Shape Preserving Surfaces for the Visualization of Positive and Convex Data using Rational Bi-quadratic Splines

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ABSTRACT
A smooth surface interpolation scheme for positive and convex data has been developed. This scheme has been extended from the rational quadratic spline function of Sarfraz [11] to a rational bi-quadratic spline function. Simple data dependent constraints are derived on the free parameters in the description of rational bi-quadratic spline function to preserve the shape of 3D positive and convex data. The rational spline scheme has a unique representation. The developed scheme is computationally economical and visually pleasant.

Keywords
Data visualization, spline, interpolation, positive, convex.

1. INTRODUCTION
Shape control [17], shape design [18], shape representation [19-20] and shape preservation [10-16] are important areas for graphical presentation of data. The problem of shape preservation has been discussed by a number of authors. In recent years, a good amount of work has been published [1-15] that focuses on shape preserving curves and surfaces. The motivation of the work, in this paper, is due to the past work of many authors. Butt and Brodlie [3] discussed the problem of positivity using the piecewise cubic interpolation. The algorithm of Butt and Brodlie [3] works by inserting one or two extra knots, wherever necessary, to preserve the shape of positive data. Brodlie, Mashwama and Butt [2] developed a scheme to preserve the shape of positive surface data by the rearrangement of data and inserted one or more knots, where ever required, to preserve the shape of the data. Piah, Goodman and Unsworth [10] discussed the problem of positivity preservation for scattered data. Nadler [9], Chang and Sederberg [4] have also discussed the problem of nonnegative interpolation. They considered nonnegative data arranged over a triangular mesh and interpolated each triangular patch using a bivariate quadratic function. Schmidt and Hess [13] discussed quadratic and rational quadratic spline and developed necessary and sufficient conditions for the positivity.

Hussain and Sarfraz [7] used the rational cubic functions to preserve the shapes of curves and surfaces over positive data. Schumaker [14] used piecewise quadratic polynomial which is very economical but the method generally inserts an extra knot in each interval to interpolate. The problem of convexity of curves using the piecewise cubic interpolation is discussed by Sarfraz and Hussain [12]. Great contributions to convexity preservation of surfaces are by Asaturyan [1], Constantini and Fontanella [5], Hussain and Maria [8], and Dodd [6]. Asaturyan [1] scheme divides each grid rectangle into nine sub rectangles to generate convexity preserving surfaces. This scheme is not local i.e. by changing data in x -location of one edge of a sub-rectangle there must be a change throughout the grid for all sub rectangle edges located at the original x - values. The scheme of Constantini and Fontanella [5] interpolates the bivariate data defined over a rectangular grid and is the extension of univariate shape preserving scheme. Tensor product of Bernstein polynomial is used as interpolant. Convexity preserving constraints are applied along grid lines. The scheme gives a C^1 convex surface but its disadvantage is that it is not local. Dodd [6] produced a quadratic spline along the boundary of each grid rectangle and used these splines to define functional and partial derivatives on the boundaries of rectangles, formed by the grid. This scheme preserves the convexity of the surface along the grid lines but fails to preserve the convexity in the interior of the grids and produces the undesirable flat spots due to vanishing of second order mixed partial derivatives. Hussain and Maria [8] discussed the convexity of surfaces. They derived simple sufficient data dependent conditions on free parameters of rational bicubic to preserve the shape of data. The scheme used for both simple data and data with derivatives. This is a local scheme and is computationally economical and visually pleasing.

This research is a contribution towards achieving shape preserving curves and surfaces for positive data. The rational quadratic spline function of Sarfraz [11], which was used to achieve monotony preserving curves for monotonic data, has been extended to a rational bi-quadratic spline function. Shapes of positivity and convexity have been considered, to preserve the positive and convex data respectively, by interpolating spline surfaces. Simple data dependent constraints are derived on the free parameters in the description of rational bi-quadratic spline function to preserve the shape of 3D positive and convex data. Unlike its cubic or rational bicubic counterparts [1-3, 5-8, 10-11], the underlying scheme is rational bi-quadratic. Hence, the proposed scheme is computationally economical. Moreover, the proposed scheme produces visually pleasant results.
The method in this paper has number of advantageous features. It produces smooth interpolant. No additional points (knots) are needed. In contrast, the quadratic spline methods of Schumaker [14] and the cubic interpolation method of Brodlie and Butt [15] require the introduction of additional knots when used as shape preserving methods. The interpolant is not concerned with an arbitrary degree as in [16]. It is a rational spline with biquadratic numerator and denominator. The rational spline curve representation is unique in its solution.

The paper begins with a definition of the rational function in Section 2 where the description of rational quadratic spline curve is made, it preserves the positivity and convexity features of the data. In Section 3, the rational quadratic spline is extended to rational bi-quadratic spline. Section 4 deals with the proposed scheme which is developed to preserve the shape of positive data to present positive surfaces. Section 5 deals with the proposed scheme which is developed to preserve the shape of convex data to present convex surfaces. Section 6 concludes the paper.

2. SHAPE PRESERVING RATIONAL QUADRATIC SPLINE

In this section, a piecewise rational quadratic spline function is introduced which was initially developed by Sarfraz [11]. Let $(x_i, f_i),\ i=1,2,...,n,$ be a given set of data points where $x_1 < x_2 < ... < x_n$. Let

$$h_i = x_{i+1} - x_i,\ \Delta_i = \frac{f_{i+1} - f_i}{h_i}.$$

In each interval $I_i = [x_i, x_{i+1}]$, a rational quadratic spline $S(x)$ may be defined as:

$$S(x) = \frac{p_i(\theta)}{q_i(\theta)}, \quad i = 1,2,...,n. \qquad (1)$$

where

$$p_i(\theta) = (1-\theta)^2 \alpha_i f_i + (\alpha_i + \beta_i)\theta(1-\theta)V_i + \theta^2 \beta_i f_{i+1},$$

$$q_i(\theta) = (1-\theta)\alpha_i + \theta\beta_i,$$

with

$$V_i = (\alpha_i d_i f_{i+1} + \beta_i d_{i+1} f_i) / (\alpha_i d_i + \beta_i d_{i+1})$$

and

$$\theta = \frac{x - x_i}{h_i}, \qquad 0 \le \theta \le 1.$$

The further analysis of the interpolant leads to the followings Theorems 1 & 2:

Theorem 1. The rational quadratic spline function (1) preserves positivity if the free parameters α_i and β_i satisfy the following conditions:

$$\beta_i > 0 \text{ and}$$

$$\alpha_i = l_i + Max\left\{0, -\beta_i \frac{f_i d_{i+1}}{f_{i+1} d_i}, -\beta_i \frac{d_{i+1}}{d_i}\right\}, l_i > 0.$$

Proof. The proof is straightforward and follows from the Bézier-Bernstein theory when we want to make all the terms in Eqn. (1) positive.

