On αψ –Compact Spaces

V. Kokilavani
Assistant Professor
Department of Mathematics
Kongunadu Arts and Science College
Coimbatore-641029, India

M.Parimala
Lecturer
Department of Mathematics
Bannari Amman Institute of Technology
Sathyamangalam-638401.

R.Devi
Principal
Kongunadu Arts and Science College
Coimbatore-641029

ABSTRACT

The objective of this paper is to obtain the properties of αψ-compact spaces by using nets, filterbase, αψ-complete accumulation points and so on. We also investigate some properties of αψ-continuous multifunctions and αψ-compact spaces in the context of multifunction.

Keywords

αψ-open sets, αψ-closed sets, αψ-accumulation point, αψ-compact spaces, αψ-continuous multifunctions

1. INTRODUCTION

It is well-known that the effects of the investigation of properties of closed bounded intervals of real numbers, spaces of continuous functions and solutions to differential equations are the possible motivations for the formation of the notion of compactness. Compactness is now one of the most important, useful, and fundamental notions of not only general topology, but also of other advanced branches of mathematics. Recently R.Devi et al. [3] introduced and investigated the concepts of αψ-US spaces, αψ-convergence, sequential αψO-compactness, sequential αψ-continuity and sequential αψ-sub-continuity. A space X is αψ-compact if every αψ-open cover of X has a finite subcover. Since every open sets is an αψ-open set, it follows that every αψ-compact space is compact.

It is the objective of this paper to give some characterizations of αψ-compact spaces in terms of nets and filterbases. We also introduce the notion of αψ- complete accumulation points by which we give some characterizations of αψ-compact spaces. By introducing the notion of 1-lower (resp. 1-upper) αψ-continuous functions and considering the known notion of 1-lower (resp. 1-upper) compatible partial orders, we investigate some more properties of αψ-compactness. We also investigate αψ-compact spaces in the context of multifunctions by introducing 1- lower (resp. 1-upper) αψ-continuous multifunctions. Lastly we also obtain some characterizations of αψ-compact spaces by using lower (resp. upper) precontinuous multifunctions due to Popa [8]. In this paper we are working in ZFC.

2. PRELIMINARIES

Throughout the present paper, (X, τ) and (Y, σ) (or simply X and Y) denote topological spaces. Let A be a subset (X, τ). We denote the interior and the closure of a set A by int(A) and cl(A), respectively. A subset A of a space X is said to be α-open [7] if $A \subseteq int(cl(int(A)))$. A subset A of a space X is said to be semi-open [6] if $A \subseteq cl(int(A))$. A subset A of a space X is said to be semi generalized-closed [2] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open. A subset A of a space X is said to be ψ-closed [9] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is sg-open. A subset A of a space X is said to be αψ-closed [4] if $\psi cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open. The union of two αψ-closed set is an αψ-closed set. The complement of a αψ-closed set is said to be αψ-open. The intersection of all αψ-closed sets of X containing A is called αψ-closure of A and is denoted by αψcl(A). The union of all αψ-open sets of X contained in A is called αψ-interior of A and is denoted by αψint(A). If $A \subseteq \alpha\psi cl(A) \subseteq cl(A)$. The collection of all αψ-closed (resp. αψ-open) subsets of X will be denoted by $\alpha\psi C(X)$ (resp. $\alpha\psi O(X)$). We set $\alpha\psi C(X, x) = \{V \in \alpha\psi C(X) : x \in V\}$ for $x \in X$. We define similarly $\alpha\psi O(X, x)$. Let p be a point of X and N be a subset of X is called an αψ-neighbourhood of p in X [3] if there exists an αψ-open set O of X such that $p \in O \subseteq N$.

Recall that a function $f : X \rightarrow Y$ is said to be αψ-continuous [4] if the inverse image of each open set in Y is αψ-open in X.

