# Some Applications of αψ-P-Open Sets

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## ABSTRACT
In this paper we introduce some new separation axioms by utilizing the notions of αψ-p-open sets and αψ-preclosure operator.

## KEYWORDS
αψ-p-open, sober (αψ, p)-$R_0$, $D_{(\alpha\psi,p)}$-set, (αψ, p)-$D_0$, (αψ, p)-$D_1$, (αψ, p)-$D_2$.

## AMS SUBJECT CLASSIFICATION (2000):
54B05, 54C08, 54C10, 54D10.

## 1. INTRODUCTION
The concept of preopen sets and precontinuous functions in topological spaces are introduced by A.S. Mashhour et al. [10]. Recently, R.Devi et al. [4] introduced the notion of αψ-open sets which are weaker than open sets. Since then, αψ-open sets have been widely used in order to introduce new spaces and functions.

In this paper, we introduce the notion of αψ-p-open sets and αψ-p-continuity in topological spaces. By utilizing these notions we introduce some weak separation axioms. Also we show that some basic properties of (αψ, p)-$T_i$, (αψ, p)-$D_i$ for i = 0, 1, 2 spaces and we ofer a new class of functions called (αψ, p)-continuous functions and a new notion of the graph of a function called an (αψ, p)-closed graph and investigate some of their fundamental properties.

## 2. PRELIMINARIES
Let A ⊆ X, the closure of A and the interior of A will be denoted by cl(A) and int(A) respectively. A is regular open if A = int(cl(A)) and A is regular closed if its complement is regular open; equivalently A is regular closed if A = cl(int(A)), see [17].

**Definition 2.1.**
A subset A of a space (X, τ) is called a

1. semi-open set [9] if A ⊆ cl(int(A)) and a semi-closed set [9] if int(cl(A)) ⊆ A,
2. α-open set [11] if A ⊆ int(cl(int(A))) and an α-closed set [11] if cl(int(cl(A))) ⊆ A,
3. pre-open set [10] if A ⊆ int(cl(A)) and pre closed set [10] if cl(int(A)) ⊆ A,
4. δ-open set [16] if for each x ∈ A, there exists a regular open set G such that x ∈ G ⊂ A and
5. pre-regular p-open set [6] if A = pint(pcl(A)).

The semi-closure (resp. α-closure) of a subset A of a space (X, τ) is the intersection of all semi-closed (resp. α-closed) sets that contain A and is denoted by scl(A) (resp. αcl(A)).

**Definition 2.2.**
A subset A of a space (X, τ) is called a

1. a semi-generalized closed (briefly sg-closed) set [1] if scl(A) ⊆ U whenever A ⊆ U and U is semi-open in (X, τ). The complement of sg-closed set is called sg-open set,
2. a ψ-closed set [15] if scl(A) ⊆ U whenever A ⊆ U and U is sg-open in (X, τ). The complement of ψ-closed set is called ψ-open set and

Let (X, τ) be a space and let A be a subset of X. A is called αψ-closed set [4] if ψcl(A) ⊆ U whenever A ⊆ U and U is α-open set of (X, τ). The complement of an αψ-closed set is called αψ-open. The intersection of all αψ-closed (resp. δ-closed) sets containing A is called the αψ-closure (resp. δ-closure) of A and is denoted by $cl_{\alpha\psi}(A)$ (resp. $cl_\delta(A)$).

**Definition 2.3.**
A subset A of a topological space (X, τ) is said to be δ-preopen [10] if A ⊆ int($cl_\delta$ (A)). A family of all δ-preopen sets in a topological space (X, τ) is denoted by δPO(X, τ).

**Definition 2.4.**
A function f : X → Y is called perfectly continuous

[12] if for each open set $A \subset Y$, $f^{-1}(A)$ is open and closed in X.

**Lemma 2.5. [7]**
If A and B are pre-regular p-open sets of the space X and Y, respectively, then $A \times B$ is a pre-regular p-open set of $X \times Y$.

**Lemma 2.6. [7]** If a space is submaximal, then any finite intersection of pre- regular p-open sets is pre-regular p-open.

## 3. αψ-P-OPEN SETS

### 3.1 Definition

A subset A of a topological space (X, τ) is said to be αψ-p-open if $A \subseteq int(cl_{\alpha\psi}(A))$.

The complement of an αψ-p-open set is said to be αψ-p-closed. The family of all αψ-p-open (resp. αψ-p-closed) sets in a topological space (X, τ) is denoted by αψPO (X, τ) (resp. αψPC(X, τ)).

### 3.2 Definition

Let A be a subset of a topological space (X, τ). The intersection of all αψ-p-closed (resp. δ-preclosed) sets containing A is called the αψ- p-closure (resp. δ-preclosure [14]) of A and is denoted by $pcl_{\alpha\psi}(A)$ (resp. $pcl_\delta(A)$).

