Anti M-Fuzzy Subgroup and its Lower Level M-Subgroups

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ABSTRACT

In this paper, we introduce the concept of an anti M-fuzzy subgroup of an M-group and lower level subset of an anti M-fuzzy subgroup and discussed some of its properties.

Keywords

M-group, fuzzy set, anti fuzzy subgroup, anti M-fuzzy subgroup of an M-group, level subset, and level M-subgroups.

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1. INTRODUCTION

The concept of fuzzy sets was initiated by Zadeh. Then it has become a vigorous area of research in engineering, medical science, social science, graph theory etc. Rosenfeld [5] gave the idea of fuzzy subgroups. Biswas .R [1] introduced the concept of anti fuzzy subgroups. N.Palaniappan , R.Muthuraj [6] discussed some of the properties of anti fuzzy group and its lower level subgroups. Author N. Jacobson [4] introduced the concept of M-group, M-subgroup. In this paper, we introduce the concept of an anti M-fuzzy subgroup of an M-group and lower level subset of an anti M-fuzzy subgroup and discussed some of its properties.

2. PRELIMINARIES

This section contains some definitions and results to be used in the sequel.

2.1 Definition

Let X be any non empty set. A fuzzy subset A of X is a function $A : X \to [0,1]$.

2.2 Definition

Let G be a group. A fuzzy subset A of G is called an anti fuzzy subgroup if for $x, y \in G$,

(i) $A(xy) \leq \max\{A(x), A(y)\}$,

(ii) $A(x^{-1}) = A(x)$.

2.1 Example

Let G be the Klein 4 – group. Then

$G = \{e, a, b, ab\}$, $a^2 = b^2 = e$ with $ab = ba$.

Define a fuzzy subset A of G by,

$A(e) = 0.2, A(a) = 0.3, A(b) = 0.4, A(ab) = 0.4.$

Clearly A is an anti fuzzy subgroup of G.

2.3 Definition

A group with operators is an algebraic system consisting of a group G , a set M and a function defined in the product set $M \times G$ and having values in G such that, if ma denotes the element in G determined by the element a of G and the element m of M, then $m(ab) = (ma)(mb)$ holds for all $a, b \in G$ and $m \in M$. We shall use the phrases "G is an M-group" to a group with operators.

A subgroup H of an M-group G is said to be an M-subgroup if $mx \in H$ for all $m \in M$ and $x \in H$.

2.4 Definition

Let G be an M-group and A be an anti fuzzy subgroup of G . Then A is called an anti M-fuzzy subgroup of G if for all $x \in G$ and $m \in M$, then $A(mx) \leq A(x)$.

2.2 Example

Let A be a fuzzy subset of an M-group G, then A is defined by

$$A(x) = \begin{cases} 0.3 & \text{if } x \in G \\ 0.8 & \text{otherwise.} \end{cases}$$

Then it is easy to verify that A is an anti M-fuzzy subgroup of G.

3. PROPERTIES OF AN ANTI M- FUZZY SUBGROUPS

In this section, we discuss some of the properties of anti M-fuzzy subgroup.

3.1 Theorem

If A is an anti M-fuzzy subgroup of an M-group G, then for any $x,y \in G$ and $m \in M$, $A(m(xy)) \leq \max\{A(mx), A(my)\}$.

Proof

Given that A is an anti M-fuzzy subgroup of an M-group G.

Then , $A(m(xy)) = A((mx)(my))$

$\leq \max \{ A(mx), A(my)\}$,

Hence, $A(m(xy)) \leq \max \{ A(mx), A(my)\}$.

3.2 Theorem

Let H be an M-subgroup of an M-group G. Define a fuzyy subset A of G by

$$A(x) = \begin{cases} t_1 & \text{if } x \in H \\ t_2 & \text{otherwise.} \end{cases}$$

For all $x \in G$ and $t_1 < t_2$, $t_1, t_2 \in [0,1]$. Then A is an anti M-fuzzy subgroup of G.