Let Λ be a directed set. Now we introduce the following notions which will be used in this paper. A net $\xi = \{x_\alpha : \alpha \in \Lambda\}$ αψ-accumulates at a point $x \in X$ if the net is frequently in every $U \in \alpha\psi O(X, x)$, i.e. for each $U \in \alpha\psi O(X, x)$ and for each $\alpha_0 \in \Lambda$, there is some $\alpha \geq \alpha_0$ such that $x_\alpha \in U$. The net ξ αψ-converges to a point x of X if it is eventually in every $U \in \alpha\psi O(X, x)$. We say that a filterbase $\Theta = \{F_\alpha : \alpha \in \Gamma\}$ αψ-accumulates at a point $x \in X$ if $x \in \cap_{\alpha\in\Gamma} \alpha\psi Cl(F_\alpha)$. Given a set S with $S \subset X$, a αψ-cover of S is a family of αψ-open subsets U_α of X for each $\alpha \in I$ such that $S \subset \cup_{\alpha\in I} U_\alpha$. A filterbase $\Theta = \{F_\alpha : \alpha \in \Gamma\}$ αψ-converges to a point x in X if for each $U \in \alpha\psi O(X, x)$, there exists an F_α in Θ such that $F_\alpha \subset U$.

Recall that a multifunction (also called multivalued function [1]) F on a set X into a set Y, denoted by $F : X \rightarrow Y$, is a relation on X into Y, i.e. $F \subset X \times Y$.

Let $F : X \rightarrow Y$ be a multifunction. The upper and lower inverse of a set V of Y are denoted by $F^+(V)$ and $F^-(V)$:

$$F^+(V) = \{x \in X : F(x) \subset V\} \text{ and}$$

$$F^{-}(V) = \{x \in X : F(x) \cap V = \varphi\}.$$

3. CHARACTERIZATIONS OF αψ-COMPACT SPACES

3.1 Definition

A point x in a space X is said to be a αψ-complete accumulation point of a subset S of X if Card (S ∩ U) = Card (S) for each U ∈ αψO(X, x), where Card (S) denotes the cardinality of S.

3.2. Example

Let X = {a, b, c} with the topology τ = {X, φ, {a, b}}. Observe that both a and c are αψ-complete accumulation points of {a}. Notice that b is not an αψ-complete accumulation point of {a}.

3.3. Definition

In a topological space X, a point x is said to be an αψ-adherent point of a filterbase Θ on X if it lies in the αψ-closure of all sets of Θ.

3.4. Theorem

A space X is αψ-compact if and only if each infinite subset of X has a αψ-complete accumulation point.

Proof.

Let the space X be αψ-compact and S an infinite subset of X. Let K be the set of points x in X which are not αψ-complete accumulation points of S. Now it is obvious that for each point x in K, we are able to find U(x) ∈ αψO(X, x) such that Card(S ∩ U(x)) = Card(S). If K is the whole space X, then Θ = {U(x) : x ∈ X} is a αψ-cover of X. By the hypothesis X is αψ-compact, so there exists a finite subcover $\Psi = \{U(x_i)\}$, where i = 1, 2, ..., n such that S ⊂ ∪ $\{U(x_i) \cap S : i = 1, 2, ..., n\}$. Then Card(S) = max{Card($U(x_i) \cap S$): i = 1, 2, ..., n} which does not agree with what we assumed. This implies that S has an αψ-complete accumulation point. Now assume that X is not αψ-compact and that every infinite subset S ⊂ X has an αψ-complete accumulation point in X. It follows that there exists an αψ-cover Ξ with no finite subcover. Set δ = min {Card(Φ) : Φ ⊂ Ξ, where Φ is an αψ-cover of X}. Fix Ψ ⊂ Ξ for which Card(Ψ) = δ and ∪{U : U ∈ Ψ} = X. Let N denote the set of natural numbers. Then by hypothesis δ ≥ Card(N). By well-ordering of Ψ by some minimal well-ordering "~"suppose that U is any member of Ψ. By minimal well-ordering "~" we have Card({V : V ∈ Ψ, V Since Ψ can not have any subcover with cardinality less than δ, then for each U ∈ Ψ we have X ≠ ∪{V : V ∈ Ψ, V~ U}. For each U ∈ Ψ, choose a point x(U) ∈ X − ∪{V ∪ {x(V)} : V ∈ Ψ, V ~U}. We are always able to do this if not one can choose a cover of smaller cardinality from Ψ. If H = {x(U) : U ∈ Ψ}, then to finish the proof we will show that H has no αψ-complete accumulation point in X. Suppose that z is a point of the space X. Since Ψ is a αψ-cover of X then z is a point of some set W in Ψ. By the fact that U~V we have that x(U) ∈ W. It follows that T = {U : U ∈ Ψ and x(U) ∈ W} ⊂ {V : V ∈ Ψ, V ~ W}. But Card(T) < δ. Therefore Card(H ∩ W) < δ. But Card(H) = δ ≥ Card(N) since, for two distinct points U and W in Ψ, we have x(U) ≠ x(W). This means that H has no αψ-complete accumulation point in X which contradicts our assumptions. Therefore X is αψ-compact.