### 3.3 Definition

Let (X, τ) be a topological space. A subset U of X is called a (αψ, p)-neighbourhood of a point $x \in X$ if there exists an αψ-p-open set V such that $x \in V \subseteq U$.

### 3.4 Theorem

For the αψ-p-closure of subsets A, B in a topological space (X, τ), the following properties hold:

(1) A is αψ-p-closed in (X, τ) if and only if $A = pcl_{\alpha\psi}(A)$,
(2) If $A \subset B$, then $pcl_{\alpha\psi}(A) \subset pcl_{\alpha\psi}(B)$,
(3) $pcl_{\alpha\psi}(A)$ is αψ-p-closed, that is $pcl_{\alpha\psi}(A) = pcl_{\alpha\psi}(pcl_{\alpha\psi}(A))$ and
(4) $x \in pcl_{\alpha\psi}(A)$ if and only if $A \cap V \neq \varphi$ for every $V \in \alpha\psi PO(X, \tau)$ containing x.

**Proof**

It is obvious

### 3.5 Theorem

For a family $\{A_\beta : \beta \in \Delta\}$ of subsets a topological space (X, τ), the following properties hold:

(1) $pcl_{\alpha\psi}\{A_\beta : \beta \in \Delta\} \subset \cap\{pcl_{\alpha\psi}(A_\beta) : \beta \in \Delta\}$
(2) $pcl_{\alpha\psi}\{A_\beta : \beta \in \Delta\} \supset \cup\{pcl_{\alpha\psi}(A_\beta) : \beta \in \Delta\}$

**Proof.**

(1) Since $\cap_{\beta \in \Delta} A_\beta \subset A_\beta$ for each $\beta \in \Delta$, by Theorem 3.4 we have $pcl_{\alpha\psi}(\cap_{\beta \in \Delta} A_\beta) \subset pcl_{\alpha\psi}(A_\beta)$ for each $\beta \in \Delta$ and hence $pcl_{\alpha\psi}(\cap_{\beta \in \Delta} A_\beta) \subset \cap_{\beta \in \Delta}(pcl_{\alpha\psi}(A_\beta))$.
(2) Since $A_\beta \subset \cup_{\beta \in \Delta} A_\beta$ for each $\beta \in \Delta$, by Theorem 3.4 we have $pcl_{\alpha\psi}(A_\beta) \subset pcl_{\alpha\psi}(\cup_{\beta \in \Delta} A_\beta)$ for each $\beta \in \Delta$ and hence $\cup_{\beta \in \Delta} pcl_{\alpha\psi}(A_\beta) \subset pcl_{\alpha\psi}(\cup_{\beta \in \Delta} A_\beta)$.

### 3.6 Theorem

Every αψ-p-open set is preopen.

**Proof**

It follows from the Definitions.

The converse of the above Theorem need not be true by the following Example.

### 3.7 Example

Let X = {a, b, c} and τ= {X, φ, {a, b}}. Here {a, c} is not αψ-p- open however it is preopen, since the αψ-p-open sets are X, φ, {a}, {b}, {a, b} and preopen sets are X, φ, {a}, {b}, {a, b}, {a, c}, {b, c}.

### 3.8 Theorem

(1) Every preopen set is δ-preopen [3].
(2) Every αψ-p-open is δ-preopen.

**Proof.**

(2) It follows from (1) and Theorem 3.6.

### 3.9 Definition

A subset A of a topological space (X, τ) is called a $D_{(\alpha\psi,p)}$- set (resp. $D_p$-set [2,5], $D_{(\delta,p)}$-set [3]) if there are two $U, V \in \alpha\psi PO(X, \tau)$ (resp. $PO(X, \tau)$, $\delta PO(X, \tau)$) such that $U \neq X$ and $A = U - V$.

It is true that every αψ-p-open (resp. preopen) set U different from X is a $(\alpha\psi,p)$ -set (resp. $D_p$ -set) if A = U and V = φ.

### 3.10 Definition

A topological space (X, τ) is said to be

(1) (αψ, p)-$D_0$ (resp. pre-$D_0$ [2,5], (δ, p)-$D_0$ [3]) if for any distinct pair of points x and y of X there exist a

$D_{(\alpha\psi,p)}$-set (resp. $D_p$ -set, $D_{(\delta,p)}$-set) of X containing x but not y or a $D_{(\alpha\psi,p)}$-set (resp. $D_p$-set, $D_{(\delta,p)}$-set) of X containing y but not x.

