b-Colouring of Central Graphs

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ABSTRACT
In this paper we discuss about the b-colouring and b-chromatic number of $C(C_n)$, $C(K_{m,n})$ and $C(P\,n)$.

Keywords
Central graph, b-colouring and b-chromatic number.

1. INTRODUCTION

Let G be a finite undirected graph with no loops and multiple edges. The central graph C(G) [10] of a graph G is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G. By definition $P_C(G) = p + q$. For any (p,q), graph there exists exactly p vertices of degree $(p-1)$ and q vertices of degree 2 in C(G).

The b-chromatic number [6] of a graph was introduced by R.W.Irving and D.F.Manlove when considering minimal proper colouring with respect to a Partial order defined on the set of all partition of vertices of graph. The b-chromatic number of a graph G, denoted by $\varphi(G)$, is the largest positive integer t such that there exists a proper coloring for G with t colors in which every color class contains at least one vertex adjacent to some vertex in all the other colour classes such a colouring is called a b-colouring.

2. THE b-COLOURING OF $C(K_{m,n})$

2.1 Theorem

For any complete bipartite graph $C(K_{m,n})$, $\varphi(C(K_{m,n})) = n + \left\lceil \frac{m}{2} \right\rceil$ where $m \le 6$.

Proof

Consider the complete bipartite graph $K_{m,n}$ with bipartation (X,Y) where $X = \{v_1, v_2, \ldots, v_n\}$ and $Y = \{u_1, u_2, \ldots, u_n\}$ in $C(K_{m,n})$. Let $v_{i,j}$ represents the newly introduced vertex in the edge joining v_i and u_j. Now assign a colouring to the vertices of $C(K_{m,n})$ as follows. Assign the colour c_i to v_i for i = 1, 2, ..., n. since $\langle v_i, i=1,2,\ldots,n \rangle$ is a complete graph, this colouring will be a b-colouring. Give the colour c_{n+i} to u_i for i = 1, 2, ..., $\left\lceil \frac{m}{2} \right\rceil$, now the vertex which has been coloured as c_{n+i} cannot realises the colour c_{n+i} to u_i. In order to overcome this, we should colour the $v_{i,j}$' s, $i \neq n$ as c_{i+1} and $v_{i,j}$'s, i = n as c_1 where $j \le \left\lceil \frac{m}{2} \right\rceil$. Again the introduction of new colours, namely c_{n+i} made the colouring of v_i, i = 1, 2, ..., n is no more b-chromatic. To make this colouring a b-chromatic one, we should colour $v_{i,j}$, $j = \left\lceil \frac{m}{2} \right\rceil + k$, k = 1, 2, ...$\left\lceil \frac{m}{2} \right\rceil$ as c_{n+k}. Thus to colour the remaining vertices in u_i, $i > \left\lceil \frac{m}{2} \right\rceil$, for this vertices we cannot assign any new colours because all the v_{ij}'s which are adjacent to any u_i is of same colour and those u_i's are not at all adjacent with any of the c_i coloured vertices. Hence, by colouring procedure the above said colouring is a b-chromatic colouring and furthermore it is the maximum colouring possible. Hence $\varphi(C(K_{m,n})) = n + \left\lceil \frac{m}{2} \right\rceil$.

