*GαO-Kernel in the Digital Plane

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ABSTRACT
Digital topology was first studied in the late 1960's by the computer image analysis researcher Azriel Rosenfeld[9]. The digital plane is a mathematical model of the computer screen. In this paper we investigate explicit forms of *GαO-kernel and *gα-closed sets in the digital plane. Also we prove that the digital plane is an $_{\alpha}T_{1/2}^{**}$ space.

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1. INTRODUCTION
The digital line, the digital plane and the three dimensional digital spaces are of great importance in the study of applications of point set topology to computer graphics. Digital topology consists in providing algorithmic tools for Pattern Recognition, Image Analysis and Image processing using a discrete formalism for geometrical objects. It is applied in image processing.

First we recall the related definitions and some properties of the digital plane. The digital line or the so called Khalimsky line is the set of the integers Z, equipped with the topology k having $\{\{2n+1,2n,2n-1\}/n\in Z\}$ as a subbase. This is denoted by (Z, k). Thus, a subset U is open in (Z, k) if and only if whenever $x\in U$ is an even integer, then $x-1, x+1\in U$. Let (Z^2, k^2) be the topological product of two digital lines (Z, k), where $Z^2 = Z\times Z$ and $k^2=k\times k$. This space is called the digital plane in the present paper (cf.[2],[3],[4],[5]). We note that for each point $x\in Z^2$ there exists the smallest open set containing x, say U(x). For the case of $x = (2n+1,2m+1)$, $U(x)=\{2n+1\}\times \{2m+1\}$; for the case of $x = (2n,2m)$, $U(x) = \{2n-1,2n,2n+1\}\times \{2m-1,2m,2m+1\}$; for the case of $x = (2n,2m+1)$, $U(x) = \{2n-1,2n,2n+1\}\times \{2m+1\}$; for the case of $x = (2n+1,2m)$, $U(x) = \{2n+1\}\times \{2m-1,2m,2m+1\}$, where $n,m\in Z$. For a subset E of (Z^2, k^2). We define the following three subsets as follows: $E_F = \{x\in E \mid x \text{ is closed in } (Z^2, k^2)\}$; $E_k^2 = \{x\in E \mid/ x \text{ is open in } (Z^2,k^2)\}$; $E_{mix}= E \mid (E_F\cup E_{k2})$. Then it is shown that $E_F = \{(2n, 2m)\in E \mid n,m \in Z\}$, $E_k^2= \{(2n+1, 2m+1)\in E \mid n,m\in Z\}$ and $E_{mix}=\{(2n, 2m+1) \in E \mid n,m \in Z\}\cup \{(2n+1, 2m) \in E \mid n,m \in Z\}$.

2. PRELIMINARIES
2.1 Definition
A subset A of a space (X, τ) is called an α-open set [8] if $A \subseteq int(cl(int((A)))$ and an α-closed set if $cl(int(cl(A))) \subseteq A$.

2.2 Definition
A subset A of a space (X, τ) is called
1. a generalized α-closed (briefly gα-closed) set [7] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in (X, τ),
2. a *gα-closed set [4] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is gα-open in (X, τ) and
3. a *gα-open set [4] if $U \subseteq int(A)$ whenever $U \subseteq A$ and U is gα-closed in (X, τ).

2.3 Definition
A subset A of a space (X, τ) is called an $_{\alpha}T_{1/2}^{**}$ space[4] if every *gα-closed set is closed.

3. *GαO-KERNEL IN THE DIGITAL PLANE
We prepare the following notations:
For a subset A of (X, τ),

$G\alpha O(X, \tau) = \{U \mid U \text{ is } g\alpha\text{-open in } (X, \tau)\}$[4];

$^{*}G\alpha O(X, \tau) = \{U \mid U \text{ is } ^{*}g\alpha\text{-open in } (X, \tau)\}$;

$ker(A) = \cap\{U \mid U\in\tau \text{ and } A \subseteq U\}$[3];

$G\alpha O\text{-}ker(A) = \cap\{U \mid U\in G\alpha O(X, \tau) \text{ and } A \subseteq U\}$[4].