(2) ($\alpha\psi$, p)-$D_1$ (resp. pre-$D_1$ [2,5], ($\delta$, p)-$D_1$ [3]) if for any distinct pair of points x and y of X there exist a $D_{(\alpha\psi,p)}$-set (resp. $D_p$ -set, $D_{(\delta,p)}$-set) of X containing x but not y and a $D_{(\alpha\psi,p)}$-set (resp. $D_p$ -set, $D_{(\delta,p)}$-set) of X containing y but not x.

(3) ($\alpha\psi$, p)-$D_2$ (resp. pre-$D_2$ [2,5], ($\delta$, p)-$D_2$ [3]) if for any distinct pair of points x and y of X there exists disjoint $D_{(\alpha\psi,p)}$-set (resp. $D_p$-set, $D_{(\delta,p)}$-set) G and E of X containing x and y, respectively.

### 3.11 Definition

A topological space (X, $\tau$) is said to be

(1) ($\alpha\psi$, p)-$T_0$ (resp. pre-$T_0$ [8,13], ($\delta$, p)-$T_0$ [3]) if for any distinct pair of points x and y of X there exist an $\alpha\psi$-p-open (resp. preopen, $\delta$-preopen) set U in X containing x but not y or an $\alpha\psi$-p-open (resp. preopen, $\delta$-open) set V in X containing y but not x.

(2) ($\alpha\psi$, p)-$T_1$ (resp. pre-$T_1$ [8,13], ($\delta$, p)-$T_1$ [3]) if for any distinct pair of points x and y of X there exist an $\alpha\psi$-p-open (resp. preopen, $\delta$-preopen) set U in X containing x but not y and an $\alpha\psi$-p-open (resp. preopen, $\delta$-preopen) set V in X containing y but not x.

(3) ($\alpha\psi$, p)-$T_2$ (resp. pre-$T_2$ [8,13], ($\delta$, p)-$T_2$ [3]) if for any distinct pair of points x and y of X there exist $\alpha\psi$-p-open (resp. preopen, $\delta$-preopen) sets U and V in X containing x and y, respectively, such that $U \cap V = \varphi$.

### 3.12 Remark

(i) If (X, $\tau$) is ($\alpha\psi$, p)-$T_i$, then it is ($\alpha\psi$, p)-$T_{i-1}$, $i = 1, 2$.
(ii) If (X, $\tau$) is ($\alpha\psi$, p)-$T_i$, then it is ($\alpha\psi$, p)-$D_i$, $i = 0, 1, 2$.
(iii) If (X, $\tau$) is ($\alpha\psi$, p)-$D_i$, then it is ($\alpha\psi$, p)-$D_{i-1}$, $i = 1, 2$.
(iv) If (X, $\tau$) is ($\alpha\psi$, p)-$D_i$, then it is pre-$T_i$, $i = 0, 1, 2$.

By Remark 3.12 and [2, Remark 3.1], we have the following diagram.

$$(\alpha\psi, p)\text{-}T_2 \rightarrow (\alpha\psi, p)\text{-}D_2 \rightarrow \text{pre-}T_2 \rightarrow (\delta, p)\text{-}T_2 \rightarrow (\delta, p)\text{-}D_2$$
$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$
$$(\alpha\psi, p)\text{-}T_1 \rightarrow (\alpha\psi, p)\text{-}D_1 \rightarrow \text{pre-}T_1 \rightarrow (\delta, p)\text{-}T_2 \rightarrow (\delta, p)\text{-}D_2$$
$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$
$$(\alpha\psi, p)\text{-}T_0 \rightarrow (\alpha\psi, p)\text{-}D_0 \rightarrow \text{pre-}T_0 \rightarrow (\delta, p)\text{-}T_2 \rightarrow (\delta, p)\text{-}D_2$$

### 3.13 Theorem

A topological space (X, $\tau$) is ($\alpha\psi$, p)-$D_1$ if and only if it is ($\alpha\psi$, p)-$D_2$.

**Proof**

Sufficiency. This follows from Remark 3.12.

Necessity. Suppose X is a ($\alpha\psi$, p)-$D_1$. Then for each distinct pair $x, y \in X$, we have $D_{(\alpha\psi,p)}$-sets $G_1$ and $G_2$ such that $x \in G_1$, $y \notin G_1$ ; $y \in G_2$, $x \notin G_2$. Let $G_1 = U_1/U_2$, $G_2 = U_3/U_4$ , where $U_1, U_2, U_3, U_4 \in \alpha\psi PO(X, \tau)$. From x $G_2$ we have either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$.

We discuss the two cases separately.

(1) $x \notin U_3$. From $y \notin G_1$ we have two sub cases:

(a) $y \notin U_1$. From x $U_1/U_2$ we have x $U_1/(U_2 \cup U_3)$ and from y $U_3/U_4$ we have y $U_3/(U_1 \cup U_4)$. It is easy to see that $(U_1/(U_2 \cup U_3)) \cap (U_3/(U_1 \cup U_4)) = \varphi$.