Proof

Assume that H be an M-subgroup of an M-group G.
Let $x,y \in G$.

i. If $x, y \in H$, then $xy \in H$.
   Clearly, $A(x) = t_1$ and $A(y) = t_1$ and $A(xy) = t_1$.
   Hence $A(xy) = t_1 = \max \{ A(x), A(y) \}$.
ii. If $x \notin H$ or $y \notin H$, then $xy \notin H$.
   Therefore, $A(x) = t_2$ or $A(y) = t_2$ and $A(xy) = t_2$.
   Hence $A(xy) = t_2 = \max \{ A(x), A(y) \}$.
iii. If $x \notin H$ and $y \notin H$, then $xy \in H$ or $xy \notin H$.
   Therefore, $A(x) = t_2$ and $A(y) = t_2$ and $A(xy) = t_1$ or $A(xy) = t_2$.
   Hence $A(xy) \leq t_2 = \max \{ A(x), A(y) \}$.
iv. If $x \in H$, then $x^{-1} \in H$.
   Hence $A(x) = t_1 = A(x^{-1})$.
v. If $x \notin H$, then $x^{-1} \notin H$.
   Hence $A(x) = t_2 = A(x^{-1})$.

Clearly, A is an anti fuzzy subgroup of G.

Since H is an M-subgroup of G, we have $mx \in H$ for all $m \in M$ and $x \in H$.Therefore, $A(mx) = t_1 = A(x)$.

If $x \notin H$, then $A(mx) = t_2 = A(x)$.

Hence A is an anti M-fuzzy subgroup of G.

3.3 Theorem

Let A be an anti M-fuzzy subgroup of an M-Group G with identity e. Then $A(m(xy^{-1})) = A(e) \Rightarrow A(mx) = A(my)$ for all x ,y in G .

Proof

Given that A is an anti M-fuzzy subgroup of an M-Group G and $A(m(xy^{-1})) = A(e)$.
Then for all x, y in G,

$$\begin{aligned} A(mx) &= A(m(x(y^{-1}y))) \\ &= A(m(xy^{-1})y)) \\ &\leq \max \{ A(m(xy^{-1})), A(my)\}, \\ &= \max \{ A(e), A(my)\} \\ &= A(my). \end{aligned}$$

That is, $A(mx) \leq A(my)$.

Now, $A(my) = A((my)^{-1})$, as A is an anti M-fuzzy subgroup of G.

$$\begin{aligned} &= A(my^{-1}), \\ &= A(m((ey^{-1}))) \\ &= A(m((x^{-1}x)y^{-1})), \\ &= A(m(x^{-1}(xy^{-1}))), \\ &\leq \max\{ A(mx^{-1}), A(m(xy^{-1}))\}, \\ &= \max \{A((mx)^{-1}), A(e)\}, \\ &= \max \{A(mx), A(e)\}, \\ &= A(mx). \end{aligned}$$

That is, $A(my) \leq A(mx)$.

Hence $A(mx) = A(my)$.

3.4 Theorem

A is an anti M- fuzzy subgroup of an M-Group if and only if $A(m(xy^{-1})) \leq \max \{A(mx), A(my)\}$, for all x, y in G.

Proof

Let A be an anti M-fuzzy subgroup of an M-group G. Then for all x, y in G,
$A(m(xy)) \leq \max \{A(mx), A(my)\}$ and $A(x) = A(x^{-1})$.

$$\begin{aligned} \text{Now,}\quad A(m(xy^{-1})) &\leq \max \{ A(mx), A(my^{-1})\} \\ &= \max \{A(mx), A((my)^{-1})\} \\ &= \max \{ A(mx), A(my)\} \end{aligned}$$

if and only if $A(m(xy^{-1})) \leq \max \{ A(mx), A(my)\}$.

3.5 Theorem

The union of any two anti M-fuzzy subgroups of an M-group G is always an anti M-fuzzy subgroup of an M-group G.

Proof

Let G be an M-group. Let A and B be any two anti M-fuzzy subgroups of G.
Clearly $(A \cup B)$ is an anti fuzzy subgroup of G.
Now, we have to prove that $(A \cup B)$ is an anti M-fuzzy subgroup of G.
Now, $(A \cup B)(mx) = \max \{ A(mx), B(mx) \}$,
$\leq \max \{ A(x), B(x) \}$, as A and B are anti M-fuzzy subgroups of G,
$= (A \cup B)(x)$.

Hence $(A \cup B)(mx) \leq (A \cup B)(x)$.

That is, $(A \cup B)$ is an anti M-fuzzy subgroup of G.