3.5. Theorem

For a space X the following statements are equivalent:

(i) X is αψ-compact;

(ii) Every net in X with a well-ordered directed set as its domain αψ-accumulates to some point of X.

Proof.

(i) ⇒ (ii): Suppose that X is αψ-compact and $\xi = \{x_\alpha : \alpha \in \Lambda\}$ a net with a well-ordered directed set Λ as domain. Assume that ξ has no αψ- adherent point in X. Then for each point x in X, there exist V(x) ∈ αψO(X, x) and an α(x) ∈ Λ such that $V(x) \cap \{x_\alpha : \alpha \geq \alpha(x)\} = \varphi$. This implies that $\{x_\alpha : \alpha \geq \alpha(x)\}$ is a subset of X − V(x). Then the collection C = {V(x) : x ∈ X} is a αψ-cover of X. By hypothesis of the theorem, X is αψ-compact and so C has a finite subfamily $\{V(x_i)\}$, where i = 1, 2, ..., n such that $X = \cup\{V(x_i)\}$. Suppose that the corresponding elements of Λ be $\{\alpha(x_i)\}$, where i = 1, 2, ..., n. Since Λ is well-ordered and $\{\alpha(x_i)\}$, where i = 1, 2, ..., n is finite, the largest element of $\{\alpha(x_i)\}$ exists. Suppose it is $\{\alpha(x_i)\}$. Then for $\gamma \geq \{\alpha(x_i)\}$, we have $\{x_\delta : \delta \geq \gamma\} \subset \bigcap_{i=1}^{n}(X - V(x_i)) = X - \bigcup_{i=1}^{n} V(x_i) = \varphi$, which is impossible. This shows that ξ has at least one αψ-adherent point in X.

(ii) ⇒ (i): Now it is enough to prove that each infinite subset has an αψ- complete accumulation point by utilizing Theorem 3.4. Suppose that S ⊂ X is an infinite subset of X. According to Zorns Lemma, the infinite set S can be well-ordered. This means that we can assume S to be a net with a domain which is a well-ordered index set. It follows that S has a αψ-adherent point z. Therefore z is an αψ-complete accumulation point of S. This shows that X is αψ-compact.

3.6. Theorem

A space X is αψ-compact if and only if each family of αψ-closed subsets of X with the finite intersection property has a nonempty intersection.

Proof
Straightforward.

3.7. Theorem
A space X is αψ-compact if and only if each filterbase in X has atleast one αψ-adherent point.
Proof.
Suppose that X is αψ-compact and $\Theta = \{F_\alpha : \alpha \in \Gamma\}$ is a filterbase in it. Since all finite intersections of F_α's are non-empty, it follows that all finite inter- sections of $\alpha\psi Cl(F_\alpha)$'s are also non-empty. Now it follows from Theorem 3.6 that $\cap_{\alpha\in\Gamma}\ \alpha\psi Cl(F_\alpha)$ is non-empty. This means that Θ has at least one αψ-adherent point. Now suppose Θ is any family of αψ-closed sets. Let each finite intersection be non-empty. The sets F_α with their finite intersection establish a filterbase Θ. Therefore, Θ αψ-accumulates to some point z in X. It follows that $z \in \cap_{\alpha\in\Gamma} F_\alpha$. Now we have, by Theorem 3.5, that X is αψ-compact.

3.8. Theorem
A space X is αψ-compact if and only if each filterbase on X, with atmost one αψ-adherent point, is αψ-convergent.
Proof.
Suppose that X is αψ-compact, x is a point of X, and Θ is a filterbase on X. The αψ-adherence of Θ is a subset of $\{x\}$. Then the αψ-adherence of Θ is equal to $\{x\}$ by Theorem 3.7. Assume that there exists a $V \in \alpha\psi O(X, x)$ such that for all $F \in \Theta, F \cap (X - V)$ is non-empty. Then $\Psi = \{F - V : F \in \Theta\}$ is a filterbase on X. It follows that the αψ-adherence of Ψ is non-empty. However, $\cap_{F\in\Theta}\ \alpha\psi Cl(F - V) \subset (\cap_{F\in\Theta}\ \alpha\psi Cl(F)) \cap (X - V) = \{x\} \cap (X - V) = \varphi$. But this is a contradiction. Hence, for each $V \in \alpha\psi O(X, x)$, there exists an $F \in \Theta$ with $F \subset V$. This shows that Θ αψ-converges to x.