$^{*}G\alpha O\text{-}ker(A) = \cap\{U \mid U\in {}^{*}G\alpha O(X, \tau) \text{ and } A \subseteq U\}$.

3.1 Theorem
Let A and E be subsets of (Z^2, k^2).

(i) If E is non - empty gα-closed set, then $E_F \neq \phi$.

(ii) If E is gα-closed and $E \subseteq B_{mix} \cup B_k^2$ holds for some subset B of (Z^2, k^2) then $E = \phi$.

(iii) The set $U (A_F) \cup A_{mix} \cup A_k^2$ is a *gα-open set containing A.

Proof
(i): Let y be a point in E. Then, $y\in cl(E)=E= E_F \cup E_{mix} \cup E_k^2$. We consider the following three cases for the point y.

Case1: $y \in E_k^2$: Let y =(2n+1,2m+1) for some n,m∈Z.

Then cl({y})={2n,2n+1,2n+2}×{2m,2m+1,2m+2} ⊆ E. Thus there exists a point x = (2n,2m)∈ E_F.

Case2: $y \in E_{mix}$: Let y =(2n+1,2m) for some n,m∈z. Then cl({y})={2n,2n+1,2n+2}×{2m}⊆ E. Thus there exists a point x = (2n,2m) ∈ E_F. The proof for y =(2n,2m+1) is similar to the above, where n,m∈Z.

Case3: $y \in E_F$: Then $E_F \neq \phi$.

Now we shown that $E_F \neq \phi$ for all the three cases.

(ii): Suppose that E ≠ ϕ. By (i), $E_F \neq \phi$. Since $E_F \subseteq (B_{mix} \cup B_k^2)_F = \phi$, we have a contradiction.

(iii): We claim that $A_{mix} \cup A_k^2$ is a *gα-open set . Let F be any non-empty gα-closed set such that $F \subseteq A_{mix} \cup A_k^2$. Then by (ii), F = ϕ. Thus, we have that $F \subseteq$ Int $(A_{mix} \cup A_k^2)$ then $A_{mix} \cup A_k^2$ is *gα - open . But we know that U (A_F) is a open set. Then $U(A_F) \cup A_{mix} \cup A_k^2$ is *gα-open (cf., Remark 3.18 [4]). But $A = A_F \cup A_{mix} \cup A_k^2$, then $A \subseteq U(A_F) \cup A_{mix} \cup A_k^2$. This implies that *gα-open set contains A.

3.2 Theorem

Let A be a subset of (Z^2, k^2). The *GαO-kernel of A and the GαO-kernel of A are obtained precisely as follows:

(i) *GαO-ker (A) = U $(A_F) \cup A_{mix} \cup A_k^2$,
where $U(A_F) = \cup\{U(x) \mid x \in A_F\}$.

(ii) GαO-ker (A) = U(A) ,
where U (A) = ∪ { U (x) | x ∈A}.

Proof

(i): Let $U_A = U(A_F) \cup A_{mix} \cup A_k^2$. By Theorem 3.1 (iii), *GαO-ker (A) ⊆ U_A. To prove $U_A \subseteq$ *GαO-ker (A), it is claimed that (∗) if there exists a *gα-open set V such that $A \subseteq V \subset U_A$ then $V = U_A$. Indeed, let x be any point of U_A. There are three cases for the point x.

Case (1):

$x \in (U_A)_F$: We note that $(U_A)_F = (U(A_F))_F \cup (A_{mix} \cup A_k^2)_F = A_F$.

Then we have that $x \in A_F \subseteq A \subseteq V$. Implies x ∈V.

