*gαc-Homeomorphisms in Topological Spaces

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ABSTRACT
R.Devi et al.[2] introduced the concepts of *gα-closed sets in topological spaces. They also introduced and studied the properties of *gα-continuous, *gα-irresolute, *gα-homeomorphism and *gαc-homeomorphism. In this paper we derive some other properties of *gαc-homeomorphism and Pasting Lemma for *gα-irresolute functions.

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1. INTRODUCTION
Research on the field of generalized closed sets was developed by many authors in the last two decades. The theory was extensively developed in the the 1990's. Several new concepts were studied and investigated.

Generalized homeomorphisms via generalized closed sets and gc-homeomorphisms in terms of preserving generalized closed sets were first introduced by Maki, Sundaram and Balachandran in [7]. Every homeomorphism is a generalized homeomorphiosm but not vice versa[6]. The two concepts coincide when both the domain and the range satisfy the weak seperation axiom $T_{1/2}$. The class of gc-homeomorphiosm is properly placed between the classes of homeomorphiosm and g-homeomorphiosm[7]. For more generalizations of homeomorphisms and relations among of them, the reader may refer to [1].

R.Devi et al.[2] introduced the concepts of *gα-closed sets in topological spaces. They also introduced and studied the properties of *gα-continuous, *gα-irresolute, *gα-homeomorphism and *gα-homeomorphism.

In this paper we study some other properties of *gαc-homeomorphism and the pasting lemma for *gα-irresolute maps.

Throught the paper (X, τ) and (Y, σ) represent non-empty topological spaces on which no seperation axioms are assumed, unless otherwise mentioned. For a subset A of (X, τ), cl(A) and int(A) represent the closure of A with respect to τ and the interior of A with respect to τ, respectively.

2. PRELIMINARIES
2.1 Definition
A subset A of a space (X, τ) is called an α-open set [8] if $A \subseteq int(cl(int((A)))$ and an α-closed set if $cl(int(cl(A))) \subseteq A$.

2.2 Definition
A subset A of a space (X, τ) is called

(i) a generalized α-closed set (briefly gα-closed) [6] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in (X, τ),

(ii) a *gα-closed set [2] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is gα-open in (X, τ) and

(iii) a *gα-open set [2] if if $U \subseteq int(A)$ whenever $U \subseteq A$ and U is gα-closed in (X, τ).

2.3 Definition
A function $f : (X, \tau) \to (Y, \sigma)$ is called

(i) *gα-continuous[2] if $f^{-1}(V)$ is *gα-closed in (X, τ) for every closed set V of (Y, σ),

(ii) *gα-irresolute[2] if $f^{-1}(V)$ is *gα-closed in (X, τ) for every *gα-closed set V of (Y, σ) and

(iii) *gαc-homeomorphism if both f and f^{-1} are *gα-irresolute and f is bijective.

3. PROPERTIES OF *gαc-HOMEOMORPHISMS
3.1 Lemma
Let $B \subseteq H \subseteq (X, \tau)$ and let $\tau\backslash H$ be the relative topology of H.

(i) If B is *gα-closed in (X,τ), then B is *gα-closed relative to H.

(ii) If B is *gα-closed in a subspace $(H, \tau\backslash H)$ and if H is clopen in (X, τ).

Note that a subset H of topological space (X, τ) is *gα-clopen if and only if H is open and *gα-closed.

We prepare some notations. Let $f : X \to Y$ be a function and H a subset of (X, τ). Let $f\backslash H : H \to Y$ be the restriction of f to H. We define a function $r_{H,K(f)} : H \to K$ by $r_{H,K}(f)(x) = f(x)$ for any $x \in H$, where $K = f(H)$. Then, $f\backslash H = jor_{H,K(f)}$ holds, where $j : K \to Y$ is an inclusion.

3.2 Theorem
Let H and K be subset of (X, τ) and (Y, σ) respectively.

(i) If $f : (X, \tau) \to (Y, \sigma)$ is *gα-irresolute and if H is a *gα-clopen subset of (X, τ), then the restriction $f\backslash H : (H, \tau\backslash H) \to (Y, \sigma)$ is *gα-irresolute.