To prove the converse, it suffices to show that each filterbase in X has at least one αψ-accumulation point. Assume that Θ is a filterbase on X with no αψ- adherent point. By hypothesis, Θ αψ-converges to some point z in X. Suppose F_α is an arbitrary element of Θ. Then for each $V \in \alpha\psi O(X, z)$, there exists an $F_\beta \in \Theta$ such that $F_\beta \subset V$ Since Θ is a filterbase, there exists a γ such that $F_\gamma \subset F_\alpha \cap F_\beta \subset F_\alpha \cap V$, where F_γ is non-empty. This means that $F_\alpha \cap V$ is non-empty for every $V \in \alpha\psi O(X, z)$ and correspondingly for each α, z is a point of $\alpha\psi Cl(F_\alpha)$. It follows that $z \in \cap_\alpha \alpha\psi Cl(F_\alpha)$. Therefore, z is a αψ-adherent point of Θ which is a contradiction. This shows that X is αψ-compact.

4. αψ-COMPACTNESS AND 1-LOWER AND 1-UPPER αψ-CONTINUOUS FUNCTIONS
In this section we further investigate properties of αψ-compactness by 1-lower and 1-upper αψ-continuous functions. We begin with the following notions and in what follows R denotes the set of real numbers.

4.1. Definition
A function $f : X \to R$ is said to be 1-lower (resp. 1-upper) αψ-continuous at the point y in X if for each $\lambda > 0$, there exists a αψ-open set $U(y) \in \alpha\psi(X, y)$ such that $f(x) > f(y) - \lambda$ (resp. $f(x) > f(y) + \lambda$) for every point x in U (y). The function fis 1-lower (resp. 1-upper) αψ-continuous in X if it has these properties for every point x of X.

4.2. Theorem
A function $f : X \to R$ is 1-lower αψ-continuous if and only if for each $\eta \in R$, the set of all x such that $f(x) \le \eta$ is αψ-closed.
Proof.
It is obvious that the family of sets $\tau = \{(\eta, \infty) : \eta \in R\} \cup R$ estab- lishes a topology on R. Then the function f is 1-lower αψ-continuous if and only if $f : X \to (R, \tau)$ is αψ-continuous. The interval $(-\infty, \eta]$ is closed in (R, τ). It follows that $f^{-1}((-\infty, \eta])$ is αψ-closed. Therefore, the set of all x such that $f(x) \le \eta$ is equal to $f^{-1}((-\infty, \eta])$ and thus, is αψ-closed.

4.3. Corollary
A subset S of X is αψ-compact if and only if the character- istic function X_S is 1-lower αψ-continuous.

4.4. Theorem
A function $f : X \to R$ is 1-upper αψ-continuous if and only if for each $\eta \in R$, the set of all x such that $f(x) \ge \eta$ is αψ-closed.

4.5. Corollary
A subset S of X is αψ-compact if and only if the character- istic function X_S is 1-upper αψ-continuous.

4.6. Theorem
If the function $F(x) = \sup_{i\in I} f_i(x)$ exists, where f_i, are 1-lower αψ-continuous functions from X into R, then F (x) is 1-lower αψ-continuous.
Proof.
Suppose that $\eta \in R$. Let $F(x) < \eta$ and therefore for every $i \in I$, $f_i(x) < \eta$. It is obvious that $\{x \in X : F(x) \le \eta\} = \cap_{i\in I} \{x \in X : f_i(x) \le \eta\}$. Since each f_i is 1-lower αψ-continuous, then each set of the form $\{x \in X : f_i(x) \le \eta\}$ is αψ-closed in X by Theorem 4.2. Since an arbitrary intersection of αψ-closed sets is αψ-closed, then F (x) is 1-lower αψ-continuous.

4.7. Theorem
If the function $G(x) = \inf_{i\in I} f_i(x)$ exists, where f_i, are 1-upper αψ-continuous functions from X into R, then G(x) is 1-upper αψ-continuous.

4.8. Theorem
Let $f : X \to R$ be a 1-lower αψ-continuous function,

where X is $\alpha\psi$-compact. Then f assumes the value $m = \inf_{x\in X} f(x)$.

Proof.