Theorem 2. The rational quadratic spline function (1) preserves the convexity if free parameters α_i and β_i satisfies the following conditions:

$$\beta_i > 0,$$

$$\alpha_i = n_i + Max\left\{0, -\beta_i \frac{d_{i+1}}{d_i}\right\}, n_i > 0.$$

Proof. One needs to derive the second derivative of Eqn. (1). While keeping all the terms in the Bézier-Bernstein form, we will want to make all the coefficient terms of the Bézier polynomials, in the numerator, positive. This will lead to the proof.

3. RATIONAL BI-QUADRATIC SPLINE

The piecewise rational quadratic spline function (1) is extended to bi-quadratic partially blended rational spline function $S(x,y)$ over rectangular domain $D = [a,b] \times [c,d]$. Let $\pi : a = x_0 < x_1 < ... < x_m = b$ be partition of $[a,b]$ and $\hat{\pi} : c = y_0 < y_1 < ... < y_n = d$ be partition of $[c,d]$. Rectangular bi-quadratic spline function is defined over each rectangular patch $[x_i, x_{i+1}] \times [y_i, y_{i+1}]$ where $i = 0,1,2,...,m-1;$ $j = 0,1,2,...,n-1$ as:

$$S(x,y) = -AFB^T, \qquad (2)$$

where

$$F = \begin{pmatrix} 0 & S(x, y_j) & S(x, y_{j+1}) \\ S(x_i, y) & S(x_i, y_j) & S(x_i, y_{j+1}) \\ S(x_{i+1}, y) & S(x_{i+1}, y_j) & S(x_{i+1}, y_{j+1}) \end{pmatrix},$$

$$A = \begin{bmatrix} -1 & a_0(\theta) & a_1(\theta) \end{bmatrix};\ B = \begin{bmatrix} -1 & b_0(\phi) & b_1(\phi) \end{bmatrix};$$

with

$$a_0 = (1-\theta)^2(1+2\theta), \qquad a_1 = \theta^2(3-2\theta),$$

$$b_0 = (1-\phi)^2(1+2\phi),\ b_1 = \phi^2(3-2\phi).$$

$$\theta = \frac{x - x_i}{h_i},\ h_i = x_{i+1} - x_i, \quad 0 \le \theta \le 1.$$

$$\phi = \frac{y - y_j}{\hat{h}_j},\ \hat{h}_j = y_{j+1} - y_j,\ 0 \le \phi \le 1.$$

$S(x, y_j)$, $S(x, y_{j+1})$, $S(x_i, y)$ and $S(x_{i+1}, y)$ are rational quadratic spline (1) defined over the boundary of rectangular patch $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ as:

$$S(x, y_j) = \frac{\sum_{i=0}^{2} (1-\theta)^{2-i} \theta^i A_i}{q_1(\theta)} \qquad (3)$$

with

$$A_0 = \alpha_{i,j} F_{i,j},$$

$$A_1 = (\alpha_{i,j} + \beta_{i,j}) V_{i,j},$$

$$A_2 = \beta_{i,j} F_{i+1,j},$$

$$V_{i,j} = \frac{\alpha_{i,j} F^x_{i,j} F_{i+1,j} + \beta_{i,j} F^x_{i+1,j} F_{i,j}}{\alpha_{i,j} F^x_{i,j} + \beta_{i,j} F^x_{i+1,j}},$$

$$q_1(\theta) = (1-\theta)\alpha_{i,j} + \theta\beta_{i,j}.$$

$$S(x, y_{j+1}) = \frac{\sum_{i=0}^{2} (1-\theta)^{2-i} \theta^i B_i}{q_2(\theta)}, \qquad (4)$$

with

$$B_0 = \alpha_{i,j+1} F_{i,j+1},$$

$$B_1 = (\alpha_{i,j+1} + \beta_{i,j+1}) V_{i,j+1},$$

$$B_2 = \beta_{i,j+1} F_{i+1,j+1},$$

$$V_{i,j+1} = \frac{\alpha_{i,j+1} F^x_{i,j+1} F_{i+1,j+1} + \beta_{i,j+1} F^x_{i+1,j+1} F_{i,j+1}}{\alpha_{i,j+1} F^x_{i,j+1} + \beta_{i,j+1} F^x_{i+1,j+1}},$$

$$q_2 \ \theta \ = \ 1-\theta \ \alpha_{i,j+1} + \theta\beta_{i,j+1}.$$

$$S \ \ x_i, y \ \ = \frac{\sum_{i=0}^{2} \ 1-\phi \ ^{2-i} \phi^i C_i}{q_3 \ \ \phi}, \qquad (5)$$

with

$$C_0 = \hat{\alpha}_{i,j} F_{i,j} \ ,$$

$$C_1 = \ \hat{\alpha}_{i,j} + \hat{\beta}_{i,j} \ \ \hat{V}_{i,j},$$

$$C_2 = \hat{\beta}_{i,j} F_{i,j+1,}$$

$$\hat{V}_{i,j} = \frac{\hat{\alpha}_{i,j} F^y_{i,j} F_{i,j+1} + \hat{\beta}_{i,j} F^y_{i,j+1} F_{i,j}}{\hat{\alpha}_{i,j} F^y_{i,j} + \hat{\beta}_{i,j} F^y_{i,j+1}} \ ,$$

$$q_3 \ \phi \ = \ 1-\phi \ \hat{\alpha}_{i,j} + \phi\hat{\beta}_{i,j}.$$

$$S \ \ x_{i+1}, y \ \ = \frac{\sum_{i=0}^{2} \ 1-\phi \ ^{2-i} \phi^i D_i}{q_4 \ \ \phi}, \qquad (6)$$