(ii) Suppose that K is a $^*g\alpha$-clopen subset of (Y,σ). A function $K : (X, \tau) \to (K, \sigma\backslash K)$ is $^*g\alpha$-irresolute if and only if $jok : (X, \tau) \to (Y, \sigma)$ is $^*g\alpha$-irresolute, where $j : (K, \sigma\backslash K) \to (Y, \sigma)$ is an inclusion.

(iii) If $f : (X, \tau) \to (Y, \sigma)$ is a $^*g\alpha c$-homeomorphism such that $f(H) = K$ and K are $^*g\alpha$-clopen subset, then $r_{H,K}(f) : (H, \tau\backslash H) \to (K, \sigma\backslash H)$ is also $^*g\alpha c$-homeomorphism.

Proof

(i) Let F be a $^*g\alpha$-closed set of (Y, σ). Since f is $^*g\alpha$-irresolute, $(f\backslash H)^{-1} = f^{-1}(F)\cap H$, H is $^*g\alpha$-closed, $(f\backslash H)^{-1}(F)$ is $^*g\alpha$-closed in $(H, \tau\backslash H)$ by Lemma 3.1(i). Therefore $f\backslash H$ is $^*g\alpha$-irresolute.

(ii) **Necessity:** Let F be a $^*g\alpha$-closed set of (Y, σ). Then $(jok)^{-1}(F) = k^{-1}(j^{-1}(F)) = k^{-1}(F\cap H)$ is a $^*g\alpha$-closed in $(K, \sigma/K)$ by Lemma 3.1(i). Therefore $jok : (X, \tau) \to (Y, \sigma)$ is $^*g\alpha$-irresolute.
Sufficiency: Let V be a $^*g\alpha$-closed set of $(K, \sigma\backslash K)$. By Lemma 3.1(ii), $(jok)^{-1}(V) = k^{-1}(j^{-1}(V)) = k^{-1}(F\cap V) = k^{-1}(F)$ is $^*g\alpha$-closed in (X, τ). Therefore k is $^*g\alpha$-irresolute.

(iii) First, it suffices to prove $r_{H,K}(f): (H, \tau\backslash H) \to (K, \sigma\backslash K)$ is $^*g\alpha$-irresolute. Let F be a $^*g\alpha$-closed subset of (Y, σ).

$$
\begin{aligned}
(jor_{H,K}(f))^{-1}(F) &= (f\backslash H)^{-1}(F)\\
&= (f\backslash H)^{-1}(F\cap K)\\
&= f^{-1}(F\cap K)\\
&= f^{-1}(F) \cap K
\end{aligned}
$$

is $^*g\alpha$-closed in $(H, \tau\backslash H)$ and hence $jor_{H,K}(f)$ is $^*g\alpha$-irresolute. By (ii) $r_{H,K}(f)$ is $^*g\alpha$-irresolute.

Next we show that $r_{H,K}(f)^{-1} : (K,\sigma\backslash K) \to (H, \tau\backslash H)$ is $^*g\alpha$-irresolute. Since $(r_{H,K}(f))^{-1} = r_{H,K}(f^{-1})$ and since f^{-1} is $^*g\alpha$-irresolute, then using the first argument above for f^{-1} we have $(r_{H,K}(f))^{-1}$ is $^*g\alpha$-irresolute. Therefore $r_{H,K}(f)$ is a $^*g\alpha c$-homeomorphism.

By using Theorem 3.2, for a $^*g\alpha$-clopen subset H of (X,τ), we have a homomorphism called restriction $(r_H)_* : {}^*g\alpha ch(X,H;\tau) \to {}^*g\alpha ch(H, \tau\backslash H)$ as follows:

$(r_H)_*(f) = r_{H,H}(f)$ for any $f\in {}^*g\alpha ch(X,H;\tau)$. To prove that $(r_H)_*$ is onto we prepare the following:

3.3 Lemma (Pasting Lemma for $^*g\alpha$-irresolute functions)

Let (X, τ) be a topological space such that $X = A\cup B$ where A and B are $^*g\alpha$-clopen subsets. Let $f : (A, \tau\backslash A) \to (Y, \sigma)$ and $g : (B, \tau\backslash B) \to (Y, \sigma)$ be $^*g\alpha$-irresolute functions such that $f(x) = g(x)$ for every $x\in A\cap B$. Then the combination $f \nabla g(x) = f(x)$ for any $x\in A$ and $f \nabla g(y) = g(y)$ for any $y\in B$.