Suppose $\eta > m$. Since f is 1-lower $\alpha\psi$-continuous, then the set $K(\eta) = \{x \in X : f(x) \leq \eta\}$ is a non-empty $\alpha\psi$-closed set in X by the infimum property. Hence, the family $\{K(\eta) : \eta > m\}$ is a collection of non-empty $\alpha\psi$-closed sets with finite intersection property in X. By Theorem 3.6, this family has non-empty intersection. Suppose $z \in \cap_{\eta>m} K(\eta)$. Therefore, $f(z) = m$ as we wished to prove.

4.9. Theorem

Let $f : X \rightarrow R$ be a 1-upper $\alpha\psi$-continuous function, where X is a $\alpha\psi$-compact space. Then f attains the value $m = \sup_{x\in X} f(x)$.

Proof.

The proof is similar to the proof of Theorem 4.7. It should be noted that if a function f at the same time satisfies conditions of Theorem 4.6 and Theorem 4.7, then f is bounded and attains its bound.

5. $\alpha\psi$-COMPACTNESS AND $\alpha\psi$-CONTINUOUS MULTIFUNCTIONS

In this section, we give some characterizations of $\alpha\psi$-compact spaces by using lower (resp. upper) $\alpha\psi$-continuous multifunctions.

5.1. Definition

A multifunction $F : X \rightarrow Y$ is said to be lower (resp. up- per) $\alpha\psi$-continuous if $X - F^{-}(S)$ (resp. $F^{-}(S)$) is $\alpha\psi$-closed in X for each open (resp. closed) set S in Y.

For the following two lemmas we shall assume that if $\alpha\psi Cl(A) = A$, then A is is $\alpha\psi$-closed.

5.2. Lemma

For a multifunction $F : X \rightarrow Y$, the following statements are equivalent:

(1) F is lower $\alpha\psi$-continuous;

(2) If $x \in F^{-}(U)$ for a point x in X and an open set $U \subset Y$, then $V \subset F^{-}(U)$ for some $V \in \alpha\psi O(x)$;

(3) If $x \notin F^{+}(D)$ for a point x in X and a closed set $D \subset Y$, then $F^{+}(D) \subset K$ for some $\alpha\psi$-closed set K with $x \notin K$;

(4) $F^{-}(U) \in \alpha\psi O(X)$ for each open set $U \subset Y$.

Proof.

(1)⇒(4): Let U be any open set in Y. By (1), $X - F^{-}(U)$ is $\alpha\psi$-closed in X and hence $F^{-}(U) \in \alpha\psi O(X)$.

(4)⇒(2): Let U be any open set of Y and $x \in F^{-}(U)$. By (4), $F^{-}(U) \in \alpha\psi O(X)$. Put $V = F^{-}(U)$. Then $V \in \alpha\psi O(X)$ and $V \subset F^{-}(U)$.

(2)⇒(3): Let D be closed in Y and x $F^{+}(D)$. Then $Y - D$ is open in Y and $x \in X - F^{+}(D) = F^{-}(X - D)$. Therefore, There exists $V \in \alpha\psi O(x)$ such that $V \subset F^{-}(U)$. Now, put $K = X - V$, then $x \notin K$ is $\alpha\psi$-closed and $K = X - V \supset X - F^{-}(Y - D) = F^{+}(D)$.

(3)⇒(1): We show that $F^{+}(H)$ is $\alpha\psi$-closed for any closed set H of Y. Let H be any closed set and $x \notin F^{+}(H)$. By (3) there exists an $\alpha\psi$-closed set K such that $x \notin K$ and $F^{+}(H) \subset K$, hence $F^{+}(H) \subset \alpha\psi Cl(F^{+}(H)) \subset K$. Since $x \notin K$, we have $x \notin \alpha\psi Cl(F^{+}(H))$. This implies that $\alpha\psi Cl(F^{+}(H)) \subset F^{+}(H)$. In general, we have $F^{+}(H) \subset \alpha\psi Cl(F^{+}(H))$ and hence $F^{+}(H) = \alpha\psi Cl(F^{+}(H))$. Hence $F^{+}(H)$ is $\alpha\psi$-closed for any closed set H of Y.

5.3. Lemma

For a multifunction $F : X \rightarrow Y$, the following statements are equivalent:

(1) F is upper $\alpha\psi$-continuous;

(2) If $x \in F^{+}(V)$ for a point x in X and an open set $V \subset Y$, then $F(U) \subset V$ for some $U \in \alpha\psi O(x)$;

(3) If $x \notin F^{-}(D)$ for a point x in X and a closed set $D \subset Y$, then $F^{-}(D) \subset K$ for some $\alpha\psi$-closed set K with $x \notin K$;

(4) $F^{+}(U) \in \alpha\psi O(X)$ for each open set $U \subset Y$.