with

$$D_0 = \hat{\alpha}_{i+1,j} F_{i+1,j,}$$

$$D_1 = \ \hat{\alpha}_{i+1,j} + \hat{\beta}_{i+1,j} \ \ \hat{V}_{i+1,j},$$

$$D_2 = \hat{\beta}_{i+1,j} F_{i+1,j+1},$$

$$\hat{V}_{i+1,j} = \frac{\hat{\alpha}_{i+1,j} F^y_{i+1,j} F_{i+1,j+1} + \hat{\beta}_{i+1,j} F^y_{i+1,j+1} F_{i+1,j}}{\hat{\alpha}_{i+1,j} F^y_{i+1,j} + \hat{\beta}_{i+1,j} F^y_{i+1,j+1}},$$

$$q_4 \ \phi \ = \ 1-\phi \ \hat{\alpha}_{i+1,j} + \phi\hat{\beta}_{i+1,j}.$$

4. POSITIVE SURFACE INTERPOLATION

Let $x_i, y_j, F_{i,j} \ : i = 1,2,...,m; \ j = 1,2,...,n$ be the positive data defined over rectangular grid $I_{i,j} = \ x_i, x_{i+1} \ \times \left[y_j, y_{j+1} \right]$, where $i = 0,1,2,...,m-1; \ \ j = 0,1,2,...,n-1$. Let us have

$$F_{i,j} > 0 \qquad \forall i, j.$$

The piecewise rational bi-quadratic spline function (2) is positive if the boundary curves $S \ \ x, y_j$, $S \ \ x, y_{j+1}$, $S \ \ x_i, y$ and $S \ \ x_{i+1}, y$ defined in (3), (4), (5) and (6) are positive. Now,

$$S \ \ x, y_j \ > 0 \ \text{if} \ \sum_{i=0}^{2} \ 1-\theta \ ^{2-i} \theta^i A_i > 0 \ \text{and} \ q_1 \ \theta \ > 0.$$

But,

$$q_1 \ \theta \ > 0 \ \text{if} \ \alpha_{i,j} > 0 \ \text{and} \ \beta_{i,j} > 0.$$

Thus,

$$\sum_{i=0}^{2} \ 1-\theta \ ^{2-i} \theta^i A_i > 0 \ \text{if} \ A_i > 0, i = 0,1,2.$$

One can easily see that

$$A_0 > 0 \ \text{if} \ \alpha_{i,j} > 0. \qquad (7)$$

$$A_1 > 0 \ \text{if} \ \alpha_{i,j} > \frac{-\beta_{i,j} F_{i,j} F^x_{i+1,j}}{F_{i+1,j} F^x_{i,j}} \ \text{and}$$

$$\alpha_{i,j} > \frac{-\beta_{i,j} F^x_{i+1,j}}{F^x_{i,j}} \qquad (8)$$

$$A_2 > 0 \ \text{if} \ \beta_{i,j} > 0. \qquad (9)$$

Similarly,

$$S \ \ x, y_{j+1} \ > 0 \quad \text{if} \quad \sum_{i=0}^{2} \ 1-\theta \ ^{2-i} \theta^i B_i > 0 \quad \text{and}$$

$$q_2 \ \theta \ > 0.$$

But,

$$q_2 \ \theta \ > 0 \ \text{if} \ \alpha_{i,j+1} > 0 \ \text{and} \ \beta_{i,j+1} > 0.$$

Thus,

$$\sum_{i=0}^{2} \ 1-\theta \ ^{2-i} \theta^i B_i > 0 \ \text{if} \ B_i > 0, i = 0,1,2.$$

One can easily see that

$$B_0 > 0 \ \text{if} \ \alpha_{i,j+1} > 0. \qquad (10)$$

$$B_1 > 0 \ \text{if} \ \alpha_{i,j+1} > -\beta_{i,j+1} \frac{F_{i,j+1} F^x_{i+1,j+1}}{F_{i+1,j+1} F^x_{i,j+1}} \ \text{and}$$

$$\alpha_{i,j+1} > -\beta_{i,j+1} \frac{F^x_{i+1,j+1}}{F^x_{i,j+1}}. \qquad (11)$$

$$B_2 > 0 \ \text{if} \ \beta_{i,j+1} > 0. \qquad (12)$$

Similarly,

$$S \ \ x_i, y \ > 0 \ \text{if} \ \sum_{i=0}^{2} \ 1-\phi \ ^{2-i} \phi^i C_i > 0 \ \text{and} \ q_3 \ \phi \ > 0.$$

But,

$$q_3 \ \phi \ > 0 \ \text{if} \ \hat{\alpha}_{i,j} > 0 \ \text{and} \ \hat{\beta}_{i,j} > 0.$$

Thus,

$$\sum_{i=0}^{2} \ 1-\phi \ ^{2-i} \phi^i C_i > 0 \ \text{if} \ C_i > 0, i = 0,1,2.$$

One can easily see that

$$C_0 > 0 \ \text{if} \ \hat{\alpha}_{i,j} > 0. \qquad (13)$$

$$C_1 > 0 \quad \text{if} \quad \hat{\alpha}_{i,j} > -\hat{\beta}_{i,j} \frac{F_{i,j} F^y_{i,j+1}}{F_{i,j+1} F^y_{i,j}} \quad \text{and}$$

$$\hat{\alpha}_{i,j} > -\hat{\beta}_{i,j} \frac{F^y_{i,j+1}}{F^y_{i,j}}. \qquad (14)$$

$$C_2 > 0 \ \text{if} \ \hat{\beta}_{i,j} > 0. \qquad (15)$$

Similarly,

$$S\left(x_{i+1},y\right)>0 \quad \text{if} \quad \sum_{i=0}^{2}\left(1-\phi\right)^{2-i}\phi^{i}D_i>0 \quad \text{and}$$

$q_4\left(\phi\right)>0.$

But,

$q_4\left(\phi\right)>0$ if $\hat{\alpha}_{i+1,j}>0$ and $\hat{\beta}_{i+1,j}>0.$

Thus,

$$\sum_{i=0}^{2}\left(1-\phi\right)^{2-i}\phi^{i}D_i>0 \ \text{if} \ D_i>0, i=0,1,2.$$