Proof

Let F be $^*g\alpha$-closed set of (Y, σ). Then $(f \nabla g)^{-1}(F) = f^{-1}(F)\cup g^{-1}(F)$, $f^{-1}(F)\in {}^*g\alpha c(X, \tau)$ by using Lemma. It follows from Theorem 3.17 [2] that $f^{-1}(F) \cup g^{-1}(F)$ is $^*g\alpha$-closed in (X, τ). Therefore $(f \nabla g)^{-1}(F)$ is $^*g\alpha$-closed in (X, τ) and hence $f \nabla g$ is $^*g\alpha$-irresolute.

3.4 Theorem

If H is a $^*g\alpha$-clopen subset of (X, τ), then $(r_H)_* : {}^*g\alpha ch(X,H;\tau) \to {}^*g\alpha ch(H, \tau/H)$ is an onto homomorphism.

Proof

Let $k\in {}^*g\alpha ch(H, \tau\backslash H)$. By Theorem 3.2, $j_1ok : (H, \tau\backslash H) \to (X, \tau)$ is $^*g\alpha$-irresolute, where $j_1 : (H, \tau\backslash H) \to (X, \tau)$ is an inclusion. Similarly it is shown that $j_2ol_{X\backslash H} : (X\backslash H, \tau\backslash(X\backslash H)) \to (X, \tau)$ is $^*g\alpha$-irresolute, where $j_2 : (X\backslash H, \tau\backslash(X\backslash H)) \to (X, \tau)$ is an inclusion.

By using Lemma 3.3, the combination $(j_1ok) \nabla (j_2ol_{X\backslash H}) : (X, \tau)\to(X, \tau)$, say k_1 is $^*g\alpha$-irresolute. It is easily shown that $k_1(x) = k(x)$ for any $x\in H$ and k_1 is bijective and $k_1^{-1} = (j_1ok^{-1}) \nabla (j_2ol_{X/H}) : (X, \tau)\to(X, \tau)$ is also $^*g\alpha$-irresolute. Therefore $k_1 : (X, \tau)\to(X, \tau)$ is the required $^*g\alpha c$-homeomorphism and $(r_H)_*(k_1) = k$ holds and hence $(r_H)_*$ is onto.

We define an equivalence relation R on $^*g\alpha ch(X,H;\tau)$ as follows:

fRh if and only if $f(x) = h(x)$ for any $x\in H$. Let [f] be the equivalence class of f. Let $H = \{f \mid f\in {}^*g\alpha ch(X,H;\tau)$ and $f(x) = x$ for any $x\in H\}$. Then, $H = \ker(r_H)_*$ and this is normal subgroup of $^*g\alpha ch(X,H;\tau)$. The factor group of $^*g\alpha ch(X,H;\tau)$ by H is $^*g\alpha ch(X,H;\tau)\backslash H = \{fH \mid f\in {}^*g\alpha ch(X,H;\tau)\}$, where $fH =\{\mu(f,k) \mid k\in H\} = [f]$.

Since $(r_H)_*$ is onto by Theorem 3.4, then the relation between the groups $^*g\alpha ch(X,H;\tau)$ is investigated as follows:

3.5 Theorem

If H is $^*g\alpha$-clopen subset of (X,τ), then $^*g\alpha ch(X,\tau\backslash H)$ is isomorphic to the factor group $^*g\alpha ch(X,H;\tau)\backslash H$.

Proof

By Theorem, $(r_H)_* : {}^*g\alpha ch(X,H;\tau) \to {}^*g\alpha ch(H, \tau\backslash H)$ is an onto homomorphism. Thus, we have the required isomorphism, $^*g\alpha ch(H, \tau\backslash H) \cong {}^*g\alpha ch(X,H;\tau)\backslash H$.