Proof.

(1)⇒(4): Let U be any open set in Y. Then $Y - U$ is closed. By (1), $F^{-}(Y - U) = X - F^{+}(U)$ is $\alpha\psi$-closed in X and hence $F^{+}(U) \in \alpha\psi O(X)$.

(4)⇒(2): Let V be any open set of Y and $x \in F^{+}(V)$. By (4), $F^{+}(V) \in \alpha\psi O(X)$. Put $U = F^{+}(V)$. Then $U \in \alpha\psi O(X)$ and $F(U) \subset V$.

(2)⇒(3): Let D be closed in Y and $x \notin F^{-}(D)$. Then $Y - D$ is open and $x \in X - F^{-}(D) = F^{+}(Y - D)$. By (2), there exists $U \in \alpha\psi O(X)$ such that $F(U) \subset Y - D$. Now, put $K = X - U$, then $x \notin K$, K is $\alpha\psi$-closed and $K = X - U \supset X - F^{+}(Y - D) = F^{-}(D)$.

(3)⇒(1): We show that $F^{-}(H)$ is $\alpha\psi$-closed for any closed set H of Y. Let H be any closed set and $x \notin F^{-}(H)$. By

(3), there exists an $\alpha\psi$-closed set K such that $x \notin K$ and $F^{-}(H) \subset K$, hence $F^{-}(H) \subset \alpha\psi Cl(F^{-}(H)) \subset K$. Since $x \notin K$, we have $x \notin \alpha\psi Cl(F^{-}(H))$. This implies that $\alpha\psi Cl(F^{-}(H)) \subset F^{-}(H)$. In general, we have $F^{-}(H) \subset \alpha\psi Cl(F^{-}(H))$ and hence $F^{-}(H) = \alpha\psi Cl(F^{-}(H))$. Hence $F^{-}(H)$ is $\alpha\psi$-closed for any closed set H of Y.

5.4. Theorem
The following two statements are equivalent for a space X:
(1) X is $\alpha\psi$-compact.
(2) Every lower $\alpha\psi$-continuous multifunction from X into the closed sets of a space assumes a minimal value with respect to set inclusion relation.

Proof.

(1)$\Rightarrow$(2): Suppose that F is a lower $\alpha\psi$-continuous multifunction from X into the closed subsets of a space Y. We denote the poset of all closed subsets of Y with the set inclusion relation " $\subseteq$ " by Λ. Now we show that $F : X \to \Lambda$ is a lower $\alpha\psi$-continuous function. We will show that $N = F^{-}(\{S \subset Y : S \in \Lambda$ and $S \subseteq C\})$ is $\alpha\psi$-closed in X for each closed set C of Y. Let $z \notin N$, then $F(z) = S$ for every closed set S of Y. It is obvious that $z \in F^{-}(Y - C)$, where $Y - C$ is open in Y. By Lemma 5.2 (2), we have $W \subset F^{-}(Y - C)$ for some $W \in \alpha\psi O(z)$. Hence $F(w) \cap (Y - C) = \varphi$, for each w in W. So for each w in W, $F(w) - C = \varphi$. Consequently, $F(w) - S = \varphi$ for every closed subset S of Y for which $S \subseteq C$. We consider that $W \cap N = \varphi$. This means that N is $\alpha\psi$-closed. Thus we observe that F assumes a minimal value.