(b) y $U_1$ and y $U_2$. We have x $U_1/U_2$, y $U_2$ and $(U_1/U_2) \cap U_2 = \varphi$. (2) x $U_3$ and x $U_4$. We have y $U_3/U_4$, x $U_4$ and $(U_3/U_4) \cap U_4 = \varphi$.

From the discussion above we know that the space X is ($\alpha\psi$, p)-$D_2$.

### 3.14 Definition

A point x X which has only X as the ($\alpha\psi$, p)-neighbourhood is called a ($\alpha\psi$, p)-neat point.

### 3.15 Theorem

If a topological spaces (X, $\tau$) is ($\alpha\psi$, p)-$D_1$, then it has no ($\alpha\psi$, p)-neat point.

**Proof.**

Since (X, $\tau$) is ($\alpha\psi$, p)-$D_1$, so each point x of X is contained in a $D_{(\alpha\psi,p)}$- set $O = U/V$ and thus in U. By definition $U \neq X$. This implies that x is not a ($\alpha\psi$, p)-neat point.

### 3.16 Definition

A topological space (X, $\tau$) is ($\alpha\psi$, p)-symmetric if x and y in X, x $pcl_{\alpha\psi}(\{y\})$ implies y $pcl_{\alpha\psi}(\{x\})$.

### 3.17 Theorem

For a topological space (X, $\tau$), the following properties hold.

(1) If $\{x\}$ is $\alpha\psi$-p-closed for each x X, then (X, $\tau$) is ($\alpha\psi$, p)-$T_1$.

(2) Every ($\alpha\psi$, p)-$T_1$ space is ($\alpha\psi$, p)-symmetric.

**Proof** Suppose $\{p\}$ is $\alpha\psi$-p-closed for every p X. Let x, y X with $x \neq y$. Now $x \neq y$ implies y $X/\{x\}$. Hence $X/\{x\}$ is an $\alpha\psi$-p-open set contained in y but not containing x. Similarly $X/\{y\}$ is an $\alpha\psi$-p-open set contained in x but not containing y. Accordingly X is a ($\alpha\psi$, p)-$T_1$ space.

(2) Suppose that y pcl$_{\alpha\psi}$({x}). Then, since x $\neq$ y, there exists an αψ-p-open set U containing x such that y U and hence x pcl$_{\alpha\psi}$({y}). This shows that x pcl$_{\alpha\psi}$({y}) implies y pcl$_{\alpha\psi}$({x}). Therefore (X, r ) is (αψ, p)-symmetric.

**3.18 Definition**

A function f : (X, τ) → (Y, σ) is said to be αψ-pre continuous if for each x ∈ X and each αψ-p-open set V containing f(x), there is an αψ-p-open set U in X containing x such that f(U) ⊆ V.

**3.19 Theorem**

If f : (X, τ ) → (Y, σ) is an αψ-pre continuous surjective function and E is a $D_{(\alpha\psi,p)}$-set in Y , then the inverse image $f^{-1}$(E) is a $D_{(\alpha\psi,p)}$-set in X.

**Proof.**

Let E be a $D_{(\alpha\psi,p)}$ set in Y . Then there are αψ-p-open sets $U_1$ and $U_2$ in Y such that E = $U_1/U_2$ and $U_1 \neq$ Y . By the αψ-precontinuity of f , $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are αψ-p-open in X . Since $U_1 \neq$ Y , we have $f^{-1}(U_1) \neq$ X. Hence $f^{-1}$(E) = $f^{-1}(U_1)/f^{-1}(U_2)$ is a $D_{(\alpha\psi,p)}$-set.

**3.20 Theorem**

If (Y, σ) is (αψ, p)-$D_1$ and f : (X, τ ) → (Y, σ) is an αψ- pre continuous bijection, then (X, r) is (αψ, p)-$D_1$ .

**Proof**

Suppose that Y is a (αψ, p)-$D_1$ space. Let x and y be any pair of distinct points in X. Since F is injective and Y is (αψ, p)-$D_1$, there exist $D_{(\alpha\psi,p)}$-sets $G_x$ and $G_y$ of Y containing f(x) and f(y), respectively, such that f(y) $G_x$ and f(x) $G_y$ . By Theorem 3.19, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are $D_{(\alpha\psi,p)}$-sets in X containing x and y, respectively, such that y $f^{-1}(G_x)$ and x $f^{-1}(G_y)$. This implies that X is a (αψ, p)-$D_1$ space.

**3.21 Theorem**

A topological space (X, τ) is (αψ, p)-$D_1$ if and only if for each pair of distinct points x, y ∈ X, there exists an αψ-pre continuous surjective function f : (X, τ ) → (Y, σ) such that f (x) and f (y) are distinct, where (Y, σ) is a (αψ, p)-$D_1$ space.