One can easily see that

$$D_0>0 \ \text{if} \ \hat{\alpha}_{i+1,j}>0. \tag{16}$$

$$D_1>0 \ \text{if} \ \hat{\alpha}_{i+1,j}>-\hat{\beta}_{i+1,j}\frac{F_{i+1,j}F^{y}_{i+1,j+1}}{F_{i+1,j+1}F^{y}_{i+1,j}} \ \text{and}$$

$$\hat{\alpha}_{i+1,j}>-\hat{\beta}_{i+1,j}\frac{F^{y}_{i+1,j+1}}{F^{y}_{i+1,j}}. \tag{17}$$

$$D_2>0 \ \text{if} \ \hat{\beta}_{i+1,j}>0. \tag{18}$$

The above discussion can be summarized as:

Theorem 3. The rational bi-quadratic spline function defined in (2) preserves the shape of positive data if in each rectangular patch $I_{i,j}=\left[x_i,x_{i+1}\right]\times\left[y_j,y_{j+1}\right]$, free parameters $\alpha_{i,j}$, $\beta_{i,j}$, $\alpha_{i,j+1}$, $\beta_{i,j+1}$, $\hat{\alpha}_{i,j}$, $\hat{\beta}_{i,j}$, $\hat{\alpha}_{i+1,j}$ and $\hat{\beta}_{i+1,j}$ satisfy the following conditions:

$$\beta_{i,j}>0, \beta_{i,j+1}>0, \hat{\beta}_{i,j}>0, \hat{\beta}_{i+1,j}>0.$$

and

$$\alpha_{i,j}>Max\left\{0,\frac{-\beta_{i,j}F_{i,j}F^{x}_{i+1,j}}{F_{i+1,j}F^{x}_{i,j}},\frac{-\beta_{i,j}F^{x}_{i+1,j}}{F^{x}_{i,j}}\right\}$$

$$\alpha_{i,j+1}>Max\left\{0,\frac{-\beta_{i,j+1}F_{i,j+1}F^{x}_{i+1,j+1}}{F_{i+1,j+1}F^{x}_{i,j+1}},\frac{-\beta_{i,j+1}F^{x}_{i+1,j+1}}{F^{x}_{i,j+1}}\right\},$$

$$\hat{\alpha}_{i,j}>Max\left\{0,\frac{-\hat{\beta}_{i,j}F_{i,j}F^{y}_{i,j+1}}{F_{i,j+1}F^{y}_{i,j}},\frac{-\hat{\beta}_{i,j}F^{y}_{i,j+1}}{F^{y}_{i,j}}\right\},$$

$$\hat{\alpha}_{i+1,j}>Max\left\{0,\frac{-\hat{\beta}_{i+1,j}F_{i+1,j}F^{y}_{i+1,j+1}}{F_{i+1,j+1}F^{y}_{i+1,j}},\frac{-\hat{\beta}_{i+1,j}F^{y}_{i+1,j+1}}{F^{y}_{i+1,j}}\right\}.$$

The above constraints can be rewritten as:

$$\beta_{i,j}>0, \beta_{i,j+1}>0, \hat{\beta}_{i,j}>0 \hat{\beta}_{i+1,j}>0.$$

and

$$\alpha_{i,j}=a_{i,j}+Max\left\{0,\frac{-\beta_{i,j}F_{i,j}F^{x}_{i+1,j}}{F_{i+1,j}F^{x}_{i,j}},\frac{-\beta_{i,j}F^{x}_{i+1,j}}{F^{x}_{i,j}}\right\}$$

$$\alpha_{i,j+1}=b_{i,j}+Max\left\{0,\frac{-\beta_{i,j+1}F_{i,j+1}F^{x}_{i+1,j+1}}{F_{i+1,j+1}F^{x}_{i,j+1}},\frac{-\beta_{i,j+1}F^{x}_{i+1,j+1}}{F^{x}_{i,j+1}}\right\}$$

$$\hat{\alpha}_{i,j}=c_{i,j}+Max\left\{0,\frac{-\hat{\beta}_{i,j}F_{i,j}F^{y}_{i,j+1}}{F_{i,j+1}F^{y}_{i,j}},\frac{-\hat{\beta}_{i,j}F^{y}_{i,j+1}}{F^{y}_{i,j}}\right\}$$

$$\hat{\alpha}_{i+1,j}=d_{i,j}+Max\left\{0,\frac{-\hat{\beta}_{i+1,j}F_{i+1,j}F^{y}_{i+1,j+1}}{F_{i+1,j+1}F^{y}_{i+1,j}},\frac{-\hat{\beta}_{i+1,j}F^{y}_{i+1,j+1}}{F^{y}_{i+1,j}}\right\}.$$

4.1 Demonstration

Let us demonstrate the devised scheme for positive data in the following examples:

Example 1: A positive data set is considered in Table 1 generated by the following function:

$$F_1\left(x,y\right)=e^{-x^2-y^2}+0.0001;-3\le x,y\le 3.$$

The data set is reported by taking the values truncated to four decimal places.

Figure 1 is produced from the data set in Table 1 using bi-quadratic spline function which looses positivity. This flaw is nicely recovered in Figure 2 using the scheme developed in Section 4 by assigning the values to free parameters $\beta_{i,j}=\beta_{i,j+1}=\hat{\beta}_{i,j}=\hat{\beta}_{i+1,j}=0.5$ and $a_{i,j}=b_{i,j}=c_{i,j}=d_{i,j}=0.5$. It is clear from the Figure 2 that the shape of positive data is preserved.

Example 2: A positive data set is considered in Table 2 generated by the following function:

$$F_2\left(x,y\right)=\left(x^2-y^2+0.5\right)^2;-3\le x,y\le 3.$$

Figure 3 is produced from the data set in Table 2 using bi-quadratic function which looses positivity. This flaw is nicely recovered in Figure 4 using the scheme developed in Section 4 by assigning the values to free parameters $\beta_{i,j}=\beta_{i,j+1}=\hat{\beta}_{i,j}=\hat{\beta}_{i+1,j}=1$ and $a_{i,j}=b_{i,j}=c_{i,j}=d_{i,j}=1$ It is clear from the Figure 4 that the shape of positive data is preserved.