(2)$\Rightarrow$(1): Suppose that X is not $\alpha\psi$-compact. It follows that we have a net $\{x_i : i \in \Lambda\}$, where Λ is a well-ordered set with no $\alpha\psi$-accumulation point by ([8], Theorem 3.2). We give Λ the order topology. Let $M_j = \alpha\psi Cl\{x_i : i \geq j\}$ for every j in Λ. We establish a multifunction $F : X \to \Lambda$ where $F(x) = \{i \in \Lambda : i \geq j_x\}$, j_x is the first element of all those js for which $x \notin M_j$. Since Λ has the order topology, $F(x)$ is closed. By the fact that $\{j_x : x \in X\}$ has no greatest element in Λ, then F does not assume any minimal value with respect to set inclusion. We now show that $F^{-}(U) \in \alpha\psi O(X)$ for every open set U in Λ. If $U = \Lambda$, then $F^{-}(U) = X$ which is $\alpha\psi$-open. Suppose that $U \subset \Lambda$ and $z \in F^{-}(U)$. It follows that $F(z) \cap U = \varphi$. Suppose $j \in F(z) \cap U$. This means that $j \in U$ and $j \in F(z) = \{i \in \Lambda : i \geq j_x\}$. Therefore $M_j \geq M_{j_x}$. Since $z = M_{j_x}$, then $z = M_j$. There exists $W \in \alpha\psi O(z)$ such that $W \cap \{x_i : i \in \Lambda\} = \varphi$. This means that $W \cap M_j = \varphi$. Let $w \in W$. Since $W \cap M_j = \varphi$, it follows that $w \notin M_j$ and since j_w is the first element for which $w \notin M_j$, then $j_w \leq j$. Therefore $j \in \{i \in \Lambda : i \geq j_w\} = F(w)$. By the fact that $j \in U$, then $j \in F(w) \cap U$. It follows that $F(w) \cap U = \varphi$ and therefore $w \in F^{-}(U)$. So we have $W \subset F^{-}(U)$ and thus $z \in W \subset F^{-}(U)$. Therefore $F^{-}(U)$ is $\alpha\psi$-open. This shows that F is lower $\alpha\psi$-continuous which contradicts the hypothesis of the theorem. So the space X is $\alpha\psi$-compact.

5.5. Theorem
The following two statements are equivalent for a space X:
(1) X is $\alpha\psi$-compact.
(2) Every upper $\alpha\psi$-continuous multifunction from X into the subsets of a T_1-space attains a maximal value with respect to set inclusion relation.

Proof.
Its proof is similar to that of Theorem 5.4.

The following result concerns the existence of a fixed point for multifunctions on $\alpha\psi$-compact spaces.

5.6. Theorem
Suppose that $F : X \to Y$ is a multifunction from an $\alpha\psi$-compact domain X into itself. Let $F(S)$ be $\alpha\psi$-closed for S being a $\alpha\psi$-closed set in X. If $F(x) = \varphi$ for every point $x \in X$, then there exists a nonempty, $\alpha\psi$-closed set C of X such that $F(C) = C$.

Proof.
Let $\Lambda = \{S \subset X : S = \varphi, S \in \alpha\psi C(X)$ and $F(S) \subset S\}$. It is evident that x belongs to Λ. Therefore $\Lambda = \varphi$ and also it is partially ordered by set inclusion. Suppose that $\{S_\gamma\}$ is a chain in Λ. Then $F(S_\gamma) \subset S_\gamma$ for each γ. By the fact that the domain is $\alpha\psi$-compact and by ([8], Theorem 3.3), $S = \cap_\gamma S_\gamma = \varphi$ and also $S \in \alpha\psi C(X)$. Moreover, $F(S) \subset F(S_\gamma) \subset S_\gamma$ for each γ. It follows that $F(S) \subset S_\gamma$. Hence $S \in \Lambda$ and $S = \inf\{S_\gamma\}$. It follows from Zorns lemma that Λ has a minimal element C. Therefore $C \subset \alpha\psi C(X)$ and $F(C) \subset C$. Since C is the minimal element of Λ, we have $F(C) = C$.

REFERENCES

[1] C. Berge, Topological Spaces, Macmillan, New York, 1963.

[2] P. Bhattacharya and B.K. Lahiri, Semi-generalized closed sets in topology, Indian J. Math., 29(3)(1987) 375-382.

[3] R. Devi, M. Parimala and Y. Beceren, On $\alpha\psi$-US space (Submitted).

[4] R. Devi, A. Selvakumar and M. Parimala, $\alpha\psi$-closed sets in Topological spaces (Submitted).

[5] M. Dunham and N. Levine, Further results on generalized closed sets in topol- ogy, Kyungpook Math. J., 20(1980), 169-175.

[6] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70(1963), 36-41.

[7] O. Njastad, On some classes of nearly open sets, Pacific J. Math., 15(1965), 961- 970.

[8] V. Popa, Some Properties of H-almost Continuous Multifunctions, Problem Mat., 10 (1988), 9-26.

[9] M.K.R.S. Veera kumar, Between semi-closed sets and semi-pre-closed sets, Rend. Istit. Mat. Univ. Trieste XXXII, (2000), 25-41.