**Proof.**

Necessity. For every pair of distinct points of X, it suffices to take the identity function on X.

Sufficiency. Let x and y be any pair of distinct points in XBy hypothesis there exists an αψ-pre continuous, surjective function f of a space X onto (αψ, p)-$D_1$ space Y such that f (x) $\neq$ f(y). By Theorem 3.13, there exist disjoint $D_{(\alpha\psi,p)}$−sets $G_x$ and $G_y$ in Y such that f(x) ∈ $G_x$ and f(y) ∈ $G_y$ . Since f is αψ-pre continuous and surjective, by Theorem 3.20, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint $D_{(\alpha\psi,p)}$-sets in X containing x and y, respectively, hence by Theorem 3.13, X is a (αψ, p)-$D_1$ space.

## 4. SOBER (αψ, P)-$R_0$ SPACES

**4.1. Definition**

Let A be a subset of a topological space (X, τ). The αψ-prekernel of A, denoted by pker$_{\alpha\psi}$(A) is defined to be the set pker$_{\alpha\psi}$(A)=∩{U∈ αψPO(X,τ) : A⊆ U}.

**4.2 Lemma**

Let (X, τ) be a topological space and x ∈X. Then pker$_{\alpha\psi}$(A) = { x ∈X: pcl$_{\alpha\psi}$({x})∩A≠ φ }.

**Proof**

Let x ∈ pker$_{\alpha\psi}$ (A) and suppose pcl$_{\alpha\psi}$ ({x}) \ A = φ. Hence x ∉ X/pcl$_{\alpha\psi}$ ({x}) which is an αψ -p-open set containing A. This is absurd, since x ∈ pker$_{\alpha\psi}$ (A). Consequently, pcl$_{\alpha\psi}$ ({x}) \ A ≠ φ. Next, let x be such that pcl$_{\alpha\psi}$ ({x}) \ A ≠ φ and suppose that x ∉ pker$_{\alpha\psi}$ (A). Then, there exists an αψ -p-open set D containing A and x ∉ D. Let y∈ pcl $_{\alpha\psi}$({x}) \ A. Hence, D is an (αψ, p)-neighbourhood of y which does not containing x. By this contradiction x ∈ pker $_{\alpha\psi}$ (A) and the claim is shown.

**4.3 Definition**

A topological space (X, τ ) is said to be sober (αψ, p)-$R_0$ (resp. sober (δ, p)-$R_0$ [3]) if $\cap_{x\in X}$ pcl$_{\alpha\psi}$({x}) = φ (resp. $\cap_{x\in X}$ pcl$_\delta$({x}) = φ).

**4.4. Theorem**

Every sober (αψ, p)-$R_0$ space is sober (δ, p)-$R_0$ space.

**Proof.**

Let (X, τ ) be a sober (αψ, p)-$R_0$ space, then $\cap_{x\in X}$ pcl$_{\alpha\psi}$({x}) = φ. There- fore, $\cap_{x\in X}$ pcl$_\delta$({x}) = φ.

**4.5. Theorem**

A topological space (X, τ) is sober (αψ, p)-$R_0$ if and only if pker$_{\alpha\psi}$({x}) ≠ X for every x ∈ X.

**Proof.**

Suppose that the space (X, τ) be sober (αψ, p)-$R_0$. Assume that there is a point y in X such that pker$_{\alpha\psi}$({y}) = X. Let x be any point of X. Then x∈ V for every αψ-p-open set V containing y and hence y ∈ pcl$_{\alpha\psi}$({x}) for any x ∈ X. This implies that y ∈ $\cap_{x\in X}$ pcl$_{\alpha\psi}$({x}).

But this is a contradiction. Now assume that $pker_{\alpha\psi}(\{x\}) = X$ for every $x \in X$. If there exists a point of X. This implies that the space X is the unique αψ-preopen set containing y. Hence $pker_{\alpha\psi}(\{y\}) \neq X$ which is a contradiction. Therefore (X, τ) is sober (αψ, p)-$R_0$ space.

**4.6. Definition**

A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called pre αψ-p-closed if the image of every αψ-p-closed subset of X is αψ-p-closed in Y.

**4.7. Theorem**

If $f : (X, \tau) \rightarrow (Y, \sigma)$ is an injective pre αψ-p-closed function and X is sober (αψ, p)-$R_0$, then Y is sober (αψ, p)-$R_0$.

**Proof.**

Since X is sober (αψ, p)-$R_0$, $\cap_{x\in X}\, pcl_{\alpha\psi}(\{x\}) = \varphi$. Since f is a pre αψ-p-closed injection, we have

$\varphi = f(\cap_{x\in X}\, pcl_{\alpha\psi}(\{x\}))$

$= \cap_{x\in X}\, f(pcl_{\alpha\psi}(\{x\}))$

$\supseteq \cap_{x\in X}\, pcl_{\alpha\psi}\, f(\{x\})$

$\supseteq \cap_{x\in X}\, pcl_{\alpha\psi}(\{y\})$.