Table 1. A positive data set.

y/x	-3	-2	-1	0	1	2	3
-3	0.0001	0.0001	0.0001	0.0002	0.0001	0.0001	0.0001
-2	0.0001	0.0004	0.0068	0.0184	0.0068	0.0004	0.0001
-1	0.0001	0.0068	0.1354	0.3680	0.1354	0.0068	0.0001
0	0.0002	0.0184	0.3680	1.0001	0.3680	0.0184	0.0002
1	0.0001	0.0068	0.1354	0.3680	0.1354	0.0068	0.0001
2	0.0001	0.0004	0.0068	0.0184	0.0068	0.0004	0.0001
3	0.0001	0.0001	0.0001	0.0002	0.0001	0.0001	0.0001

Table 2. A positive data set.

y/x	-3	-2	-1	0	1	2	3
-3	0.2500	30.2500	72.2500	90.2500	72.2500	30.2500	0.2500
-2	20.2500	0.2500	12.2500	20.2500	12.2500	0.2500	20.2500
-1	56.2500	6.2500	0.2500	2.2500	0.2500	6.2500	56.2500
0	72.2500	12.2500	0.2500	0.2500	0.2500	12.2500	72.2500
1	56.2500	6.2500	0.2500	2.2500	0.2500	6.2500	56.2500
2	20.2500	0.2500	12.2500	20.2500	12.2500	0.2500	20.2500
3	0.2500	30.2500	72.2500	90.2500	72.2500	30.2500	0.2500

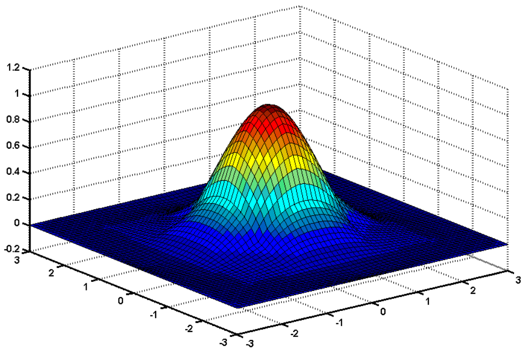

Fig 1: Bi-quadratic Surface.

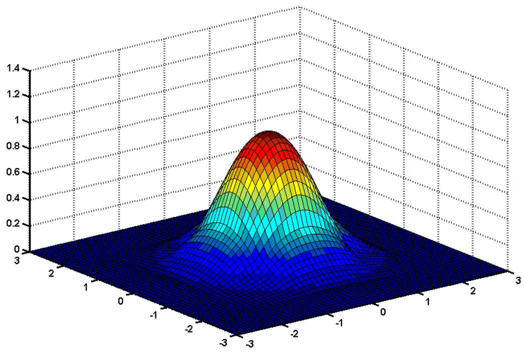

Fig 2: Positive Rational Bi-quadratic Surface.

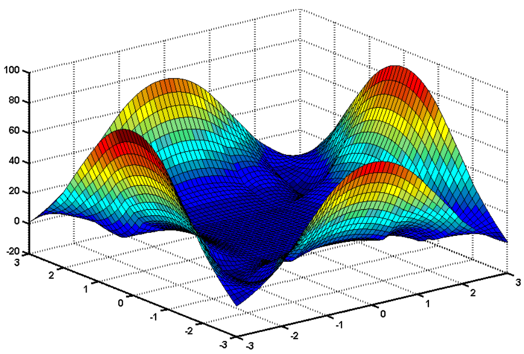

Fig 3: Bi-quadratic Surface.

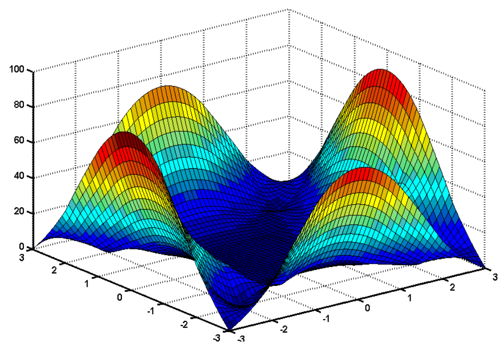

Fig 4: Positive Rational Bi-quadratic Surface.

5. CONVEX SURFACE INTERPOLATION

Let $\{(x_i, y_j, F_{i,j}) : i = 1,2,...,m;\ j = 1,2,...,n\}$ be convex data defined over rectangular grid $I_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $i = 0,1,2,...,m-1;\ \ j = 0,1,2,...,n-1.$ such that:

$$F_{i,j}^x \le \Delta_{i,j} \le F_{i+1,j}^x, \quad F_{i,j+1}^x \le \Delta_{i,j+1} \le F_{i+1,j+1}^x,$$

$$F_{i,j}^y \le \hat{\Delta}_{i,j} \le F_{i,j+1}^y, \quad F_{i+1,j}^y \le \hat{\Delta}_{i+1,j} \le F_{i+1,j+1}^y.$$

where

$$\Delta_{i,j} = \frac{F_{i+1,j} - F_{i,j}}{h_i}, \ \hat{\Delta}_{i,j} = \frac{F_{i,j+1} - F_{i,j}}{\hat{h}_j},$$

$$\Delta_{i,j+1} = \frac{F_{i+1,j+1} - F_{i,j+1}}{h_i}, \ \hat{\Delta}_{i+1,j} = \frac{F_{i+1,j+1} - F_{i+1,j}}{\hat{h}_j}.$$

Now rational bi-quadratic spline function defined in (2) will be convex in each rectangular patch $I_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, if each of the boundary curves $S(x, y_j)$, $S(x, y_{j+1})$, $S(x_i, y)$ and $S(x_{i+1}, y)$ defined in (3), (4), (5) and (6) are convex.

Now, $S(x, y_j)$ will be convex if $S^{(2)}(x, y_j) > 0$. That is

$$S^{(2)}(x, y_j) = \frac{\sum_{i=0}^{2} (1-\theta)^{2-i}\theta^i R_i}{h_i q_1^3(\theta)\left(\alpha_{i,j}F_{i,j}^x + \beta_{i,j}F_{i+1,j}^x\right)} > 0,$$

with

$$R_0 = 2\alpha_{i,j}^2\beta_{i,j}^2\Delta_{i,j}\left(F_{i+1,j}^x - F_{i,j}^x\right),$$

$$R_1 = 4\alpha_{i,j}^2\beta_{i,j}^2\Delta_{i,j}\left(F_{i+1,j}^x - F_{i,j}^x\right),$$

$$R_2 = 2\alpha_{i,j}^2\beta_{i,j}^2\Delta_{i,j}\left(F_{i+1,j}^x - F_{i,j}^x\right),$$

$$q_1(\theta) = (1-\theta)\alpha_{i,j} + \theta\beta_{i,j}.$$

Thus, $S^{(2)}(x, y_j) > 0$ if

$\sum_{i=0}^{2} (1-\theta)^{2-i}\theta^i R_i > 0$, $\left(\alpha_{i,j}F_{i,j}^x + \beta_{i,j}F_{i+1,j}^x\right) > 0$ and $q_1^3(\theta) > 0$.

