On Upper and Lower Faintly αψ-Continuous Multifunctions

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ABSTRACT
In this paper we introduce the notion of upper and lower faintly αψ-continuous multifunctions. The basic properties and characterizaions of such functions are established.

Keywords

αψ-open sets, αψ-closed sets, faintly αψ-continuous multifunctions, αψ-θ-closed.

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1. INTRODUCTION

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good number of them have been extended to the setting of multifunctions. This implies that both functions and multifunctions are important tools for studying properties of spaces and for constructing new spaces from previously existing ones. R.Devi et all introduced the con-cept of αψ-closed sets [1] in topological spaces. In this paper, we introduce and study upper and lower faintly αψ-continuous multifunctions in topological spaces. The main purpose of this paper is to define faintly αψ-continuous multifunctions and to obtain several characterizations and basic properties of such multifunctions. A subset A of X is called regular open (resp. regular closed) if and only if A = int(cl(A)) (resp. A = cl(int(A))). The family of all regular open subsets of (X, τ) form a base for a smaller topology τ_s on X.

2. PRELIMINARIES

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) mean topological spaces on which no separation axioms are assumed unless explicitly stated. For any subset A of X, the closure and the interior of A are denoted by cl(A) and int(A), respectively. A point x ∈ X is called a θ-cluster point of A if cl(V)∩A ≠ φ for every open subset V of X containing x. The set of all θ-cluster points of A is called the θ-closure of A and is denoted by $cl_\theta(A)$. If A = $cl_\theta(A)$, then A is said to be θ-closed [3]. The complement of a θ-closed set is said to be θ-open. Clearly, A is θ-open if and only if for each x ∈ A, there exists an open set U such that x ∈ U ⊂ cl(U) ⊂ A. A subset A of (X, τ) is said to be αψ-closed [1] if ψcl(A) ⊂ U, whenever A ⊂ U and U is α-open . The complement of a αψ-closed set is called αψ-open. The family of all αψ-open subsets of (X, τ) will be denoted by αψO(X). By a multifunction F : X → Y , we mean a point to-set correspondence from X into Y , also we always assume that F (x) ≠ φ for all x ∈ X. For a multifunction F : X → Y , the upper and lower inverse of any subset A of Y are denoted by $F^+(A)$ and $F^-(A)$ respectively, where $F^+(A)$ ={ x∈ X : F (x) ⊂ A} and $F^-(A)$ = { x∈ X : F (x) ∩ A ≠ φ }. In particular, $F^-(y)$ = x∈ X : y∈ F (x) for each point y ∈ Y . A multifunction F : X → Y is said to be surjective if F (X) = Y . A multifunction F : (X, τ) → (Y, σ) is said to be lower αψ-continuous (resp. upper αψ-continuous) multifunction if $F^-(V)$ ∈ αψO(X) (resp. $F^+(V)$ ∈ αψO(X)) for every V ∈ σ.

3. FAINTLY AΨ-CONTINUOUS MULTIFUNCTIONS

3.1 Definition

A multifunction F : X → Y is said to be:

(i) upper faintly αψ-continuous at x ∈ X if for each θ-open subset V of Y containing F (x), there exists U ∈ αψO(X) containing x such that F (U) ⊂ V ;

(ii) lower faintly αψ-continuous at x ∈ X if for each θ-open subset V of Y such that F (x) ∩ V ≠ φ, there exists U ∈ αψO(X) containing x such that F (u) ∩ V ≠ φ for every u ∈ U;

(iii) upper (resp. lower) faintly αψ-continuous if it is upper (resp. lower) faintly αψ-continuous at each point of X.

3.2 Remark

Since every θ-open set is open, it is clear that every upper (lower) αψ-continuous multifunction is upper (lower) faintly αψ-continuous. However, the converse is not true as the following simple example shows.

3.3 Example

Let X = {a, b, c} , τ = {X, φ, {b}} , σ = {X, φ, {a}}. Then the multifunction F : (X, τ) → (X, σ) defined by F (x) = {x} is upper (lower) faintly αψ-continuous (observe that the only θ-open subsets of (X, τ) are X and φ, so we may take U = X in the definition of an upper (lower) faintly αψ-continuous multifunction). However, F is not upper (lower) αψ-continuous (observe that $F^+(\{a\}) = F^-(\{a\})$ = {a} is not αψ-open in (X, τ)).

