is a minimum partition of X into X-independent sets. Therefore, $\chi_X(G) = a$.

The partition $\{\{x_{11}, x_{21}, \ldots, x_{(b-a+1)}\}, \ldots, \{x_{i1}\}, \{x_{i2}\}, \ldots, \{x_{ia}\}, \ldots, \{x_{1a}, \ldots, x_{(b-a+1)a}\}\}$ is a minimum X-dominator, X-colouring of G. $\chi_X d(G) = a-1+b-a+1 = b$ and $|Y| = b-a+1+b-a = 2(b-a)+1$. Therefore, $c \geq 2(b-a)+1$.

Theorem 4.11: Given a positive integer k, there exists a graph with $\chi_X d(G) - \chi_h(G) = k$.

Proof: Let the vertices of X be $\{x_{11}, x_{12}, x_{13}, \ldots, x_{(k-1)1}, x_{(k-1)2}, x_{(k-1)3}, x_{(k-1)4}\}$. x_{i1}, x_{i2}, x_{i3}, x_{i4} are adjacent to y_i i=1 to k-1. x_{i4} and $x_{(i+1)1}$ are adjacent to $y_{i(i+1)}$ i = 1 to k-2.

$\Pi = \{\{x_{11}, x_{12}, \ldots, x_{(k-1)3}\}, \{x_{14}, x_{24}, \ldots, x_{(k-1)4}\}\}$ is a partition of X into hyper independent sets. Therefore, $\chi_h(G) = 2$.

$\Pi = \{\{x_{11}, x_{21}, \ldots, x_{(k-1)1}\}, \{x_{12}, x_{22}, \ldots, x_{(k-1)2}, x_{13}, x_{23}, \ldots, x_{(k-1)3}\}, \{x_{14}\}, \ldots, \{x_{(k-1)4}\}\}$ is a partition of X into X-dominator X-colouring of G. Therefore, $\chi_X d(G) = k+2$. Hence, $\chi_X d(G) - \chi_h(G) = k$.

5. BIPARTITE THEORY OF DOMINATOR COLOURING

The Bipartite graph VE(G) constructed from an arbitrary graph G=(V,E) is defined as in [2]. VE(G) = (V,E,F) is defined by the edges F={(u,e): e=(u,v) in E}. VE(G) ≅ S(G), where S(G) denotes the subdivision graph of G.

Theorem 5.1: For any graph G, $\chi_X d(VE(G)) = \chi_d(G)$.

Proof: Let $\chi_X d(VE(G)) = k$. There exists a partition of X, $\Pi = \{X_1, X_2, \ldots, X_k\}$ of X-independent sets such that every vertex x in X X-dominates some colour class in Π. In G, $\Pi^1 = \{X_1, X_2, \ldots, X_k\}$ is a partition of V into independent sets such that every v in V dominates some colour class in Π^1. Therefore, Π^1 is a dominator colouring of G. Hence, $\chi_d(G) \le k = \chi_X d(VE(G))$.

Conversely, let $\chi_d(G) = r$. Let $\Pi = \{V_1, V_2, \ldots, V_k\}$ is a partition of V into independent sets such that every vertex v in V dominates some colour class in Π. In VE(G), $\Pi^1 = \{V_1, V_2, \ldots, V_k\}$ is a partition of X in to X-independent sets and every x in X X-dominates some colour class in Π^1. Therefore, Π^1 is a X-dominator, X-colouring of VE(G). Hence, $\chi_X d(VE(G)) \le r = \chi_d(G)$. Therefore, $\chi_X d(VE(G)) = \chi_d(G)$.

The graph EV(G)=(E,V,K) is defined by edges K={(e,u) : e=uv in E}.

Theorem 5.2: For any graph G, $\chi_X d(EV(G)) = \chi^1_d(G)$.

Proof: Let $\chi_X d(EV(G)) = k$. There exists a partition of X, $\Pi = \{X_1, X_2, \ldots, X_k\}$ of X-independent sets such that every vertex x in X, X-dominates some colour class in Π. In G, $\Pi^1 = \{X_1, X_2, \ldots, X_k\}$ is a partition of E into independent sets such that every e in E dominates some colour class in Π^1. Therefore, Π^1 is an edge dominator colouring of G. Hence, $\chi_d(G) \le k = \chi_X d(EV(G))$.

Conversely, let $\chi_d^{\,1}(G) = r$. Let $\Pi = \{E_1, E_2, \ldots, E_k\}$ is a partition of E into independent sets such that every edge e in E dominates some colour class in Π. In EV(G), $\Pi^1 = \{E_1, E_2, \ldots, E_k\}$ is a partition of X in to X-independent sets and every x in X X-dominates some colour class in Π^1. Therefore, Π^1 is a X-dominator, X-colouring of EV(G). Hence, $\chi_X d(EV(G)) \le r = \chi^1{}_d(G)$. Therefore, $\chi_X d(EV(G)) = \chi^1{}_d(G)$.