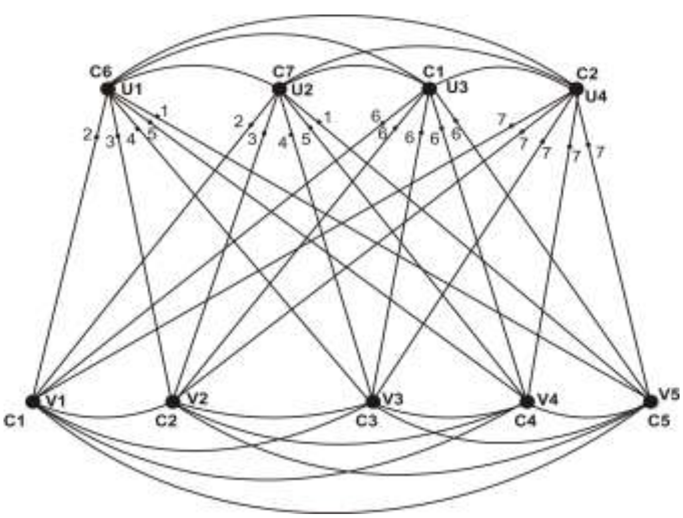

Figure 1 : $\varphi[C(K_{4,5})] = 7$

3.THE b-COLOURING OF [C(Cn)]

3.1 Theorem

For any cycle C_n of length $n \ge 5, n = 5x + r$,

$$\varphi[C(C_n)] = \begin{cases} n - x + 1 & \text{when r} \neq 0 \\ n - x & \text{when r} = 0 \end{cases}$$

Proof

Let C_n be any cycle of length n with vertices $v_1, v_2, \ldots v_n$. Let v_{ij} represents the newly introduced vertex in the edge connecting v_i and v_j. Now in $C(C_n)$ we can note that the vertex v_i is adjacent with all the vertices except the vertices except v_{i+1} and v_{i-1} for i = 2, 3, 4, 5, ... n-1. v_1 is adjacent with all the vertices except v_2 and v_n and v_n is adjacent with all the vertices except v_{n-1} and v_1. Consider a blind colouring of $C(C_n)$ as follows. Assign the colour c_i to v_i for i = 1, 2, ..., n. Due to the above said non-adjacency of v_i's this colouring will not produce a b-colouring. Thus to make it a b-colouring, we should assign a proper colour to v_{ij}'s. Consider an arbitary vertex v_i, but v_i is not adjacent with v_{i+1} and v_{i-1}. To realize the colour c_i we should colour $v_{i,i+1}$ as c_{i-1} and $v_{i,i-1}$ as c_{i+1}. Thus v_i will realise the colour c_i. Now take the vertex v_{i+1}, which is coloured as c_{i+1}. In order to realise the colour c_{i+1}, we should colour two neighbours of v_{i+1} as c_{i+1} and c_i but the previous colouring of v_i had left out only one vertex namely $v_{i+1,\ i+2}$ to be coloured. Thus realisation of c_{i+1} is not possible. Similar situation will occur if we are proceeding with v_{i-1} too. This shows that assigning different colours to v_i's is not possible. i.e. there should be repetation of colours. A close examination will reveal that there should be minimum of $\left\lceil \frac{n}{5} \right\rceil$ repetations. Thus we will assign a colouring to $C(C_n)$ as follows.

Case: 1

When r = 0, assign the colour $c_{i-\left\lceil \frac{i}{5} \right\rceil}$ to the vertex v_i for i = 1, 2, ..., n. Here only the repeated colour vertex realises its own colour but for the remaining vertex it is not possible. So the above colouring does not produce a b-colouring. To make it a b-colouring, we assign a proper colouring $v_{i,j}$'s as follows. For i = 1, 2, ..., n-1 and i $\equiv$ 2, 3, 4(mod 5) assign the colour $c_{i-\left(\left\lceil \frac{i}{5} \right\rceil -1\right)}$ to the vertex $v_{i,\ i+1}$ otherwise assign the colour $c_{i-\left(\left\lceil \frac{i}{5} \right\rceil +2\right)}$ to $v_{i,\ i+1}$. Now all v_i's for i = 1, 2, ..., n realises its own colour c_i. Hence by the colouring procedure it is the maximum colouring.