3.4 Theorem

For a multifunction $F : X \to Y$, the following are equivalent:

(i) F is upper faintly $\alpha\psi$-continuous;

(ii) For each $x \in X$ and for each θ-open set V such that $x \in F^{+}(V)$, there exists a $\alpha\psi$-open set U containing x such that $U \subset F^{+}(V)$;

(iii) For each $x \in X$ and for each θ-closed set V such that $x \in F^{+}(Y - V)$, there exists a $\alpha\psi$-closed set H such that $x \in X - H$ and $F^{-}(V) \subset H$;

(iv) $F^{+}(V)$ is $\alpha\psi$-open for any θ-open subset V of Y;

(v) $F^{-}(V)$ is $\alpha\psi$-closed for any θ-closed subset V of Y;

(vi) $F^{-}(Y - V)$ is $\alpha\psi$-closed for any θ-open subset V of Y;

(vii) $F^{+}(Y - V)$ is $\alpha\psi$-open for any θ-closed subset V of Y.

Proof

(i) $\Leftrightarrow$ (ii): Clear.

(ii) $\Leftrightarrow$ (iii): Let $x \in X$ and V be a θ-closed subset of Y such that $x \in F^{+}(Y - V)$. By (ii), there exists a $\alpha\psi$-open set U containing x such that $U \subset F^{+}(Y - V)$. Thus $F^{-}(V) \subset X - U$. Take $H = X - U$. Then $x \in X - H$ and H is $\alpha\psi$-closed. The converse is similar.

(i) $\Leftrightarrow$ (iv): Let $x \in F^{+}(V)$ and V be a θ-open subset of Y. By (i), there exists a $\alpha\psi$-open set U_x containing x such that $U_x \subset F^{+}(V)$. Thus, $F^{+}(V) = \cup_{x \in F^{+}(V)} U_x$. Since any union of $\alpha\psi$-open sets is $\alpha\psi$-open, $F^{+}(V)$ is $\alpha\psi$-open. The converse is clear.

(iv) $\Leftrightarrow$ (vii) and (v) $\Leftrightarrow$ (vi): Clear.

(iv) $\Leftrightarrow$ (vi): Follows from the fact that $F^{-}(V) = X - F^{+}(Y - V)$.

3.5 Theorem

For a multifunction $F : X \to Y$, the following are equivalent:

(i) F is lower faintly $\alpha\psi$-continuous;

(ii) For each $x \in X$ and for each θ-open set V such that $x \in F^{-}(V)$, there exists a $\alpha\psi$-open set U containing x such that $U \subset F^{-}(V)$;

(iii) For each $x \in X$ and for each θ-closed set V such that $F^{-}(Y - V)$, there exists a $\alpha\psi$-closed set H such that $x \in X - H$ and $F^{+}(V) \subset H$;

(iv) $F^{-}(V)$ is $\alpha\psi$-open for any θ-open subset V of Y;

(v) $F^{+}(V)$ is $\alpha\psi$-closed for any θ-closed subset V of Y;

(vi) $F^{+}(Y - V)$ is $\alpha\psi$-closed for any θ-open subset V of Y;

(vii) $F^{-}(Y - V)$ is $\alpha\psi$-open for any θ-closed subset V of Y.

Proof.

Similar to that of Theorem 3.4.

3.6 Theorem

Suppose that (X, τ) and (X_i, τ_i) are topological spaces where $i \in I$. Let $F: X \to \Pi_{i \in I} X_i$ be a multifunction from X to the product space $\Pi_{i \in I} X_i$ and let $P_i: \Pi_{i \in I} X_i \to X_i$ be a projection multifunction for each $i \in I$ which is defined by $P_i((x_i)) = \{x_i\}$. If F is upper (lower) faintly $\alpha\psi$-continuous, then $P_i \circ F$ F is upper (lower) faintly $\alpha\psi$-continuous for each $i \in I$.