Case (2):

$x \in (U_A)_k^2$: We note that

$(U_A)_k^2 = (U(A_F)_k^2) \cup (A_{mix})_k^2 \cup (A_k^2)_k^2 = (U(A_F))_k^2 \cup A_k^2$.

Firstly suppose that $x \in U(A_F)$, then x ∈U(y) for some $y \in A_F$. Since $y \in A_F \subseteq A \subseteq V$ and V is *gα-open, we have {y} ⊆ Int (V). Then U(y) ⊆ Int (V), because Int (V) is open. Thus we have that x ∈ V.

Secondly, suppose $x \in A_k^2$, then we have x ∈ V, because $x \in A_k^2 \subseteq A \subseteq V$.

Case (3):

$x \in (U_A)_{mix}$: We note that

$(U_A)_{mix} = (U(A_F))_{mix} \cup (A_k^2)_{mix} \cup (A_{mix})_{mix}$
$= (U(A_F))_{mix} \cup A_{mix}$

First, suppose that $x \in U(A_F)$. Then x ∈ U(y) for some $y \in A_F$. Then y be a gα-closed since every closed point is gα-closed. Since $y \in A_F \subseteq A \subseteq V$, {y} is gα-closed and V is *gα - open set, we have {y} ⊆ Int (V). Then U (y) ⊆ Int (V) ⊆ V and so x∈ V.

Secondly, suppose that $x \in A_{mix}$. Then $x \in A_{mix} \subseteq A \subseteq V$ implies x ∈ V.

For all cases we assume that $x \in U_A$ then we show that x ∈ V, then $U_A \subseteq V$. But we know that $V \subseteq U_A$. From the above cases we conclude that $V = U_A$. Thus we shown (∗).

Let *GαO(A) be the family of all *gα-open sets containing A. Then, we have that $U_A \subseteq W$ for each W∈ *GαO(A), using (∗) above and properties that $A \subseteq W \cap U_A \subseteq U_A$ and $W \cap U_A$ is *gα -open set.[4, Theorem 3.17].

Hence, we show that $U_A \subseteq \cap\{W \mid W \in$ *GαO(A) and A⊆W }= *GαO-ker (A).

That is $U_A \subseteq$ *GαO-ker (A). Therefore *GαO-ker (A) = U_A.

(ii): Since U(A) is open, GαO-ker (A) ⊆ U(A) holds. To prove U(A) ⊆ GαO-ker (A), it is prove that (**) if there exists an open set V such that A ⊆ V⊆ U(A). Let W be any *gα-open sets containing A. Then, we have that U(A) ⊆ W, using (**) above and properties that A ⊆ W∩U(A) ⊆ U(A) and W∩U(A) is gα-open. Hence, we show that U(A) ⊆ ∩{W | W∈ GαO(A) and A⊆W}= GαO-ker (A). Therefore GαO-ker (A) = U(A).

3.3 Theorem

Let E be a subset of (Z^2, k^2).

(i) If E is a non-empty *gα-closed set, then $E_F \neq \phi$.

(ii) If E is *gα-closed set and $E \subseteq B_{mix} \cup B_k^2$ holds for some subset B of (Z^2, k^2), then E = ϕ.

Proof

(i): We recall that a subset E is *gα-closed if and only if cl(E) ⊆ GαO-ker (E) [4, Theorem 3.21]. Let y be a point of E.

We consider the following three cases for the point y.

Case (1):

$y \in E_k^2$: Let y = (2n+1, 2m+1) for some n,m ∈ Z. Then cl({y})= {2n, 2n+1, 2n+2} × {2m, 2m+1, 2m+2}⊆ cl(E) ⊆ GαO-ker(E). Thus there exists a point (2n, 2m)∈GαO-ker (E), say y_1 = (2n, 2m). Using Theorem 3.2(ii), we have that $y_1 \in U(z)$ for some z∈E.

If $z \in E_{mix}$, say z = (2s+1, 2t) for some s,t∈Z, then U(z) = {2s+1}× {2t-1,2t,2t+1} and $y_1 \notin U(z)$. This is a contradiction.