3.6 Theorem

The intersection of any two anti M-fuzzy subgroups of an M-group G is always an anti M-fuzzy subgroup of an M-group G.

Proof

Let G be an M-group. Let A and B be any two anti M-fuzzy subgroups of G.

Clearly $(A \cap B)$ is an anti fuzzy subgroup of G.

Now, we have to prove that $(A \cap B)$ is an anti M-fuzzy subgroup of G.

Now, $(A \cap B)(mx) = \min \{ A(mx), B(mx) \}$,

$\leq \min \{ A(x) , B(x) \}$, as A and B are anti M-fuzzy subgroups of G,

$= (A \cap B)(x)$.

Hence $(A \cap B)(mx) \leq (A \cap B)(x)$.

That is, $(A \cap B)$ is an anti M-fuzzy subgroup of G.

REMARK

Arbitrary union and arbitrary intersection of anti M-fuzzy subgroups are anti M-fuzzy subgroups.

4. PROPERTIES OF LOWER LEVEL SUBSETS OF AN ANTI M-FUZZY SUBGROUP OF AN M-GROUP

In this section, we introduce the concept of lower level subset of an anti M-fuzzy subgroup of an M-group and discuss some of its properties.

4.1 Definition

Let A be a fuzzy subset of S. For $t \in [0, 1]$, the lower level subset of A is the set, $\bar{A}_t = \{ x \in S : A(x) \leq t \}$.

4.1 Theorem

Let A be a fuzzy subset of an M-group G. If A is an anti M-fuzzy subgroup of G, then the lower level subsets $\bar{A}_t$, $t \in$ Im(A) are M-subgroups of G.

Proof

Let $t \in$ Im(A) and $x , y \in \bar{A}_t$.

Then $A(x) = t$ and $A(y) = t$.

Given that A is an anti M-fuzzy subgroup of G.

Therefore, A is an anti fuzzy subgroup of G.

Hence $A(xy) \leq \max \{ A(x) , A(y) \} = t$.

That is, $A(xy) \leq t$.

That is, $xy \in \bar{A}_t$.

Moreover, if $x \in \bar{A}_t$, then $A(x^{-1}) = A(x) \leq t$.

Then, $x^{-1} \in \bar{A}_t$.

Hence $\bar{A}_t$ is a subgroup of G.

Now, for any $x \in \bar{A}_t$ and $m \in M$, then

$A(mx) \leq A(x) \leq t$.

Hence $mx \in \bar{A}_t$.

Hence $\bar{A}_t$ is an M-subgroup of G.

4.2 Theorem

Let A be a fuzzy subset of an M-group G. If the lower level subsets $\bar{A}_t$, $t \in$ Im(A) are M-subgroups of G, then A is an anti M-fuzzy subgroup of G.

Proof

Let the lower level M-subsets $\bar{A}_t$, $t \in$ Im(A) are M-subgroups of G.

If there exist $x_0 , y_0 \in G$ such that

$A(x_0y_0) \geq \max \{A(x_0) , A(y_0)\}$.

Let $t_0 = (A(x_0y_0) + \min\{A(x_0) , A(y_0)\}) / 2$,

we have $A(x_0y_0) > t_0 > \max \{A(x_0) ,A(y_0)\}$.

It follows that $x_0 , y_0 \in \bar{A}_{t0}$,but $x_0y_0 \notin \bar{A}_{t0}$.

Which is a contradiction to $\bar{A}_{t0}$ is an M-subgroup of G.

Hence $A(xy) \leq \max \{ A(x) , A(y)\}$.

Similarly, we have $A(x^{-1}) = A(x)$.

Hence A is an anti fuzzy subgroup of G.

Now, suppose , for $m \in M$ and $x \in G$, $A(mx) > A(x)$.

Let $t_0 = (A(mx) + A(x)) / 2$.

Then, $A(mx) > t_0 > A(x)$.

That is, for $m \in M$ and $x \in G$, then $x \in \bar{A}_{t0}$, but $mx \notin \bar{A}_{t0}$.

Which is a contradiction to $\bar{A}_{t0}$ is an M-subgroup of G.

Hence $A(mx) \leq A(x)$.

Hence A is an anti M-fuzzy subgroup of G.

4.2 Definition

Let A be an anti M-fuzzy subgroup of an M-group G. Then the M-subgroups $\bar{A}_t$, for $t \in [0,1]$ and $t \geq A(e)$, are called lower level M-subgroups of A.