Proof

Let V_i be a θ-open set in (X_i, τ_i). Then $(Pi \circ F)^{+}(V_i) = F^{+}(P^{+}_i(V_i)) = F^{+}(V_i \times \Pi_{j \neq i} X_j)$ (resp. $(P_i \circ F)^{-}(V_i) = F^{-}(P^{-}_i(V_i)) = F^{-}(V_i \times \Pi_{j \neq i} X_j)$). Since F is upper (lower) faintly $\alpha\psi$-continuous and since $V_i \times \Pi_{j \neq i} X_j$ is a θ-open set, it follows from Theorems 3.4 and 3.5 that $F^{+}(V_i \times \Pi_{j \neq i} X_j)$ (resp. $F^{-}(V_i \times \Pi_{j \neq i} X_j)$) is a $\alpha\psi$-open set in (X, τ). Hence again by Theorems 3.4 and 3.5, $P_i \circ F$ is upper (lower) faintly $\alpha\psi$-continuous for each $i \in I$.

3.7 Corollary

Let $F : X \to Y$ be a multifunction. If the graph multifunc-tion G_F of F is upper (lower) faintly $\alpha\psi$-continuous, then F is upper (lower) faintly $\alpha\psi$-continuous, where $G_F : X \to X \times Y$, $G_F(x) = \{x\} \times F(x)$.

3.8 Corollary

Suppose that (X, τ), (Y, σ), (Z, η) are topological spaces and $F_1 : X \to Y$, $F_2 : X \to Z$ are multifunctions. Let $F_1 \times F_2 : X \to Y \times Z$ be the multifunction defined by $F_1 \times F_2(x) = F_1(x) \times F_2(x)$ for each $x \in X$. If $F_1 \times F_2$ is upper (lower) faintly $\alpha\psi$-continuous, then F_1 and F_2 are upper (lower) faintly $\alpha\psi$-continuous.

The following lemma can be easily established.

3.9 Lemma

If $A \times B \in \alpha\psi O(X \times Y)$, then $A \in \alpha\psi O(X)$ and $B \in \alpha\psi O(Y)$.

3.10 Theorem

Suppose that (X_i, τ_i) and (Y_i, σ_i) are topological spaces for each $i \in I$. Let $F_i : X_i \to Y_i$ be a multifunction for each $i \in I$ and let $F : \Pi_{i \in I} X_i \to \Pi_{i \in I} Y_i$ be the multifunction defined by $F((x_i)) = \Pi_{i \in I} F_i(x_i)$. If F is upper (lower) faintly $\alpha\psi$-continuous, then F_i is upper (lower) faintly $\alpha\psi$-continuous for each $i \in I$.

Proof

Let V_i be a θ-open subset of Y_i. Then $V_i \times \Pi_{j \neq i} X_j$ is a θ-open set. Since F is upper (lower) faintly $\alpha\psi$-continuous, it follows from Theorems 3.4 and 3.5 that $F^{+}(V_i \times \Pi_{j \neq i} Y_j) = F_i^{+}(V_i) \times \Pi_{j \neq i} X_j$ (resp. $F^{-}(V_i \times \Pi_{j \neq i} Y_j) = F_i^{-}(V_i) \times \Pi_{j \neq i} X_j$). Consequently, it follows from Lemma 3.9 that $F_i^{+}(V_i)$ (resp. $F^{-}(V_i)$) is a $\alpha\psi$-open set. Thus again by Theorems 3.4 and 3.5, F_i is upper (lower) faintly $\alpha\psi$-continuous for each $i \in I$.

3.11 Corollary

Suppose that $F_1 : X_1 \to Y_1$, $F_2 : X_2 \to Y_2$ are multifunctions. If $F_1 \times F_2$ is upper (lower) faintly αψ-continuous, then F_1 and F_2 are upper (lower) faintly αψ-continuous, where $F_1 \times F_2$ is the product multifunction defined as follows: $F_1 \times F_2 : X_1 \times X_2 \to Y_1 \times Y_2$, $F_1 \times F_2((x_1, x_2)) = F_1(x_1) \times F_2(x_2)$, where $x_1 \in X_1$ and $x_2 \in X_2$.