Case: 2

When r $\neq$ 0, for i = 1, 2, ..., n – 1 assign the colour $c_{i-\left\lceil \frac{i}{5} \right\rceil}$ to the vertex v_i. Here also only the vertex with repeated colour realises its own colour. Thus to make the colouring a b-chromatic one, we assign a proper colouring to $v_{i,j}$'s as follows. For i = 1, 2, ..., n and i $\equiv$ 2, 3, 4, (mod 5) assign the colour $c_{i-\left(\left\lceil \frac{i}{5} \right\rceil -1\right)}$ to the vertex $v_{i,\ i+1}$ otherwise assign the colour $c_{i-\left(\left\lceil \frac{i}{5} \right\rceil +2\right)}$ to $v_{i,,i+1.}$ Now the only vertex remaining to be coloured is v_n. Suppose we assign a new colour to the vertex v_n, the vertex does not realises the new colour, because v_n is not adjacent with v_{n-1} and v_i. Thus to realise the new colour we should colour the two neighbours of v_n as c_{n-1} and c_1, but by previous colouring, no vertex is left to be coloured. Thus introducing a new colour to the vertex v_n is not possible. Note that any rearrangement of colours to the graph also fails to accomodate the new colour. Hence by colouring procedure this is a b-chromatic colouring and furthermore it is the maximum colouring possible.

Example

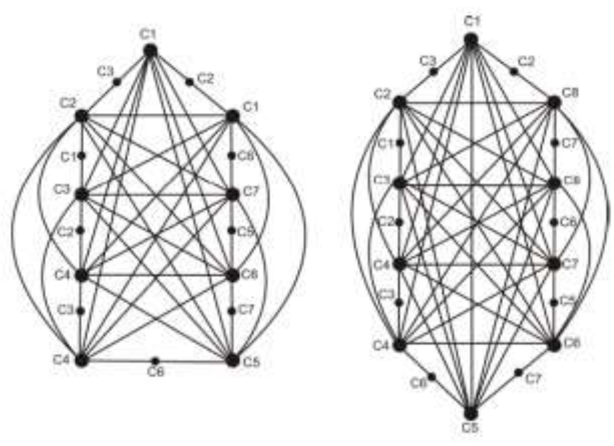

Figure 2 : $\varphi[C(C_9)] = 7$ $\varphi[C(C_{10})] = 8$

4. THE b-COLOURING OF $C(P_n)$

4.1. Theorem

For any path P_n of length n $\geq$ 5, n = 5*x* + r

$$\varphi[C(P_n)] = \begin{cases} n - x + 1 & \text{where r} = 4 \\ n - x & \text{otherwise} \end{cases}$$

Proof

Let P_n be any path of length n – 1 with vertices $v_1, v_2, \ldots, v_n$. Let v_{ij} represents the newly introduced vertex in the edge connecting v_i and v_j. Now in $C(P_n)$ we can see that the vertex v_i is adjacent with all the vertices except the vertices v_{i+1} and v_{i-1} for i = 2, 3, ..., n – 1. v_n is adjacent with all the vertices except v_{n-1} and v_1 is adjacent with all the vertices except v_2. Now consider a blind colouring of $C(P_n)$ as follows. Assign the colour c_i to v_i for i = 1, 2, ..., n due to the above mentioned non-adjacency of v_i's this colouring will not be a b-colouring. Thus to make it a b-colouring, we should assign a proper colouring to v_{ij}'s. Consider an internal vertex v_i of P_n, but v_i is not adjacent with v_{i+1} and v_{i-1}. Thus to realise the colour c_i we should colour $v_{i,i+1}$ as c_{i-1} and $v_{i,i-1}$ as c_{i+1}. Thus v_i will realise the colour c_i. Now take the vertex v_{i+1}, which is coloured as c_{i+1}. In order to realise the colour c_{i+1}, we should colour the two neighbours of v_{i+1} as c_{i+2} and c_i, but by the previous colouring v_i had left out only one vertex namely $v_{i+2,i+3}$ to be coloured. Thus realisation of c_{i+1} is not possible. Similarly this will occur for v_{i-1} too. This shows that assigning different colours to v_i is not possible i.e. there should be repetation of colours. For $n \equiv 0, 1, 2, 3 \pmod 5$ there are $\left[\frac{n}{5}\right]+1$ repetitions otherwise $\left[\frac{n}{5}\right]+1$ repetitions.