4.3 Theorem

Let A be an anti M-fuzzy subgroup of a group G. If two lower level subgroups A_{t1}, A_{t2}, for, $t_1, t_2 \in [0,1]$ and $t_1 , t_2 \geq A(e)$ with $t_1 < t_2$ of A are equal then there is no x in G such that $t_1 < A(x) \leq t_2$.

Proof

Let $A_{t1} = A_{t2}$.

Suppose there exists a $x \in G$ such that $t_1 < A(x) < t_2$ then $A_{t1} \subseteq A_{t2}$.

Then $x \in A_{t2}$,but $x \notin A_{t1}$,which contradicts the assumption that, $A_{t1} = A_{t2}$.

Hence there is no x in G such that $t_1 < A(x) \leq t_2$.

Conversely let, there is no x in G such that $t_1 < A(x) \leq t_2$.

Since $t_1 < t_2$, then $A_{t1} \subseteq A_{t2}$.

But there is no x in G such that $t_1 < A(x) \leq t_2$, $A_{t2} \subseteq A_{t1}$.

Hence $A_{t1} = A_{t2}$.

4.4 Theorem

A fuzzy subset A of G is an anti M-fuzzy subgroup of an M- group G if and only if the lower level subsets $\bar{A}_t$, $t \in Im(A)$ are M-subgroups of G.

Proof It is clear.

4.5 Theorem

Any M-subgroup H of a M-group G can be realized as a lower level M-subgroup of some anti M-fuzzy subgroup of G.

Proof

Let A be a fuzzy subset and $x \in G$.

Define,

$$A(x) = \begin{cases} 0 & \text{if } x \in H \\ t & \text{if } x \notin H \text{, where } t \in (0,1]. \end{cases}$$

We shall prove that A is an anti M-fuzzy subgroup of G.

Let x , y ∈ G.

i. Suppose $x, y \in H$, then $xy \in H$ and $xy^{-1} \in H$.

$A(x) = 0$, $A(y) = 0$, $A(xy) = 0$ and $A(xy^{-1}) = 0$.

Hence $A(xy^{-1}) \leq \max \{ A(x) , A(y) \}$.

ii. Suppose $x \in H$ and $y \notin H$, then $xy \notin H$ and $xy^{-1} \notin H$.

$A(x) = 0$, $A(y) = t$ and $A(xy^{-1}) = t$

Hence $A(xy^{-1}) \leq \max \{ A(x) , A(y) \}$.

iii. Suppose $x, y \notin H$, then $xy^{-1} \in H$ or $xy^{-1} \notin H$.

$A(x) = t$, $A(y) = t$ and $A(xy^{-1}) = 0$ or t.

Hence $A(xy^{-1}) \leq \max \{ A(x) , A(y) \}$.

Thus in all cases, A is an anti fuzzy subgroup of G.

i. Now, for all $m \in M$ and $x \in H$, then $mx \in H$.

$A(x) = 0$ and $A(mx) = 0$.

Hence $A(mx) \leq A(x)$.

ii. Now, for all $m \in M$ and $x \notin H$, then $mx \in H$ or $mx \notin H$.

$A(x) = t$ and $A(mx) = 0$ or t.

Then, $A(mx) \leq A(x)$.

Thus in all cases, A is an anti M-fuzzy subgroup of G.

For this anti M- fuzzy subgroup, $\bar{A}_t = H$.

REMARK

As a consequence of the Theorem 4.3 and 4.4, the lower level M-subgroups of an anti M-fuzzy subgroup A of an M-group G form a chain. Since $A(me) = A(m) \leq A(mx)$ for all x in G and m in M and $A(m) = t_0$, we have the chain :

$\{e\} = A_{t0} \subset A_{t1} \subset A_{t2} \subset \ldots \subset A_{tn} = G$, where $t_0 < t_1 < t_2 < \ldots\ldots < t_n$.

5. CONCLUSION

In this paper , we define a new algebraic structure of anti M-fuzzy subgroup of an M-group and lower level subset of an anti M-fuzzy subgroup and studied some of its properties. Futher, we wish to define the anti M-fuzzy normal subgroup of an M-group and its lower level subsets and also the same in Intuitionistic fuzzy and other some groups are in progress.

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