Therefore, Y is sober (αψ, p)-$R_0$.

**4.8 Theorem**

If a topological space X is sober (αψ, p)-$R_0$ and Y is any topo logical space, then the product $X \times Y$ is sober (αψ, p)-$R_0$.

**Proof.**

We show that $\cap_{(x,y)\in X\times Y}\, pcl_{\alpha\psi}(\{(x, y)\}) = \varphi$. We have

$\cap_{(x,y)\in X\times Y}\, pcl_{\alpha\psi}(\{(x, y)\})$

$\subseteq \cap_{(x,y)\in X\times Y}\, (pcl_{\alpha\psi}(\{x\}) \times pcl_{\alpha\psi}(\{y\}))$

$= \cap_{x\in X}\, pcl_{\alpha\psi}(\{x\}) \times \cap_{y\in Y}\, pcl_{\alpha\psi}(\{y\})$

$\subseteq \varphi\times Y$

$= \varphi$.

# 5. (αψ, p)-CONTINUOUS FUNCTIONS AND (αψ, p)-CLOSED GRAPHS

**5.1. Definition**

A function $f : X \rightarrow Y$ is said to be (αψ, p)-continuous if for every open set V of Y, $f^{-1}(V)$ is (αψ, p)-open in X.

**5.2. Theorem**

The following are equivalent for a function $f : X \rightarrow Y$:

(i) f is (αψ, p)-continuous,

(ii) The inverse image of every closed set in Y is (αψ, p)-closed in X,

(iii) For each subset A of X, $f(\alpha\psi cl_p(A)) \subset cl(f(A))$,

(iv) For each subset B of Y, $\alpha\psi cl_p(f^{-1}(B)) \subset f^{-1}(cl(B))$.

**Proof.**

(i) ⇔ (ii): Obvious.

(iii) ⇔ (iv): Let B be any subset of Y. Then by (iii), we have $f(\alpha\psi cl_p(f^{-1}(B))) \subset cl(f(f^{-1}(B))) \subset cl(B)$. This implies $\alpha\psi cl_p(f^{-1}(B)) \subset f^{-1}(f(\alpha\psi cl_p(f^{-1}(B)))) \subset f^{-1}(cl(B))$.

Conversely, let $B = f(A)$ where A is a subset of X. Then, by (iv), we have, $\alpha\psi cl_p(A) \subset \alpha\psi cl_p(f^{-1}(f(A))) \subset f^{-1}(cl(f(A)))$. Thus, $f(\alpha\psi cl_p(A)) \subset cl(f(A))$. (ii) ⇒ (iv): Let $B \subset Y$. Since $f^{-1}(cl(B))$ is (αψ, p)-closed and $f^{-1}(B) \subset f^{-1}(cl(B))$, then $\alpha\psi cl_p(f^{-1}(B)) \subset f^{-1}(cl(B))$.

(iv) ⇒ (ii): Let $K \subset Y$ be a closed set. By (iv), $\alpha\psi cl_p(f^{-1}(K)) \subset f^{-1}(cl(K)) = f^{-1}(K)$. Thus, $f^{-1}(K)$ is (αψ, p)-closed.

Recall that for a function $f : X \rightarrow Y$, the subset $\{(x, f(x)) : x \in X\}$ of the product space $X \times Y$ is called the graph of f and is denoted by G(f).

**5.3. Definition**

For a function $f : X \rightarrow Y$, the graph $G(f) = \{(x, f(x)) : x \in X\}$ is said to b (αψ, p)-closed if for each $(x, y) \in X \times Y - G(f)$, there exist $U \in \alpha\psi PO(X, x)$ and an open set V of Y containing y such that $(U \times V) \cap G(f) = \varphi$.

**5.4. Lemma**

Let $f : X \rightarrow Y$ be a function. Then the graph G(f) is (αψ, p)- closed in $X \times Y$ if and only if for each point $(x, y) \in X \times Y - G(f)$, there exist a (αψ, p)-open set U and an open set V containing x and y, respectively, such that $f(U) \cap V = \varphi$.

**Proof.**

It follows readily from the above definition.

**5.5. Theorem**

If $f : X \rightarrow Y$ is an injective function with the (αψ, p)-closed graph, then X is (αψ, p)-$T_1$.

**Proof.**

Let x and y be two distinct points of X. Then $f(x) \neq f(y)$. Thus there exist an (αψ, p)-open set U and an open set V containing x and f(y), respectively, such that $f(U) \cap V = \varphi$. Therefore $y \notin U$ and it follows that X is (αψ, p)-$T_1$.