Case: 1

When r = 4, i = 1, 2, ..., n and $i \equiv 0, 1, 2 \pmod 5$ assign the colour $c_{i-\left[\frac{i}{5}\right]}$ to the vertex v_i otherwise assign the colour $c_{i-\left(\left[\frac{i}{5}\right]+1\right)}$ to the vertex v_i. Here also only the vertex with repeated colours realises its own colour. Thus to make the colouring a b-chromatic one, we assign a proper colouring to v_{ij}'s as follows. For i = 2, 3, ..., n – 2 and $i \equiv 0, 1, 2, \pmod 5$ assign the colour $c_{i-\left(\left[\frac{i}{5}\right]+1\right)}$ to the vertex v_{ij} otherwise assign the colour $c_{i+1-\left[\frac{i}{5}\right]}$ to the vertex v_{ij}. For remaining v_{ij}'s we can assign any already assigned colours. Now all v_i's for i = 1, 2, ..., n realises its own colour c_i. Hence by colouring procedure it is the maximum colouring.

Case: 2

When $r \neq 4$, for i = 1, 2, ..., n and $i \equiv 0, 1, 2 \pmod 5$ assign the colour $c_{i-\left[\frac{i}{5}\right]}$ to the vertex v_i otherwise assign the colour $C_{i-\left(\left[\frac{i}{5}\right]\right)+1}$ to the vertex v_i. Here also only the vertex with repeated colours realises its own colour. Thus to make the colouring a b-chromatic one we assign a proper colouring to v_{ij}'s as follows. For i = 2, 3, ..., n – 3 and $i \equiv 0, 1, 2 \pmod 5$ assign the colour $C_{i-\left(\left[\frac{i}{5}\right]\right)+1}$ to the vertex v_{ij} otherwise assign the colour $C_{i+1-\left(\left[\frac{i}{5}\right]\right)}$ to the vertex v_{ij} and the remaining v_{ij} otherwise assign the colour $c_{i+1-\left(\left[\frac{i}{5}\right]\right)}$ to the vertex v_{ij} and the remaining v_{ij}'s can be coloured with already used colours. Now the only vertex remaining is to colour v_n. Suppose we assign a new colour to the v_n, the vertex does not realises the new colour because u_n is not adjacent with v_{n-1}. Thus to realise the new colour we should colour the neighbour of v_n as c_{n-1}, which is not possible by colouring procedure. Thus introducing a new colour to the vertex v_n is not possible. Note that any rearrangement of the colours to the graph also fails to accomodate the new colour. Hence by colouring procedure this is a b-chromatic colouring and furthermore it is the maximum colouring possible.

Example

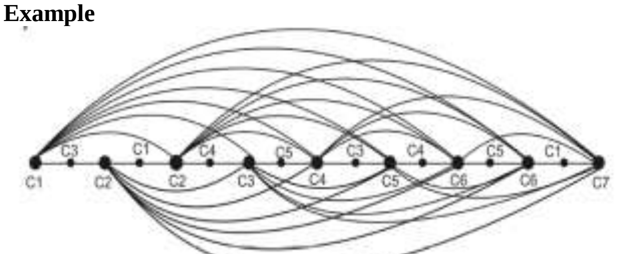

Figure 3 : $\varphi[C(P_9)] = 7$

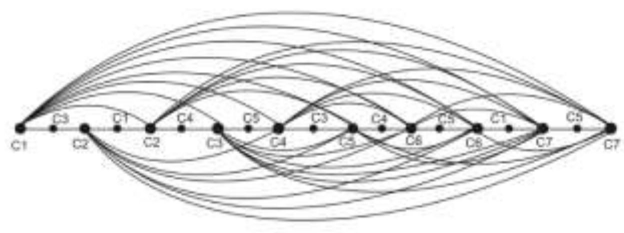

Figure 4 : $\varphi[C(P_{10})] = 7$

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