3.6 Theorem

If $\alpha : (X, \tau) \to (Y, \sigma)$ is a $^*g\alpha c$-homoemorphism such that $\alpha(H) = K$, then there is an isomorphism, $\alpha_* : {}^*g\alpha ch(X,H;\tau) \to {}^*g\alpha ch(H, H;\tau)$.

Proof

The isomorphism α_* is defined by $\alpha_*(f) = \alpha \circ f \circ \alpha^{-1}$. Let $(X\backslash R, \tau\backslash R)$ be the quotient topological space of (X, τ) by an equivalence relation R on X and let $\pi : (X, \tau) \to (X\backslash R, \tau\backslash R)$ be the canonical projection.

3.7 Definition [3]

A space (X, τ) is called $^*g\alpha$-connected if X cannot be expressed as the disjoint union of two non-empty $^*g\alpha$-closed sets.

3.8 Definition [2]

A function $f : (X, \tau) \to (Y, \sigma)$ is called $^*g\alpha$-closed if f(F) is $^*g\alpha$-closed in (Y, σ) for every closed set F of (X, τ).

3.9 Definition [2]

A space (X, τ) is called ${}_{\alpha}T_{1/2}^{**}$ if every $^*g\alpha$-closed set is closed.

3.10 Theorem

Let F be a subset of (X\R, τ\R). If π : (X, τ) → (X\R, τ\R) is *gα-closed function and π^{-1}(F) is *gα-closed in (X, τ), then F is *gα-closed, where (X, τ) is a $_{\alpha}T_{1/2}^{**}$ space.

Proof

Let π^{-1}(F) is *gα-closed in (X, τ). Then π^{-1}(F) is closed in (X, τ), since (X, τ) is $_{\alpha}T_{1/2}^{**}$ space. Then $\pi(\pi^{-1}(F))$ = F is *gα-closed in (X\R, τ\R),since π is a *gα-closed map.

3.11 Theorem

If π : (X, τ) → (X\R, τ\R) is *gα-continuous and the subset F is *gα-closed (X\R, τ\R), then π^{-1}(F) is *gα-closed in (X, τ).

Proof

Let F be a closed set in (X\R, τ\R). Then F is *gα-closed in (X\R, τ\R). Since π is *gα-continuous, Then π^{-1}(F) is *gα-closed in (X, τ).

3.12 Theorem

If the bijective map π : (X, τ) → (X\R, τ\R) is *gα-continuous and (X, τ) is *gα-connected, then (X\R, τ\R) is *gα-connected.

Proof

Suppose that (X\R, τ\R) is not *gα-connected. Therefore X\R = A ∪ B, where A and B are *gα-closed set. Then π^{-1}(A) and π^{-1}(B) are *gα-closed in (X, τ) such that X = π^{-1}(A) ∪ π^{-1}(B). Therefore (X, τ) is not *gα-connected. It is a contradiction. Therefore (X\R, τ\R) is *gα-connected.

4. CONCLUSION

In this paper we derived some properties of *gαc-homeomorphism and Pasting Lemma for *gα-irresolute functions.

5. REFERENCES

[1] R.Devi, K.Balachandran and H.Maki, Semi generalized homeomorphisms and generalized semi-homeomorphisms in topological spaces, Indian J. Pure.Appl.Math, 26(3)(1995),271-284.

[2] R.Devi and M.Vigneshwaran, On GαO-kernel in the digital plane(submitted).

[3] R.Devi and M.Vigneshwaran, On *gα-compact and *gα-connected spaces (submitted).

[4] J.Dontchev and H.Maki, Groups of θ-generalized homeomorphisms and the digital line,Topology and its Applications 95(1999) 113-128.

[5] N.Levine, Generalized closed sets in topology, Rend.Circ.Mat.Palermo 19(2) (1970) 89-96.

[6] H.Maki,R.Devi and K.Balachandran, Generalized α-closed sets in topology, Bull. Fukuoka Univ. Ed. Part III, 42(1993),13-21.

[7] H.Maki, P.Sundaram and K.balachandran, On generalized homeomorphisms in topological spaces, Bull.Fukuoka Univ.Ed.III 40 (1991) 13-21.

[8] O.Njastad, On some classes of nearly open sets, Pacific J. Math, 15(1965),961- 970.