But, $q_1^3(\theta) > 0$ if $\alpha_{i,j} > 0$ and $\beta_{i,j} > 0.$ (19)

$$\left(\alpha_{i,j}F_{i,j}^x + \beta_{i,j}F_{i+1,j}^x\right) > 0 \text{ if } \alpha_{i,j} > -\beta_{i,j}\frac{F_{i+1,j}^x}{F_{i,j}^x} \quad (20)$$

Thus,

$$\sum_{i=0}^{2} (1-\theta)^{2-i}\theta^i R_i > 0 \text{ if } R_i > 0, i = 0,1,2.$$

One can see easily that

$$R_i > 0 \text{ if } \alpha_{i,j} > 0 \text{ and } \beta_{i,j} > 0. \quad (21)$$

Similarly, $S(x, y_{j+1})$ will be convex if $S^{(2)}(x, y_{j+1}) > 0$. That is

$$S^{(2)}(x, y_{j+1}) = \frac{\sum_{i=0}^{2} (1-\theta)^{2-i}\theta^i S_i}{h_i q_2^3(\theta)\left(\alpha_{i,j+1}F_{i,j+1}^x + \beta_{i,j+1}F_{i+1,j+1}^x\right)} > 0,$$

with

$$S_0 = 2\alpha_{i,j+1}^2\beta_{i,j+1}^2\Delta_{i,j+1}\left(F_{i+1,j+1}^x - F_{i,j+1}^x\right),$$

$$S_1 = 4\alpha_{i,j+1}^2\beta_{i,j+1}^2\Delta_{i,j+1}\left(F_{i+1,j+1}^x - F_{i,j+1}^x\right),$$

$$S_2 = 2\alpha_{i,j+1}^2\beta_{i,j+1}^2\Delta_{i,j+1}\left(F_{i+1,j+1}^x - F_{i,j+1}^x\right),$$

$$q_2(\theta) = (1-\theta)\alpha_{i,j+1} + \theta\beta_{i,j+1}.$$

Thus, $S^{(2)}(x, y_{j+1}) > 0$ if

$\sum_{i=0}^{2} (1-\theta)^{2-i}\theta^i S_i > 0$, $\left(\alpha_{i,j+1}F_{i,j+1}^x + \beta_{i,j+1}F_{i+1,j+1}^x\right) > 0$ and $q_2^3(\theta) > 0.$

But,

$q_2^3(\theta) > 0$ if $\alpha_{i,j+1} > 0$ and $\beta_{i,j+1} > 0$ (22)

$$\left(\alpha_{i,j+1}F_{i,j+1}^x + \beta_{i,j+1}F_{i+1,j+1}^x\right) > 0 \text{ if}$$

$$\alpha_{i,j+1} > -\beta_{i,j+1}\frac{F_{i+1,j+1}^x}{F_{i,j+1}^x}. \quad (23)$$

Therefore,

$$\sum_{i=0}^{2} (1-\theta)^{2-i}\theta^i S_i > 0 \text{ if } S_i > 0, i = 0,1,2.$$

One can see easily that

$$S_i > 0 \text{ if } \alpha_{i,j+1} > 0 \text{ and } \beta_{i,j+1} > 0. \quad (24)$$

Similarly, $S(x_i, y)$ will be convex if $S^{(2)}(x_i, y) > 0$. That is,

$$S^{(2)}(x_i, y) = \frac{\sum_{i=0}^{2} (1-\phi)^{2-i}\phi^i T_i}{\hat{h}_j q_3^3(\phi)\left(\hat{\alpha}_{i,j}F_{i,j}^y + \hat{\beta}_{i,j}F_{i,j+1}^y\right)} > 0,$$

with

$$T_0 = 2\hat{\alpha}_{i,j}^2\hat{\beta}_{i,j}^2\hat{\Delta}_{i,j}\left(F_{i,j+1}^y - F_{i,j}^y\right),$$

$$T_1 = 4\hat{\alpha}_{i,j}^2\hat{\beta}_{i,j}^2\hat{\Delta}_{i,j}\left(F_{i,j+1}^y - F_{i,j}^y\right),$$

$$T_2 = 2\hat{\alpha}_{i,j}^2\hat{\beta}_{i,j}^2\hat{\Delta}_{i,j}\left(F_{i,j+1}^y - F_{i,j}^y\right),$$

$$q_3(\phi) = (1-\phi)\hat{\alpha}_{i,j} + \phi\hat{\beta}_{i,j}.$$

Thus, $S^{(2)}(x_i, y) > 0$ if

$\sum_{i=0}^{2} (1-\phi)^{2-i}\phi^i T_i > 0$, $\left(\hat{\alpha}_{i,j}F_{i,j}^y + \hat{\beta}_{i,j}F_{i,j+1}^y\right) > 0$ and $q_3^3(\phi) > 0.$

But, $q_3^3(\phi) > 0$ if $\hat{\alpha}_{i,j} > 0$, and $\hat{\beta}_{i,j} > 0.$ (25)

$$\left(\hat{\alpha}_{i,j}F_{i,j}^y + \hat{\beta}_{i,j}F_{i,j+1}^y\right) > 0 \text{ if } \hat{\alpha}_{i,j} > -\hat{\beta}_{i,j}\frac{F_{i,j+1}^y}{F_{i,j}^y} \quad (26)$$

Therefore,

$$\sum_{i=0}^{2} (1-\phi)^{2-i}\phi^i T_i > 0 \text{ if } T_i > 0, i = 0,1,2.$$

One can see easily that

$$T_i > 0 \text{ if } \hat{\alpha}_{i,j} > 0 \text{ and } \hat{\beta}_{i,j} > 0. \quad (27)$$