Let V^1 be a copy of the vertices V of G. (a) The graph $VV(G)=(V,V^1,E^1)$ is defined by the edges $E^1=\{(u,v^1):(u,v) \text{ in } E\}$.

From a graph G=(V,E) the graph G2 and G^2 can be constructed as follows: G2 and G^2 have the same vertex set as G, with two vertices u and v adjacent to G2 if and only if they have a common neighbor in G, and adjacent in G^2 if and only if $d(u,v) \le 2$ in G.

Theorem 5.3: For any graph G, $\chi_X d(VV(G)) = \chi_d(G2)$.

Proof: Let $\chi_X d(VV(G)) = k$. There exists a partition of X, $\Pi = \{X_1, X_2, \ldots, X_k\}$ of X-independent sets such that every vertex x in X X-dominates some colour class in Π. Any two vertices in the same partition are not X-adjacent. In G2, any two vertices in X_i are not adjacent. $\Pi^1 = \{X_1, X_2, \ldots, X_k\}$ is a partition of V(G2) into independent sets such that every v in V(G2) dominates some colour class in Π^1. Therefore, Π^1 is a dominator colouring of G2. Hence, $\chi_d(G2) \le k = \chi_X d(VV(G))$.

Conversely, let $\chi_d(G2) = r$. Let $\Pi = \{V_1, V_2, \ldots, V_k\}$ is a partition of V(G2) into independent sets such that every vertex v in V(G2) dominates some colour class in Π. In G, any two vertices in the same partition do not have a common neighbor. In VV(G), any two vertices in V_i, i=1 to k are not X-adjacent. Hence, in VV(G), $\Pi^1 = \{V_1, V_2, \ldots, V_k\}$ is a partition of X in to X-independent sets and every x in X X-dominates some colour class in Π^1. Therefore, Π^1 is a X-dominator, X-colouring of VV(G). Hence, $\chi_X d(VV(G)) \le r = \chi_d(G2)$. Therefore, $\chi_X d(VV(G)) = \chi_d(G2)$.

The graph $VV^+(G)=(V,V^1,E^+)$ contains the edges E^1 of the graph VV together with the edges $\{(u,u^1):u \text{ in } V\}$.

Theorem 5.4: For any graph G, $\chi_X d(VV^+(G)) = \chi_d(G^2)$.

Proof: Let $\chi_X d(VV^+(G)) = k$. There exists a partition of X, $\Pi = \{X_1, X_2, \ldots, X_k\}$ of X-independent sets such that every vertex x in X, X-dominates some colour class in Π. In G^2, $\Pi^1 = \{X_1, X_2, \ldots, X_k\}$ is a partition of $V(G^2)$ into independent sets such that every v in $V(G^2)$ dominates some colour class in Π^1. Therefore, Π^1 is a dominator colouring of G^2. Hence, $\chi_d(G^2) \le k = \chi_X d(VV^+(G))$.

Conversely, let $\chi_d(G^2) = r$. Let $\Pi = \{V_1, V_2, \ldots, V_r\}$ is a partition of $V(G^2)$ into independent sets such that every vertex v in $V(G^2)$ dominates some colour class in Π. In $VV^+(G)$, $\Pi^1 = \{V_1, V_2, \ldots, V_r\}$ is a partition of X in to X-independent sets and every x in X X-dominates some colour class in Π^1. Therefore, Π^1 is a X-dominator, X-colouring of $VV^+(G)$. Hence, $\chi_X d(VV^+(G)) \le r = \chi_d(G^2)$. Therefore, $\chi_X d(VV^+(G)) = \chi_d(G^2)$.

Corollary 5.5: For any graph G,

(i) $\mathrm{Max}\{\gamma(G), \chi(G)\} \le \chi_d(G) \le \chi(G) + \gamma(G)$

(ii) $\mathrm{Max}\{\gamma^1(G), \chi^1(G)\} \le \chi^1{}_d(G) \le \gamma^1(G) + \chi^1(G)$.

6. CONCLUSION

The bounds of X-chromatic number and hyper independent number are given. We introduce the bipartite theory of dominator colouring of a graph G. Given any three positive integers a, b, c such that $c \ge 2(b-a)+1$, we have proved the existence of a bipartite graph G such that $\chi_X(G)=a$, $\chi_X d(G)=b$ and $|Y|=c$.

7. REFERENCES

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