Recall that a space X is said to be $T_1$ if for each pair of distinct points x and y of X, there exist an open set U containing x but not y and an open set V containing y but not x.

**5.6. Theorem**
If $f : X \to Y$ is an surjective function with the (αψ, p)-closed graph, then Y is $T_1$.

**Proof.**

Let $y_1$ and $y_2$ be two distinct points of Y. Since f is surjective, there exist a point x in X such that $f(x) = y_2$. Therefore $(x, y_1) \notin G(f)$. By lemma 5.4., there exist an (αψ, p)-open set U and an open set V containing x and $y_1$, respectively, such that $f(U) \cap V = \varphi$. It follows that $y_2 \notin V$. Hence Y is $T_1$.

**5.7. Definition**
A function $f : X \to Y$ is said to be (αψ, p)-W-continuous if for each $x \in X$ and each open set V of Y containing f(x), there exists an (αψ, p)- open set U in X containing x such that $f(U) \subset cl(V)$.

**5.8. Theorem**
If $f : X \to Y$ is (αψ, p)-W-continuous and Y is Hausdorff, then G(f) is (αψ, p)-closed.

**Proof.**
Suppose that $(x, y) \notin G(f)$, then $f(x) \neq y$. By the fact that Y is Hausdorff, there exist open sets W and V such that $f(x) \in W$, $y \in V$ and $V \cap W = \varphi$. It follows that $cl(W) \cap V = \varphi$. Since f is (αψ, p)-W-continuous, there exists $U \in \alpha\psi PO(X, x)$ such that $f(U) \subset cl(W)$. Hence, we have $f(U) \cap V = \varphi$. This means that G(f) is (αψ, p)-closed.

**5.9. Corollary**
If $f : X \to Y$ is (αψ, p)-W-continuous and Y is Hausdorff, then G(f) is (αψ, p)-closed in $X \times Y$.

**5.10. Definition**
A subset A of a space X is said to be (αψ, p)-compact relative to X if every cover of A by (αψ, p)-open sets of X has a finite subcover.

**5.11. Theorem**
Let $f : X \to Y$ have a (αψ, p)-closed graph. If K is (αψ, p)- compact relative to X, then f(K) is closed in Y.

**Proof.**
Suppose that $y \notin f(K)$. For each $x \in K$, $f(x) = y$. By lemma 5.4., there exist $U_x \in \alpha\psi PO(X, x)$ and an open neighbourhood $V_x$ of y such that $f(U_x) \cap V_x = \varphi$. The family $\{U_x : x \in K\}$ is a cover of K by (αψ, p)-open sets of X and there exists a fnite subset $K_0$ of K such that $K \subset \cup\{U_x : x \in K_0\}$. Put $V = \cap\{V_x : x \in K_0\}$. Then V is an open neighbourhood of y and $f(K) \cap V = \varphi$. This means that f(K) is closed in Y.

**5.12. Theorem**
If $f : X \to Y$ has an (αψ, p)-closed graph G(f) and $g : Y \to Z$ is a perfectly continuous function, then the set $\{(x, y) : f(x) = g(y)\}$ is (αψ, p)- closed in $X \times Y$.

**Proof.**
Let $A = \{(x, y) : f(x) = g(y)\}$ and $(x, y) \in (X \times Y) - G(f)$. Since f has an (αψ, p)-closed graph G(f), there exist an (αψ, p)-open set U and an open set V containing x and g(y), respectively, such that $f(U) \cap V = \varphi$. This implies that there exists a pre-regular p-open set N containing x such that $N \subset U$ and $f(N) \cap V = \varphi$. Since g is a perfectly continuous function, then there exist an open and closed set G containig y such that $g(G) \subset V$. We have $f(U) \cap g(G) = \varphi$. This implies that $(N \times G) \cap A = \varphi$. Since $N \times G$ is pre-regular p-open, then $(x, y) \notin \alpha\psi cl_p(A)$. Thus, E is (αψ, p)-closed in $X \times Y$.

**5.13. Corollary**
If $f : X \to Z$ is an (αψ, p)-continuous function and $g : Y \to Z$ is a perfectly continuous function and Z is Hausdorff, then the set $\{(x, y) : f(x) = g(y)\}$ is (αψ, p)-closed in $X \times Y$

**Proof.**
It follows from Corollary 5.9 and Theorem 5.12.

**5.14. Theorem**
If $f : X \to Y$ is an (αψ, p)-continuous function and Y is Hausdorff, then the set $\{(x, y) \in X \times Y : f(x) = f(y)\}$ is (αψ, p)-closed in $X \times X$.