Similarly, $S(x_{i+1}, y)$ will be convex if $S^{(2)}(x_{i+1}, y) > 0$. That is,

$$S^{(2)}(x_{i+1}, y) = \frac{\sum_{i=0}^{2}(1-\phi)^{2-i}\phi^i U_i}{\hat{h}_j q_4^3(\phi)\left(\hat{\alpha}_{i+1,j}F^y_{i+1,j} + \hat{\beta}_{i+1,j}F^y_{i+1,j+1}\right)} > 0,$$

with

$$U_0 = 2\hat{\alpha}^2_{i+1,j}\hat{\beta}^2_{i+1,j}\hat{\Delta}_{i+1,j}\left(F^y_{i+1,j+1} - F^y_{i+1,j}\right),$$

$$U_1 = 4\hat{\alpha}^2_{i+1,j}\hat{\beta}^2_{i+1,j}\hat{\Delta}_{i+1,j}\left(F^y_{i+1,j+1} - F^y_{i+1,j}\right),$$

$$U_2 = 2\hat{\alpha}^2_{i+1,j}\hat{\beta}^2_{i+1,j}\hat{\Delta}_{i+1,j}\left(F^y_{i+1,j+1} - F^y_{i+1,j}\right),$$

$$q_4(\phi) = (1-\phi)\hat{\alpha}_{i+1,j} + \phi\hat{\beta}_{i+1,j}.$$

Thus, $S^{(2)}(x_{i+1}, y) > 0$ if

$$\sum_{i=0}^{2}(1-\phi)^{2-i}\phi^i U_i > 0,$$

$$\left(\hat{\alpha}_{i+1,j}F^y_{i+1,j} + \hat{\beta}_{i+1,j}F^y_{i+1,j+1}\right) > 0 \text{ and } q_4^3(\phi) > 0.$$

But,

$$q_4^3(\phi) > 0 \text{ if } \hat{\alpha}_{i+1,j} > 0 \text{ and } \hat{\beta}_{i+1,j} > 0. \quad (28)$$

$$\left(\hat{\alpha}_{i+1,j}F^y_{i+1,j} + \hat{\beta}_{i+1,j}F^y_{i+1,j+1}\right) > 0 \text{ if}$$

$$\hat{\alpha}_{i+1,j} > -\hat{\beta}_{i+1,j}\frac{F^y_{i+1,j+1}}{F^y_{i+1,j}}. \quad (29)$$

Therefore,

$$\sum_{i=0}^{2}(1-\phi)^{2-i}\phi^i U_i > 0 \text{ if } U_i > 0, i = 0,1,2.$$

One can see easily that

$$U_i > 0 \text{ if } \hat{\alpha}_{i+1,j} > 0 \text{ and } \hat{\beta}_{i+1,j} > 0. \quad (30)$$

The above discussion can be summarized as follows:

Theorem 4. The rational bi-quadratic spline function defined in (2) preserves the shape of convex data if in each rectangular patch $I_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, free parameters $\alpha_{i,j}, \beta_{i,j}$, $\alpha_{i,j+1}$, $\beta_{i,j+1}$, $\hat{\alpha}_{i,j}$, $\hat{\beta}_{i,j}$, $\hat{\alpha}_{i+1,j}$ and $\hat{\beta}_{i+1,j}$ satisfy the following conditions:

$$\beta_{i,j} > 0, \beta_{i,j+1} > 0, \hat{\beta}_{i,j} > 0 \hat{\beta}_{i+1,j} > 0.$$

and

$$\alpha_{i,j} > Max\left\{0, -\beta_{i,j}\frac{F^x_{i+1,j}}{F^x_{i,j}}\right\},$$

$$\alpha_{i,j+1} > Max\left\{0, -\beta_{i,j+1}\frac{F^x_{i+1,j+1}}{F^x_{i,j+1}}\right\},$$

$$\hat{\alpha}_{i,j} > Max\left\{0, -\hat{\beta}_{i,j}\frac{F^y_{i,j+1}}{F^y_{i,j}}\right\},$$

$$\hat{\alpha}_{i+1,j} > Max\left\{0, -\hat{\beta}_{i+1,j}\frac{F^y_{i+1,j+1}}{F^y_{i+1,j}}\right\}.$$

The above constraints can be rewritten as:

$$\beta_{i,j+1} > 0,\ \hat{\beta}_{i,j} > 0,\ \hat{\beta}_{i+1,j} > 0.$$

$$\alpha_{i,j} = u_{i,j} + Max\left\{0, -\beta_{i,j}\frac{F^x_{i+1,j}}{F^x_{i,j}}\right\}, u_{i,j} > 0,$$

$$\alpha_{i,j+1} = v_{i,j} + Max\left\{0, -\beta_{i,j+1}\frac{F^x_{i+1,j+1}}{F^x_{i,j+1}}\right\}, v_{i,j} > 0,$$

$$\hat{\alpha}_{i,j} = w_{i,j} + Max\left\{0, -\hat{\beta}_{i,j}\frac{F^y_{i,j+1}}{F^y_{i,j}}\right\}, w_{i,j} > 0,$$

$$\hat{\alpha}_{i+1,j} = x_{i,j} + Max\left\{0, -\hat{\beta}_{i+1,j}\frac{F^y_{i+1,j+1}}{F^y_{i+1,j}}\right\}, x_{i,j} > 0.$$

5.1 Demonstration

Let us demonstrate the devised scheme for convex data in the following examples:

Example 3: A convex data set is considered in Table 3 generated by the following function:

$$F_3(x, y) = x^2 + y^2.$$

Figure 5 is produced from the data set in Table 3 using bi-quadratic spline function that looses convexity. This flaw is nicely recovered in Figure 6 using the scheme developed in Section 5 by assigning the values to free parameters $\beta_{i,j} = \beta_{i,j+1} = \hat{\beta}_{i,j} = \hat{\beta}_{i+1,j} = 0.5$ and $u_{i,j} = v_{i,j} = w_{i,j}, = x_{i,j} = 1$. It is clear from Figure 6 that the shape of convex data is preserved.