Recall that a multifunction $F : X \to Y$ is said to be punctually closed if for each $x \in X$, $F(x)$ is closed. Recall also that a space X is called θ-normal if for any disjoint closed subsets F_1, F_2 of X, there exist two disjoint θ-open subsets V_1, V_2 of X containing F_1, F_2, respectively.

3.12 Definition

A topological space (X, τ) is said to be αψ-T_2 (resp. θ-T_2 [5]) if for each pair of distinct points x and y of X, there exist disjoint αψ-open (resp. θ-open) subsets U and V of X containing x and y, respectively.

3.13 Theorem

Let $F : X \to Y$ be an upper faintly αψ-continuous multi-function and punctually closed from a topological space X into a θ-normal space Y such that $F(x) \cap F(y) = \varphi$ for each pair of distinct points x and y of X. Then X is αψ-T_2.

Proof

Let x and y be any two distinct points of X. Then $F(x) \cap F(y) = \varphi$. Since Y is θ-normal and F is punctually closed, there exist disjoint θ-open sets U and V containing F (x) and F (y), respectively, but F is upper faintly αψ-continuous, so it follows from Theorem 3.4 that $F^+(U)$ and $F^+(V)$ are disjoint αψ-open subsets of X containing x and y, respectively. Hence, X is αψ-T_2.

3.14 Definition

A topological space (X, τ) is said to be θ-compact [5] (resp. αψ-compact) if every θ-open (resp. αψ-open) cover of X has a finite subcover. A subset A of a topological space X is said to be θ-compact relative to X if every cover of A by θ-open subsets of X has a finite subcover of A.

3.15 Theorem

Let $F : X \to Y$ be an upper faintly αψ-continuous surjec-tive multifunction such that F (x) is θ-compact relative to Y for each $x \in X$. If X is αψ-compact, then Y is θ-compact.

Proof

Let $V_\alpha : \alpha \in \Lambda$ be a θ-open cover of Y . Since F (x) is θ-compact relative to Y for each $x \in X$, there exists a finite subset $\Lambda(x)$ of Λ such that $F(x) \subset \cup_{\alpha\in\Lambda(x)} V_\alpha$. Put $V(x) = \cup_{\alpha\in\Lambda(x)} V_\alpha$. Then V (x) is a θ-open subset of Y containing F (x). Since F is upper faintly αψ-continuous, it follows from The-orem 3.4 that $F^+(V(x))$ is a αψ-open subset of X containing {x}. Thus the family $\{F^+(V(x)) : x \in X\}$ is a αψ-open cover of X, but X is αψ-compact, so there exist $x_1, x_2, ..., x_n \in X$ such that $X = \cup^n_{i=1} F^+(V(x_i))$. Hence, $Y = F(\cup^n_{i=1} F^+(V(x_i))) = \cup^n_{i=1} F(F^+(V(x_i))) \subset \cup^n_{i=1} V(x_i) = \cup^n_{i=1} \cup_{\alpha\in\Lambda(x_i)} V_\alpha$. Hence, Y is θ-compact.

For a given multifunction $F : X \to Y$, the graph multifunction $G_F : X \to X \times Y$ is defined as $G_F(x) = \{x\} \times F(x)$ for every $x \in X$. In [4], it was shown that for a multifunction $F : X \to Y$, $G^+_F(A \times B) = A \cap F^+(B)$ and $G^-_F(A \times B) = A \cap F^-(B)$ where $A \subseteq X$ and $B \subseteq Y$. A multifunction $F : X \to Y$ is said to be a point closed if and only if for each $x \in X$, F (x) is closed in Y .

3.16 Definition

Let $F : X \to Y$ be a multifunction. The multigraph $G(F) = \{(x, y) : y \in F(x), x \in X\}$ of F is said to be αψ-θ-closed if for each $(x, y) \in (X \times Y) - G(F)$, there exist a αψ-open set U and a θ-open set V containing x and y, respectively, such that $(U \times V) \cap G(F) = \varphi$, i.e. $F(U) \cap V = \varphi$.