**Proof.**
Let $\{(x, y) : f(x) = f(y)\}$ and let $\{(x, y) \in (X \times Y) - A\}$. It follows tat $f(x) \neq f(y)$. Since Y is Hausdorff, there exist open set U and V containing f(x) and f(y), respectively, such that $U \cap V = \varphi$. Since f is (αψ, p)-continuous, there exist pre-regular p-open set in $X \times X$ containing (x, y). Hence, A is (αψ, p)-closed in $X \times X$.

**5.15. Definition**
A function $f : X \to Y$ is called contra (αψ, p)-open if the image of every (αψ, p)-open set in X is closed in Y.

**5.16. Theorem**
If $f : X \to Y$ is a contra (αψ, p)-open funcion such that the inverse image of each opoint of Y is (αψ, p)-closed, then f has an (αψ, p)-closed graph G(f).

**Proof.**

Let $(x, y) \in X - G(f)$. We have $x \notin f^{-1}(y)$. Since $f^{-1}(y)$ is (αψ, p)-closed, there exists a pre-regular p-open set A containing x such that $A \cap f^{-1}(y) = \varphi$. Since, f is contra (αψ, p)-open, then f(A) is closed. This implies that there exist an open set B in Y containing y such that $f(A) \cap B = \varphi$. Hence, $f$ has an $(\alpha\psi, p)$-closed graph $G(f)$.

**5.17. Theorem**

If $f : X \to Y$ has an $(\alpha\psi, p)$-closed graph $G(f)$, then for each $x \in X$, $\{f(x)\} = \cap_{x \in A \in \alpha\psi PO(X,\tau)} cl(f(A))$.

**Proof.**

Suppose that $y \neq f(x)$ and $y \in \cap_{x \in A \in \alpha\psi PO(X,\tau)} cl(f(A))$. Then $y \in cl(f(A))$ for each $x \in A \in \alpha\psi PO(X, \tau)$. This implies that for each open set B containing y, $B \cap f(A) \neq \varphi$. Since $(x, y) \notin G(f)$ and $G(f)$ is an $(\alpha\psi, p)$- closed graph, this is a contradiction.

**5.18. Definition**

A function $f : X \to Y$ is called an $(\alpha\psi, p)$-open if the image of every $(\alpha\psi, p)$-open set in X is open in Y.

**5.19. Theorem**

If $f : X \to Y$ is a surjective $(\alpha\psi, p)$-open function with an$(\alpha\psi, p)$-closed graph $G(f)$, then Y is $T2$.

**Proof.**

Let $y_1$ and $y_2$ be any two distinct points of Y. Since $f$ is surjective $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in (X \times Y) - G(f)$. This implies that there exist an $(\alpha\psi, p)$-open set A of X and an open set B of Y such that $(x, y_2) \in (A \times B)$ and $(A \times B) \cap G(f) = \varphi$. We have $f(A) \cap B = \varphi$. Since $f$ is $(\alpha\psi, p)$-open, then $f(A)$ is open such that $f(x) = y_1 \in f(A)$. Thus, Y is $T_2$.

**5.20 Theorem**

If $f : X \to Y$ is an $(\alpha\psi, p)$-continuous injective function and Y is $T_2$, then X is $(\alpha\psi, p)$-$T_2$.

**Proof.**

Let x and y in X be any pair of distinct points, then there exist disjoint open sets A and B in Y such that $f(x) \in A$ and $f(y) \in B$. Since $f$ is $(\alpha\psi, p)$-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are $(\alpha\psi, p)$-open in X containing x and y respectively, we have $f^{-1}(A) \cap f^{-1}(B) = \varphi$. Thus, X is $(\alpha\psi, p)$-$T_2$.

**5.21. Theorem**

If $f, g : X \to Y$ are $(\alpha\psi, p)$-continuous functions, X is sub- maximal and Y is Hausdorff, then the set $\{x \in X : f(x) = g(x)\}$ is $(\alpha\psi, p)$-closed in X.

**Proof.**

Let $A = \{x \in X : f(x) = g(x)\}$. Take $x \in X - A$. We have $f(x) \neq g(x)$. Since Y is Hausdorff, then there exist open sets U and V in Y containing $f(x)$ and $g(x)$, respectively, such that $U \cap V = \varphi$. Since $f$ and g are $(\alpha\psi, p)$-continuous, then $f^{-1}(U)$ and $g^{-1}(V)$ are $(\alpha\psi, p)$-open in X with $x \in f^{-1}(U)$ and $x \in g^{-1}(V)$. Then there exist pre-regular p-open sets G and H such that $x \in G \subset f^{-1}(U)$ and $x \in H \subset g^{-1}(V)$. Take $K = G \cap H$. By lemma 2.6, K is pre-regular p-open. Thus, $f(K) \cap g(K) = \varphi$ and hence $x \notin \alpha\psi cl_p(A)$. This shows that A is $(\alpha\psi, p)$-closed in X.

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