Example 4: A convex data set is considered in Table 4 generated by the following function:

$$F_4(x, y) = y^2 + 9x^2 - 16.$$

Figure 7 is produced from the data set in Table 4 using bi-quadratic spline function that looses convexity. This flaw is nicely recovered in Figure 8 using the scheme developed in Section 5 by assigning the values to free parameters $\beta_{i,j} = \beta_{i,j+1} = \hat{\beta}_{i,j} = \hat{\beta}_{i+1,j} = 0.5$ and $u_{i,j} = v_{i,j} = w_{i,j} = x_{i,j} = 1$. It is clear from the Figure 8 that the shape of convex data is preserved.

Table 3. A convex data set.

y/x	-2	-1.5	-1	-0.5	-0.001	0.001	0.5	1	1.5	2
-2	8	6.25	5	4.25	4.0001	4.0001	4.25	5	6.25	8
-1.5	6.25	4.50	3.25	2.50	2.2501	2.2501	2.50	3.25	4.50	6.25
-1	5	3.25	2	1.25	1.0001	1.0001	1.25	2	3.25	5
-0.5	4.25	2.50	1.25	0.50	0.2501	0.2501	0.50	1.25	2.50	4.25
-0.001	4.0001	2.2501	1.0001	0.2501	0.0002	0.0002	0.2501	1.0001	2.2501	4.0001
0.001	4.0001	2.2501	1.0001	0.2501	0.0002	0.0002	0.2501	1.0001	2.2501	4.0001
0.5	4.25	2.50	1.25	0.50	0.2501	0.2501	0.50	1.25	2.50	4.25
1	5	3.25	2	1.25	1.0001	1.0001	1.25	2	3.25	5
1.5	6.25	4.50	3.25	2.50	2.2501	2.2501	2.50	3.25	4.50	6.25
2	8	6.25	5	4.25	4.0001	4.0001	4.25	5	6.25	8

Table 4. A convex data set

y/x	-2	-1.5	-1	-0.5	-0.001	0.001	0.5	1	1.5	2
-2	24	22.25	21	20.25	20.000	20.000	20.25	21	22.25	24
-1.5	8.25	6.50	5.25	4.50	4.250	4.250	4.50	5.25	6.50	8.25
-1	-3	-4.75	-6	-6.75	-6.999	-6.999	-6.75	-6	-4.75	-3
-0.5	-9.75	-11.50	-12.75	-13.50	-13.749	-13.749	-13.50	-12.75	-11.50	-9.75
-0.001	-11.999	-13.749	-14.999	-15.749	-15.999	-15.999	-15.749	-14.999	-13.749	-11.999
0.001	-11.999	-13.749	-14.999	-15.749	-15.999	-15.999	-15.749	-14.999	-13.749	-11.999
0.5	-9.75	-11.50	-12.75	-13.50	-13.749	-13.749	-13.50	-12.75	-11.50	-9.75
1	-3	-4.75	-6	-6.75	-6.9999	-6.999	-6.75	-6	-4.75	-3
1.5	8	6.50	5.25	4.50	4.2501	4.250	4.50	5.25	6.50	8
2	24	22.25	21	20.25	20.000	20.000	20.25	21	22.25	24

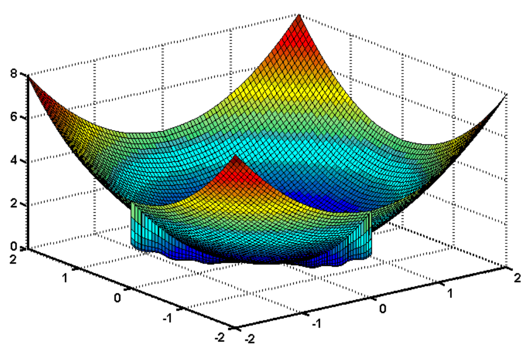

Fig 5: Bi-quadratic Surface.

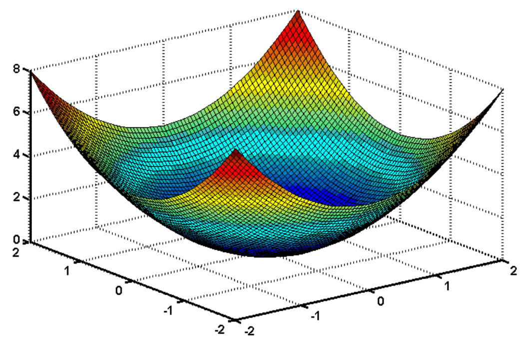

Fig 6: Convex Rational Bi-quadratic Surface.

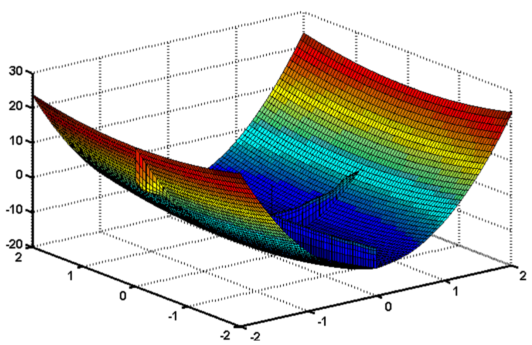

Fig 7: Bi-quadratic Surface.

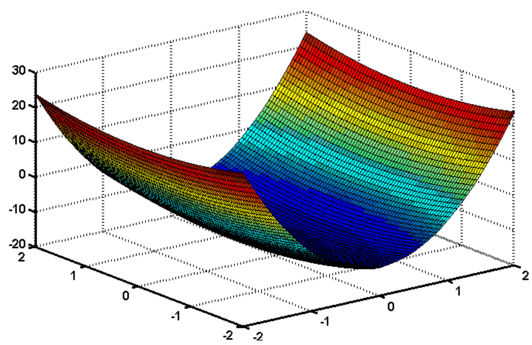

Fig 8: Convex Rational Bi-quadratic Surface.

6. CONCLUSION

In this paper, we have derived the data dependent constraints on the free parameters in the description of rational bi-quadratic spline function to preserve the shape of data. Shape preserving surfaces have been produced to visualize the positive and convex data. The choice of derivatives is left free for the user. The developed methods are verified with some examples of data. The rational spline scheme has a unique representation. The developed schemes are very easy to implement, computationally economical, and visually pleasant. However, this paper does not deal with another important shape of the data called monotony. This work is left as a future research and hopefully will appear soon as a continuity of the work in a subsequent paper.

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