Next if $z \in E_k^2$, say z = (2s+1, 2t+1) for some s,t∈Z, then U(z) = {(2s+1, 2t+1)} and $y_1 \notin U(z)$. This is also a contradiction.

Thus we have that $z \in E_F$ and hence $E_F \neq \phi$ for case 1.

Case (2):

$y \in E_{mix}$.: Let y = (2n+1, 2m) for some n,m ∈ Z. Then, cl({y})= {2n, 2n+1, 2n+2} × {2m}⊆ cl(E) ⊆ GαO-ker(E). Thus there exists a point (2n, 2m)∈ GαO-ker(E), say y_1 = (2n, 2m). Using Theorem 3.2(ii), we have that $y_1 \in U(z)$ for some z∈E.

If $z \in E_{mix}$, say z = (2s+1, 2t) for some s,t∈Z, then U(z) = {2s+1}× {2t-1,2t,2t+1} and $y_1 \notin U(z)$. This is a contradiction.

Next if $z \in E_k^2$, say z = (2s+1, 2t+1) for some s,t∈Z, then U(z) = {(2s+1, 2t+1)} and $y_1 \notin U(z)$. This is also a contradiction.

Thus we have that $z \in E_F$ and hence $E_F \neq \phi$ for case 2.

Case (3):

$y \in E_F$. Then $E_F \neq \phi$.

We shown that $E_F \neq \phi$ for all the three cases.

(ii): Suppose that $E \neq \phi$. Then using (i) we have that $E_F \neq \phi$. It follows from assumption and definition that $E_F \subseteq (B_{mix} \cup B_k{}^2)_F = \phi$. We have a contradiction.

3.4 Theorem

The digital plane (Z^2, k^2) is an ${}_{\alpha}T_{1/2}{}^{**}$ space.

Proof

Let x be a point of (Z^2, k^2).

Case (1): x = (2m+1,2n+1), where n,m$\in$Z: The singleton {x} is open in (Z^2, k^2).

Case (2): x = (2m,2n),where n,m$\in$Z: The singleton {x} is closed in (Z^2, k^2) and also it is gα-closed set[7].

Case (3): x = (2m,2n+1) where n,m$\in$Z : Let be U any α-open set containing {x}, that is {x}$\subseteq$ U. Then αcl({x})={x}$\cup$cl(int(cl(x))) = {x}$\cup$ cl(int({2m} $\times${2n,2n+1,2n+2})) = {x}$\cup$cl(ϕ) = {x}$\subseteq$ U. Therefore {x} is gα-closed set in (Z^2, k^2).

Case (4): x = (2m+1,2n), where n,m$\in$Z : The proof is similar that of case(3).

Thus we prove that every singleton {x} of (Z^2, k^2) is either gα-closed or open in all the four cases.

Therefore the statement "If (X, τ) is ${}_{\alpha}T_{1/2}{}^{**}$ space if every singleton of {x} is either gα-closed set or open" [4, Theorem 4.2] is true.

Therefore the digital plane (Z^2, k^2) is an ${}_{\alpha}T_{1/2}{}^{**}$ space.

3.5 Theorem

Let E be a subset of (Z^2, k^2). If E is *gα-closed and dense set in (Z^2, k^2), then (Z^2, k^2) is the only gα-open set containing E.

Proof

Let U be a gα-open set containing E. Then, cl(E) $\subseteq$ U, since E is *gα-closed set in (Z^2, k^2). Then $(Z^2, k^2) \subseteq$ U, since E is dense in (Z^2, k^2). Therefore (Z^2, k^2) is the only gα-open set containing E.

4. CONCLUSION

In this work we derived some of the properties of *GαO-kernel and *gα-closed sets in the digital plane. Also we have proved that the digital plane is an ${}_{\alpha}T_{1/2}{}^{**}$ space.

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