3.17 Theorem

If the graph multifunction $F : X \to Y$ is upper (lower) faintly αψ-continuous, then F is upper (lower) faintly αψ-continuous.

Proof

We shall only prove the case where F is upper faintly αψ-continuous. Let $x \in X$ and V be a θ-open set in Y such that $x \in F^+(V)$. Then $G_F(x) \cap (X \times Y) = (\{x\} \times F(x)) \cap (X \times Y) = \{x\} \times (F(x) \cap V) \neq \varphi$ and $X \times V$ is θ-open in $X \times Y$ by Theorem 5 in [3]. Since the graph multifunction G_F is upper faintly αψ-continuous, there exists an open set U containing x such that $z \in U$ implies that $G_F(z) \cap (X \times V) \neq \varphi$. Therefore, we obtain $U \subseteq G^+_F(X \times V) = F^+(V) \in \alpha\psi O(X)$ from the above equalities. Consequently, F is upper faintly αψ-continuous.

3.18 Theorem

Let $F : X \to Y$, be a point closed multifunction. If F is upper faintly αψ-continuous and assume that Y is regular, then G(F) is θ-closed with respect to X.

Proof

Suppose $(x, y) \notin G(F)$. Then we have $y \notin F(x)$. Since Y is regular, there exist disjoint open sets V_1, V_2 of Y such that $y \in V_1$ and $F(x) \in V_2$. By regularity of Y , V_2 is also θ-open in Y . Since F is upper faintly αψ-continuous at x, there exists an αψ-open set U in X containing x such that $F(U) \subseteq V_2$. Therefore, we obtain $x \in U$, $y \in V_1$ and $(x, y) \in U \times V_1 \subseteq (X \times Y) - G(F)$. So G(F) is θ-closed with respect to X.

3.19 Theorem

Let $F : (X, \tau) \to (Y, \sigma)$ be a point closed set and upper faintly αψ-continuous multifunction. If F satisfies $x_1 \neq x_2 \Rightarrow F(x_1) \neq F(x_2)$ and Y is regular space, then X will be Hausdorff.

Proof

Let x_1, x_2 be two distinct points belong to X, then

$F(x_1) \neq F(x_2)$. Since F is point closed and Y is regular, for all $y \in F(x_1)$ with $y \notin F(x_2)$, there exists θ-open sets V_1, V_2 containing y and $F(x_2)$ respectively such that $V_1 \cap V_2 = \varphi$. Since F is upper faintly αψ-continuous and $F(x_2) \subseteq V_2$, there exists an open open set U containing x_2 such that $F(U) \subseteq V_2$. Thus $x_1 \notin U$. Therefore, U and X − U are disjoint open sets separating x_1 and x_2.

3.20 Theorem

If a multifunction $F : X \rightarrow Y$ is upper faintly αψ-continuous such that F (x) is θ-compact relative to Y for each $x \in X$ and Y is θ-T_2, then the multigraph G(F) of F is αψ-θ-closed.

Proof

Let $(x, y) \in (X \times Y) - G(F)$. Then $y \in Y - F(x)$. Since Y is θ-T_2, for each $z \in F(x)$, there exist disjoint θ-open subsets U(z) and V (z) of Y containing z and y, respectively. Thus $\{U(z) : z \in F(x)\}$ is a θ-open cover of F (x), but F (x) is θ-compact relative to Y , so there exist $z_1, z_2, ..., z_n \in F(x)$ such that $F(x) \subset \cup^{n}_{i=1} U(z_i)$. Put $U = \cup^{n}_{i=1} U(z_i)$ and $V = \cap^{n}_{i=1} V(z_i)$. Then U and V are θ-open subsets of Y such that $F(x) \subset U$, $y \subset V$ and $U \cap V = \varphi$. Since F is upper faintly αψ-continuous, it follows from Theorem 3.4 that $F^{+}(U)$ is a αψ-open subset of X. Also $x \in F^{+}(U)$ since $F(x) \subset U$ and $F(F^{+}(U)) \cap V = \varphi$ since $U \cap V = \varphi$. Hence, G(F) is αψ-θ-closed.

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